HOMOTOPY THEORY OF CLASSIFYING SPACES OF COMPACT LIE GROUPS

by Stefan Jackowski, James McClure, and Bob Oliver

The basic problem of homotopy theory is to classify spaces and maps between spaces, up to homotopy, by means of invariants like cohomology. In the last decade some striking progress has been made with this problem when the spaces involved are classifying spaces of compact Lie groups. For example, it has been shown, for G connected and simple, that if two self maps of BG agree in rational cohomology then they are homotopic. It has also been shown that if a space X has the same mod p cohomology, cup product, and Steenrod operations as a classifying space BG then (at least if p is odd and G is a classical group) X is actually homotopy equivalent to BG after mod p completion. Similar methods have also been used to obtain new results on Steenrod's problem of constructing spaces with a given polynomial cohomology ring. The aim of this paper is to describe these results and the methods used to prove them.

The study of maps between classifying spaces goes back to Hurewicz [Hur, p. 219], who in 1935 showed that

$$[X, Y] \cong \operatorname{Hom}(\pi_1 X, \pi_1 Y)/\operatorname{Inn}(\pi_1 Y)$$

for any pair of aspherical spaces X and Y. This result might suggest a hope that all maps between the classifying spaces of any pair of compact Lie groups should be induced by homomorphisms. Much later however, in 1970, Sullivan [Su] provided the first counterexamples, by constructing maps (called "unstable Adams operations") which did not agree even in rational cohomology with any map induced by a homomorphism. Sullivan's work then led to a careful investigation by Hubbuck, Mahmud, and Adams ([Hub1], [AM], [Ad], [AM2]) of the effect that maps between classifying spaces could have in rational cohomology.

It was the proof of the Sullivan conjecture by Miller ([Mil], [DMN]) and Carlsson [Ca], and subsequent work by Lannes ([La1], [La2]) which provided the basis for the large amount of current activity and progress in this subject. It led, for example,

Second author partially supported by NSF grant DMS–9002525.

to the theorems of Dwyer & Zabrodsky [DZ] and Notbohm [No], which describe $[BP, BG]$ for any p-toral group P and any compact Lie group G. And that, together with a decomposition, due to the authors, of BG as a homotopy direct limit of BP's for certain p-toral $P \subseteq G$, has now yielded more general results about mapping sets $[BG, BG']$. In particular, we obtained a complete description of $[BG, BG]$ when G is connected and simple by showing that homotopy between maps is detected by rational cohomology in this case.

Decompositions of BG have also led to proofs of uniqueness theorems for classifying spaces in many cases. More precisely, for certain compact Lie groups G and certain primes p, any p-complete space X for which $H^*(X; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p)$ (as algebras over the Steenrod algebra) is homotopy equivalent to $BG\hat{_p}$. Results of this form have been proven by Dwyer, Miller & Wilkerson ([DMW1], [DMW2]), and more recently by Notbohm (unpublished).

Classifying spaces of compact connected Lie groups are notable by (in many cases) having polynomial algebras as their mod p cohomology, and in having loop spaces with the homotopy type of finite complexes. Hence the study of classifying spaces is closely connected to the problem of constructing other spaces with finite loop space, or with given polynomial algebras (with given Steenrod operations) as their mod p cohomology. The first "exotic" examples of such spaces were p-complete spaces whose $\mathbb{F}_p$-cohomology is polynomial on one generator of dimension $2n$ (i.e., deloopings of S^{2n-1} at p), constructed whenever $n|(p-1)$ by Sullivan [Su]. More examples were then given by Quillen [Q2, §10], Clark & Ewing [CEw], and Zabrodsky [Za]. The general decomposition of classifying spaces constructed in [JM2] then helped to motivate the development, by Lannes, Aguade [Ag], and Dwyer & Wilkerson [DW2], of more systematic ways of constructing new spaces of this type as homotopy direct limits of familiar spaces.

CONTENTS OF THE PAPER:

1. Prerequisites on homotopy direct limits

In this section we recall the construction of homotopy colimits of diagrams of spaces, a construction which plays a central role throughout this survey. The first approach is based on a construction of Segal [Se]. Let $\mathcal{C}$ be a small category (which can be a topological category, as defined in [Se]), and let $F : \mathcal{C} \to \mathcal{T}op$ be a covariant functor into the category of (compactly generated) topological spaces. Define a new category $\mathcal{C}_F$, whose objects are pairs (C, x) for $C \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(C)$, and where

$$\mathrm{Mor}_{\mathcal{C}_F}((C,x),(C',x')) = \{f \in \mathrm{Mor}_{\mathcal{C}}(C,C') : f(x) = x'\}.$$

We regard $\mathcal{C}_F$ as a topological category with the compact-open topology. Then the *homotopy colimit* of F, $\underrightarrow{\mathrm{hocolim}}\,(F)$, is defined to be the nerve (geometric realization) $B\mathcal{C}_F$. This construction is clearly functorial with respect to continuous natural transformations of functors on $\mathcal{C}$, and with respect to pullbacks of functors via continuous functors $\mathcal{C}' \to \mathcal{C}$. The natural transformation from F to the point functor induces a map $p_F : \underrightarrow{\mathrm{hocolim}}\,(F) \to B\mathcal{C}$.

More concretely, $\underrightarrow{\mathrm{hocolim}}\,(F)$ is a kind of bar construction: an identification space of the form

$$\underset{\mathcal{C}}{\underrightarrow{\mathrm{hocolim}}}(F) = \Bigl(\coprod_{n\geq 0}\ \coprod_{x_0\to\cdots\to x_n} F(x_0)\times\Delta^n\Bigr)/\sim,$$

where each face or degeneracy map between the $x_0 \to \cdots \to x_n$ gives rise to an obvious identification between the corresponding $F(x_0)\times\Delta^n$. Regarded in this fashion, the projection $p_F : \underrightarrow{\mathrm{hocolim}}\,(F) \to B\mathcal{C}$ is just induced by projecting each $F(x_0)\times\Delta^n$ to its second factor. Note that each fiber (point inverse) of p_F has the form $F(x)$ for some $x \in \mathrm{Ob}(\mathcal{C})$.

A different but equivalent approach to defining $\underrightarrow{\mathrm{hocolim}}\,(F)$ is given by Bousfield & Kan in [BK]. They consider functors $F : \mathcal{C} \to \mathcal{S}_*$ defined on an arbitrary small category $\mathcal{C}$ with values in the category of simplicial sets, and regard homotopy colimits as a special case of the more general construction of a balanced product (tensor product) $X \times_{\mathcal{C}} F$ of F with a contravariant functor $X : \mathcal{C} \to \mathcal{S}_*$. An elegant treatment of this viewpoint on homotopy colimits and limits is given in Hopkins [Ho].

Important examples of homotopy colimits include:

(a) **Double mapping cylinders** Let $\mathcal{C}$ be the "push-out" category:

$$\mathcal{C} = (y_1 \leftarrow x \to y_2).$$

Then the homotopy colimit of a functor $F : \mathcal{C} \to \mathcal{T}op$ is just the double mapping cylinder of the maps $F(y_1) \leftarrow F(x) \to F(y_2)$.

(b) **Mapping telescopes** Let $\mathcal{C} = \mathbb{N}$ be the category obtained from the directed set of natural numbers. Then for any functor $F : \mathcal{C} \to \mathcal{T}op$, there is a homotopy equivalence $\underrightarrow{\mathrm{hocolim}}\,(F) \simeq \mathrm{Tel}(F(1) \to F(2) \to \dots)$ (cf. [BK, XII.3.5]). (Note however that this is not a homeomorphism.)

(c) **The Borel construction** Fix a group G, and let $\mathcal{O}_1(G)$ be the category of G: the category with single object $*$, and with $\mathrm{End}(*) \cong G$. (This notation is motivated by that used later for the orbit categories $\mathcal{O}(G)$ and $\mathcal{O}_p(G)$.) Then a functor $F : \mathcal{O}_1(G) \to \mathcal{T}op$ is just a G-space $F(*)$; and (with an appropriate definition of EG) $\underrightarrow{\mathrm{hocolim}}\,(F)$ is homeomorphic to the Borel construction $EG \times_G F(*)$. If we allow topological categories, then the last equivalence remains true for continuous actions of topological groups.

As noted above, the natural transformation into the constant (point) functor induces a map $p_F : \underrightarrow{\mathrm{hocolim}}\,(F) \to B\mathcal{C}$, whose fibers all have the form $F(-)$. When $\mathcal{C}$ is a discrete category, the Leray spectral sequence for p_F provides a means of calculating the cohomology of $\underrightarrow{\mathrm{hocolim}}\,(F)$. If $f : X \to Y$ is any map (satisfying certain simple conditions), then the Leray spectral sequence for f is a spectral sequence of the form

$$E_2^{pq} = H^p(Y; \mathcal{H}^q(f)) \Rightarrow H^{p+q}(X).$$

Here, $\mathcal{H}^q(f)$ is a sheaf over Y, whose stalk over $y \in Y$ is $H^q(f^{-1}y)$ (cf. [Br, §IV.6] for details). When this is applied to the map p_F, then the cohomology groups for these sheaves over $B\mathcal{C}$ are isomorphic to the higher derived functors of inverse limits of the (contravariant) functors

$$H^q(F(-)) : \mathcal{C} \to \mathcal{A}b.$$

(This can be shown using Lemma 4.2 below.) We thus get the following theorem.

Theorem 1.1. *For every covariant functor $F : \mathcal{C} \to \mathcal{T}op$, there is a spectral sequence*

$$E_2^{pq} = \underset{\mathcal{C}}{\underleftarrow{\lim}}{}^p(H^q(F(-))) \Rightarrow H^{p+q}(\underrightarrow{\mathrm{hocolim}}(F)).$$

For a more explicit statement, and a proof of Theorem 1.1, see [BK, XII.4.5]. Higher derived functors of inverse limits play an important role throughout this subject, and will be discussed in more detail in Section 4. Note that in example (a) the spectral sequence reduces to the usual Meyer-Vietoris exact sequence; in (b) to the Milnor lemma [Mln]; and in (c) to the Serre spectral sequence of the fibration $F(*) \to EG \times_G F(*) \to BG$.

Our own interest in homotopy colimits stems from their use as a tool for constructing maps and homotopies. For example, we will be constructing and classifying maps $BG \to X$, for appropriate compact Lie groups G, by decomposing BG as a homotopy colimit of simpler spaces, and then comparing maps defined on the homotopy colimit with maps defined on the individual spaces. This is done using the next two theorems.

Theorem 1.2. *Fix a discrete category $\mathcal{C}$, and a functor $F : \mathcal{C} \to \mathcal{T}op$. Let X be any other space, and consider the map*

$$R : [\underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}(F), X] \to \underset{\overleftarrow{c\in\mathcal{C}}}{\lim}[F(c), X]$$

defined by restriction. Fix an element $\hat{f} = (f_c)_{c\in\mathcal{C}} \in \underleftarrow{\lim}[F(-), X]$; and define contravariant functors $\alpha_n : \mathcal{C} \to \mathcal{A}b$ (all $n \geq 1$) by setting

$$\alpha_n(c) = \pi_n(\mathrm{map}(F(c), X)_{f_c})$$

for all $c \in \mathrm{Ob}(\mathcal{C})$. Then $\hat{f} \in \mathrm{Im}(R)$ if the groups $\underleftarrow{\lim}^{n+1}(\alpha_n)$ vanish for all $n \geq 1$; and $R^{-1}(\hat{f})$ contains at most one element if the groups $\underleftarrow{\lim}^{n}(\alpha_n)$ vanish for all $n \geq 1$.

Theorem 1.2 can be proved by a standard obstruction-theory argument in which one inducts over the simplicial filtration of the homotopy colimit. However, one must be quite careful about basepoints, and about what happens when $\alpha_1(c)$ is not abelian. These points are handled in detail by Wojtkowiak in [Wo].

Theorem 1.2 is a special case of the following spectral sequence, constructed by Bousfield & Kan ([BK, XII.4.1 & XI.7.1] and [Bo]). As we will show later, this can often be used for explicit calculations involving $R^{-1}(\hat{f})$, even when the obstructions of Theorem 1.2 do not all vanish.

Theorem 1.3. *Let F, R, and $\hat{f}$ be as in Theorem 1.2. Let $\mathrm{map}(\underrightarrow{\mathrm{hocolim}}\,(F), X)_{\hat{f}}$ be the union of the components of the mapping space lying in $R^{-1}(\hat{f})$. Define $\alpha_n : \mathcal{C} \to \mathcal{A}b$ by setting*

$$\alpha_n(c) = \pi_n(\mathrm{map}(F(c), X)_{f_c})$$

for all c. Then there is a spectral sequence of the form

$$E^2_{-p,q} = \underset{\overleftarrow{\mathcal{C}}}{\lim}{}^p(\alpha_q) \Rightarrow \pi_{q-p}\Big(\mathrm{map}(\underrightarrow{\mathrm{hocolim}}(F), X)_{\hat{f}}\Big)$$

which converges strongly if there is an N such that $\underleftarrow{\lim}^p(\alpha_q)$ vanishes for all q and all $p > N$.

Bousfield & Kan construct the spectral sequence of Theorem 1.3 as a special case of a spectral sequence for a homotopy inverse limit (and with a less restrictive convergence condition: see [BK, XI.7.1 and IX.5.3]). A more direct proof can be given as follows. The skeletal filtration of the homotopy colimit induces a tower of fibrations of mapping spaces, and the spectral sequence of Theorem 1.3 is the spectral sequence of this tower of fibrations. There is an explicit chain complex which calculates $\varprojlim^i$ (see Lemma 4.2 below), and by using arguments similar to Wojtkowiak's in [Wo] one can show that the E_2-term in the spectral sequence is the homology of this chain complex.

We have seen above that, for any discrete group G, BG is the homotopy direct limit of the functor $F : \mathcal{O}_1(G) \to \mathcal{T}op$ which sends the single object $*$ to a point. Theorem 1.3 therefore gives a spectral sequence

$$E^2_{-p,q} = \varprojlim_{\mathcal{O}_1(G)}{}^p\, (\pi_q(X)) \cong H^p(G; \pi_q(X)) \Rightarrow \pi_{q-p}(\mathrm{map}(BG, X)),$$

and one might hope that this would be a useful tool for proving results about maps out of BG. The problem is that this spectral sequence will in general have lots of differentials, and these seem to be impossible to compute even in the simplest cases: e.g., when $G = \mathbb{Z}/2$ and $X = BSO(3)$. Our solution to this problem is to consider more complicated decompositions of BG for which the spectral sequence of Theorem 1.3 collapses.

2. Homotopy decompositions of classifying spaces

Most recent results involving the homotopy theory of the classifying spaces of all but the simplest compact Lie groups are based on decompositions of BG as direct limits of simpler spaces—usually the classifying spaces of other groups. The first such decompositions were given by mapping telescopes; while later results involved taking homotopy colimits of more complicated diagrams. In particular, there are now two general ways to decompose an arbitrary BG, at a given prime p, in such a way that the pieces are classifying spaces of subgroups of G. These two general decompositions have both proved quite useful for answering different types of questions about BG.

The idea of decomposing the classifying space of a Lie group into a telescope of the classifying spaces of its subgroups goes back to Adams [Ad], who approximated the classifying spaces of p-toral groups via their finite p-subgroups. This idea was developed further by Feshbach [Fe], who in his work on the Segal conjecture for compact Lie groups was interested in extensions of tori by finite groups, and used their finite subgroups for a similar approximation. More precisely, Feshbach showed that for any compact Lie group G whose identity component G_0 is a torus, there exists a chain of finite subgroups $F_1 \subseteq F_2 \subseteq \cdots \subseteq G$ such that the natural map $\mathrm{Tel}(BF_i) \to BG$ induces an isomorphism in cohomology with arbitrary finite coefficients.

The approach of Adams and Feshbach was extended to arbitrary compact Lie groups by Friedlander & Mislin ([FM1], [FM2], [FM3]). For any compact Lie group G and any prime p not dividing the order of $\pi_0(G)$, they found a chain of finite groups $F_1 \subseteq F_2 \subseteq \dots$ (not necessary subgroups of G), and a compatible family of maps $f_i : BF_i \to BG$ (not necessarily induced by homomorphisms), such that the limit map $f : \mathrm{Tel}(BF_i) \to BG$ induces an isomorphism in cohomology with coefficients in any finite $\pi_0(G)$-module of order not divisible by p. For example, when $G = U(n)$ (and p is any prime), then this takes the form of a map

$$\mathrm{Tel}(BGL_n(\mathbb{F}_{p^m})) \simeq BGL_n(\bar{\mathbb{F}}_p) \to BU(n)$$

(where $\bar{\mathbb{F}}_p$ denotes the algebraic closure of $\mathbb{F}_p$).

The use of homotopy colimit decompositions other than telescopes was first introduced by Dwyer, Miller, & Wilkerson for the particular case of $G = SO(3)$. Consider the following two diagrams, where $D(8)$ denotes the dihedral subgroup of order 8, Σ_4 the octahedral group, and $O(2)$ the normalizer of the maximal torus $SO(2)$:

$$\begin{array}{ccc} SO(3)/D(8) & \longrightarrow & SO(3)/O(2) \\ \downarrow & & \\ SO(3)/\Sigma_4 & & \end{array} \qquad (1) \qquad\qquad \begin{array}{ccc} BD(8) & \longrightarrow & BO(2) \\ \downarrow & & \\ B\Sigma_4 & & \end{array} \qquad (2)$$

It is not hard to check that the homotopy pushout of diagram (1) is $\mathbb{F}_2$-acyclic. Diagram (2) is obtained by applying the Borel construction $EG \times_G -$ to (1); and hence its homotopy pushout is $\mathbb{F}_2$-homology equivalent to $EG \times_G pt = BG = BSO(3)$.

There is another, closely related, decomposition of $BSO(3)$ at the prime $p = 2$, which is taken over a more complicated category, but more useful because the pieces are simpler. Consider the category $\mathcal{C}$ which consists of two objects, 1 and 2, and where $\mathrm{End}(1) \cong \Sigma_3$ (the symmetric group of order 6), $\mathrm{End}(2) \cong 1$, $|\,\mathrm{Mor}(1,2)| = 3$ (the maps being permuted by $\mathrm{End}(1)$ in the obvious way), and $\mathrm{Mor}(2,1) = \emptyset$. Consider the functor $F : \mathcal{C} \to \mathcal{T}op$ given by

$$F(1) = ESO(3)/((\mathbb{Z}/2)^2) \simeq B(\mathbb{Z}/2)^2 \text{ and } F(2) = ESO(3)/O(2) \simeq BO(2)$$

(with the obvious induced maps). In other words, F defines a diagram of the following form:

$$F = \Big(\Sigma_3 \circlearrowleft ESO(3)/((\mathbb{Z}/2)^2) \Rrightarrow ESO(3)/O(2)\Big) \qquad (3)$$

A cohomology computation, using Theorem 1.1, shows that there is an $\mathbb{F}_2$-homology equivalence $\underrightarrow{\mathrm{hocolim}}\,(F) \to BSO(3)$. Alternatively, it can be shown directly that

$\operatorname{hocolim}_{\longrightarrow}(F)$ must be homotopy equivalent to the homotopy pushout of Dwyer, Miller, & Wilkerson in (2) above. For more information on converting homotopy colimits to homotopy pushouts in certain cases, cf. Słomińska [Sl].

This approximation of $BSO(3)$ at the prime 2 can, unlike the homotopy-pushout decomposition, be generalized to all compact Lie groups and all primes. In fact, this can be done in two essentially different ways, which will be described in the next two subsections.

2a. Approximation via p-toral subgroups.

Recall that a compact Lie group P is p-toral if its identity component P_0 is a torus and P/P_0 is a p-group. The p-toral groups play a similar role for compact Lie groups to that played by p-groups for finite groups. For example, the "Sylow theorem" says that any compact Lie group G contains a unique conjugacy class of maximal p-toral subgroups, that any p-toral subgroup is contained in a maximal one, and that a p-toral subgroup $P \subseteq G$ is maximal if and only if $p \nmid \chi(G/P)$. For any given maximal torus $T \subseteq G$ and any Sylow p-subgroup W_p of the Weyl group $W = N(T)/T$, the extension

$$1 \to T \to N_p(T) \to W_p \to 1$$

is a maximal p-toral subgroup.

For any G, let $\mathcal{O}(G)$ be the orbit category: the (topological) category whose objects are the orbits G/H for closed subgroups $H \subseteq G$, and where $\operatorname{Mor}_{\mathcal{O}(G)}(G/H, G/K)$ is the space of all G-maps. Let $\mathcal{O}_p(G)$ be the full subcategory of orbits G/P for p-toral $P \subseteq G$. We want to approximate BG as a homotopy colimit taken over $\mathcal{O}_p(G)$ or some appropriate subcategory.

When G is finite, then this is motivated in part by the theorem of Cartan & Eilenberg [CE, Theorem XII.10.1], which says that for any $\mathbb{Z}_{(p)}[G]$-module A,

$$H^*(BG; A) \cong \lim_{\substack{\longleftarrow \\ G/P \in \mathcal{O}_p(G)}} H^*(BP; A). \tag{4}$$

More precisely, $H^*(BG; A)$ is the inverse limit of the functor $H^*(EG \times_G -; A)$ (defined on $\mathcal{O}_p(G)$). Formula (4) suggests that BG is $\mathbb{F}_p$-homology equivalent to the homotopy direct limit of the BP's—and this can be confirmed by showing that the higher limits of the system in (4) all vanish (see [Mis, §2]).

The obvious way to extend this to the case of compact Lie groups is to take the limit of the BP's taken over all p-toral $P \subseteq G$ (i.e., taken over $\mathcal{O}_p(G)$). The main problem with doing this is that we want to work over a discrete category, and preferably a finite one. For example, we saw that when $G = SO(3)$ and $p = 2$, it suffices to consider the 2-toral subgroups $(\mathbb{Z}/2)^2$ and $O(2)$.

Definition 2.1. *A closed subgroup $P \subseteq G$ is called* p-stubborn *if it is p-toral, if $N(P)/P$ is finite, and if $N(P)/P$ does not contain any nontrivial normal p-subgroup. We let $\mathcal{R}_p(G)$ denote the full subcategory of $\mathcal{O}_p(G)$ whose objects are those G/P for p-stubborn $P \subseteq G$.*

For any G and p, the category $\mathcal{R}_p(G)$ is finite in the sense that all morphism sets are finite, and there are finitely many isomorphism classes of objects (i.e., finitely many conjugacy classes of p-stubborn subgroups). When $G = SO(3)$, it is easy to see that the only 2-stubborn subgroups are the groups $(\mathbb{Z}/2)^2$ and $O(2)$ already mentioned above. When $p \nmid |W|$ ($W = N(T)/T$), then the only p-stubborn subgroups of G are the maximal tori: i.e., $\mathcal{R}_p(G)$ is equivalent to the category with one object with automorphism group W.

For finite G, the p-stubborn subgroups are exactly the same as the p-radical subgroups which play a role in the classification of finite simple groups. They have also been used by Puig [Pu] for computing cohomology groups. In fact, Puig showed that $H^*(G; A)$ (for any $\mathbb{Z}_{(p)}[G]$-module A) is the inverse limit of the $H^*(P; A)$ taken over all essential p-subgroups $P \subseteq G$: where the essential subgroups form a proper subclass of the radical (stubborn) subgroups.

The next theorem is our main approximation result for classifying spaces using p-stubborn subgroups. Here, G-$\mathcal{T}op$ denotes the category of spaces with G-action.

Theorem 2.2. *Fix any G and p, and let $\mathcal{I} : \mathcal{R}_p(G) \hookrightarrow G\text{-}\mathcal{T}op$ be the inclusion. Then the map*

$$\underset{\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}(EG \times_G \mathcal{I}) \cong EG \times_G \Big(\underset{\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}}(\mathcal{I})\Big) \longrightarrow BG$$

induces an isomorphism of cohomology with $Z_{(p)}$-coefficients.

Note, in Theorem 2.2, that the individual terms in the homotopy colimit have the form $EG \times_G (G/P) \cong EG/P \simeq BP$ for p-stubborn subgroups $P \subseteq G$.

When $p \nmid |W|$, then by the above remarks (G/T is the only object in $\mathcal{R}_p(G)$), the homotopy colimit has the form $EW \times_W BT \simeq BN(T)$. So in this case, Theorem 2.2 recovers a well known result of Borel.

As discussed earlier (diagrams (1) and (2) above), the original decomposition of $BSO(3)$ as a homotopy pushout of certain orbits was found by first constructing an $\mathbb{F}_2$-acyclic $SO(3)$-complex as a homotopy pushout, and then applying the Borel construction. Similarly, when proving Theorem 2.2, it suffices to show that the homotopy colimit $\underrightarrow{\mathrm{hocolim}}_{\mathcal{R}_p(G)}(\mathcal{I})$ is $Z_{(p)}$-acyclic. The difficult part of doing this is the following result about transformation groups:

Theorem 2.3. *For any compact Lie group G and any prime p, there exists a finite* *dimensional $\mathbb{F}_p$-acyclic G-CW-complex X such that all isotropy groups G_x for $x \in X$ are p-stubborn.*

Theorem 2.2 follows from Theorem 2.3 by first constructing a map $X \to \operatorname{hocolim}_{\longrightarrow}(\mathcal{I})$, and then showing that it is an $\mathbb{F}_p$-homology equivalence by showing that it restricts to an $\mathbb{F}_p$-homology equivalence on all fixed point sets of isotropy subgroups. The key observation when doing this is that for any full subcategory $\mathcal{C} \subseteq \mathcal{O}(G)$, $(\operatorname{hocolim}_{\longrightarrow \mathcal{C}}(\mathcal{I}))^H$ is contractible for all G/H in $\mathcal{C}$. For the details of these arguments, see [JMO, Section 1].

The G-CW-complex X of Theorem 2.3 is constructed in two steps. One first constructs a G-CW-complex all of whose isotropy subgroups are p-toral, and then one eliminates those which are not p-stubborn (this is the origin of the term p-stubborn). The proof parallels Oliver's construction of fixed-point free actions on acyclic complexes [Ol1], and also uses some arguments from his proof of the Conner conjecture [Ol2].

2b. Approximation via centralizers of elementary abelian p-subgroups.

Diagram (3) above admits a second generalization to arbitrary compact Lie groups. For any G, we let $\mathcal{A}_p(G)$ be the category introduced by Quillen [Q1], whose objects are nontrivial elementary abelian p-subgroups, and whose morphisms are restrictions of inner automorphisms of G. Note that $SO(3)$ has precisely two conjugacy classes of nontrivial elementary abelian subgroups, and that $C(\mathbb{Z}/2) \cong O(2)$ and $C((\mathbb{Z}/2)^2) \cong (\mathbb{Z}/2)^2$. Here, $C(-)$ denotes the centralizer of a subgroup in $SO(3)$. Thus, diagram (3) above can be considered as being indexed over $\mathcal{A}_2(SO(3))$, and defined by the functor which sends every elementary abelian 2-subgroup to the classifying space of its centralizer. This viewpoint has been generalized in [JM2] to arbitrary compact Lie groups, as follows.

On the opposite category $\mathcal{A}_p^o(G)$, we define a functor

$$EG \times_G (G/C(-)) : \mathcal{A}_p^o(G) \to \mathcal{T}op$$

which assigns to every elementary abelian subgroup $A \in \mathcal{A}_p^o(G)$ the space $EG \times_G (G/C(A)) \simeq BC(A)$. Projection maps induce a natural transformation a from the functor $EG \times_G (G/C(-))$ to the constant functor BG.

Theorem 2.4. [JM2, Theorem 1.3] *The projection maps $EG \times_G (G/C(-)) \twoheadrightarrow BG$ induce a map*

$$\alpha_G : \operatorname*{hocolim}_{\substack{\longrightarrow \\ \mathcal{A}_p^o(G)}}(EG \times_G (G/C(-))) \longrightarrow BG$$

which is an $\mathbb{F}_p$-cohomology isomorphism.

The proof in [JM2] of Theorem 2.4 is based on the Bousfield-Kan spectral sequence (Theorem 1.1 above) for computing the cohomology of the homotopy colimit. One first proves, using the Becker-Gottlieb transfer and Feshbach's double coset formula, that

$$\mathrm{Res} : H^*(BG; \mathbb{F}_p) \to \lim_{\substack{\longleftarrow \\ \mathcal{A}_p^o(G)}} H^*(EG \times_G G/C(-); \mathbb{F}_p)$$

is an isomorphism. It then remains to show that the higher inverse limits

$$\lim_{\substack{\longleftarrow \\ \mathcal{A}_p^o(G)}}{}^i \left(H^*(EG \times_G G/C(-); \mathbb{F}_p)\right)$$

all vanish for $i \geq 1$. This is again a consequence of the existence of transfer maps for the functor $H^*(EG \times_G G/C(-); \mathbb{F}_p)$. It turns out to be a Mackey functor in a sense close to the one introduced by Dress (cf. [Dr, §1]). A general, algebraic theorem is proven in [JM2] (Theorem 4.6 below), which says that all higher inverse limits vanish for such functors.

Theorem 2.4 was the starting point for a generalization obtained by Dwyer & Wilkerson [DW1]. The key observation for their generalization was a theorem of Dwyer & Zabrodsky [DZ], stated as Theorem 3.1 below. In particular, their theorem says that when A is an elementary abelian p-group and G is a compact Lie group, then conjugacy classes of homomorphisms $A \to G$ can be identified with homotopy classes of maps $BA \to BG$. Inclusions correspond to the maps $f : BA \to BG$ such that the induced homomorphism $H^*(f^*; \mathbb{F}_p)$ makes $H^*(BA; \mathbb{F}_p)$ into a finitely generated $H^*(BG; \mathbb{F}_p)$-module. Moreover, for any inclusion $i : A \hookrightarrow G$, there is a natural map $e_i : BC_G(A) \to \mathrm{map}(BA, BG)_{Bi}$ into the connected component containing the map Bi; and e_i becomes a homotopy equivalence after p-completion.

These remarks suggest a definition of the category of "elementary abelian subgroups of an arbitrary space". For any X, let $\mathcal{A}_p^o(X)$, be the category whose objects are homotopy classes of those maps $f : BA \to X$, for which A is a nontrivial elementary abelian p-group and $H^*(f; \mathbb{F}_p)$ makes $H^*(BA; \mathbb{F}_p)$ into a finitely generated $H^*(X; \mathbb{F}_p)$-module. Morphisms in $\mathcal{A}_p^o(X)$ are homotopy classes $B\varphi : BA' \to BA$ of maps over X. A functor $\mathcal{A}_p^o(X) \to \mathcal{T}op$ is defined, which assigns to every map $f : BA \to X$ its connected component $\mathrm{map}(BA, X)_f$ in the mapping space. When $X = BG$ for a compact Lie group G, then this functor is naturally $\mathbb{F}_p$-homology equivalent (under the equivalence of categories $\mathcal{A}_p^o(BG) = \mathcal{A}_p^o(G)$) to the functor $EG \times_G G/C(-)$ defined above. This motivates the following theorem:

Theorem 2.5. (Dwyer & Wilkerson [DW1]) *Let X be a space whose cohomology $H^*(X; \mathbb{F}_p)$, as an unstable algebra over the Steenrod algebra, is isomorphic to the* *subalgebra of elements fixed by an action of a finite group on $H^*(BT;\mathbb{F}_p)$ for some torus T. Then the evaluation map induces an $\mathbb{F}_p$-homology equivalence:*

$$\alpha_X : \underset{\mathcal{A}_p^o(G)}{\operatorname{hocolim}_{\longrightarrow}}(\operatorname{map}(B-,X)_-) \to BG$$

The proof of Theorem 2.5 follows from a purely algebraic result which involves unstable modules over the Steenrod algebra. Similarly to the proof in [JM2] of Theorem 2.4 above, the proof of Dwyer & Wilkerson is based on the existence of certain transfer maps.

Dwyer & Wilkerson found various other conditions on X which imply that α_X is an $\mathbb{F}_p$-homology equivalence. They also noted (at the end of [DW1]) that this is not the case when $p = 2$ and $X = B\mathbb{Z}/2 \vee B\mathbb{Z}/2$. It would be quite interesting to have a characterization of the spaces X for which α_X is an $\mathbb{F}_p$-homology equivalence, since such spaces could be expected to have many of the good properties of classifying spaces.

3. Maps between classifying spaces

The title of this section was first used by Adams & Mahmud [AM], and since then by many authors. At the beginning of their paper, Adams and Mahmud explained their interest in the subject as follows: "Let G and G' be compact connected Lie groups ... The object of this paper is to study maps $f : BG \to BG'$, since this seems to be a case of the homotopy classification problem which is both particularly interesting and particularly favourable." This section provides an update of this problem, and a confirmation of the prediction of Adams and Mahmud.

The results of Adams & Mahmud, both in [AM] and in their later papers [Ad] and [AM2], classified maps $BG \to BG'$ only up to what was detected by rational cohomology or K-theory, and constructed maps only after inverting certain primes in BG'. The main tools which now make it possible to study such maps more precisely are a series of consequences of the proof of the Sullivan conjecture ([Mil], [Ca], [La2]). The results of principal importance are those of Dwyer & Zabrodsky [DZ], and their extension by Notbohm [No].

As mentioned in the introduction, the construction by Sullivan of unstable Adams operations destroyed the hope that all maps $BG{\to}BG'$ might be homotopic to maps induced by homomorphisms. In contrast, the next theorem says that in the special case when G is p-toral (i.e., an extension of a torus by a finite p-group), then not only are all maps induced by homomorphisms, but all homotopies between maps are induced by conjugations, and the higher homotopies also have a group theoretic description.

For any G and G', define $\operatorname{Rep}(G,G') = \operatorname{Hom}(G,G')/\operatorname{Inn}(G')$; i.e., the set of homomorphisms $\rho : G \to G'$ modulo conjugation in G'. Since $B\alpha \simeq \operatorname{Id}_{BG'}$ for any

$\alpha \in \mathrm{Inn}(G')$, there is a well defined map

$$B : \mathrm{Rep}(G, G') \to [BG, BG']$$

defined by sending ρ to $B\rho$.

Theorem 3.1. (Dwyer & Zabrodsky [DZ], Notbohm [No]) *Fix a p-toral group P, and any compact Lie group G. Then*

$$B : \mathrm{Rep}(P, G) \xrightarrow{\cong} [BP, BG]$$

is a bijection, and the completion map

$$[BP, BG] \rightarrowtail [BP, BG\hat{_p}]$$

is injective. Furthermore, for any $\rho : P \to G$,

$$\pi_* \left(\mathrm{map}(BP, BG\hat{_p})_{B\rho}\right) \cong \pi_* \left(\mathrm{map}(BP, BG)_{B\rho}\right)\hat{_p} \cong \pi_* \left(BC_G(\mathrm{Im}(\rho))\right) \otimes \hat{\mathbb{Z}}_p.$$

Theorem 3.1 was shown by Dwyer & Zabrodsky when P is a finite p-group, and extended by Notbohm to the case when P is p-toral.

The homotopy isomorphisms in Theorem 3.1 are induced by the map $BC_G(\mathrm{Im}(\rho)) \to \mathrm{map}(BP, BG)$, which is in turn the adjoint to the map $B\left(\mathrm{Im}(\rho) \times P \to G\right)$.

The following corollary to Theorem 3.1 will also be useful in the discussion below.

Corollary 3.2. (Notbohm [No]) *Let T be a torus, and let G be any compact connected Lie group. Then two maps $f, f' : BT \to BG$ are homotopic if and only if $H^*(f; \mathbb{Q}) = H^*(f'; \mathbb{Q})$.*

Again let G and G' be compact Lie groups, where G' is connected. For each prime p and each p-toral subgroup $P \subseteq G$ (in particular, for each p-stubborn subgroup), Theorem 3.1 describes the set of homotopy classes of maps $BP \to BG'$, as well as the homotopy type of each component of $\mathrm{map}(BP, BG'\hat{_p})$. This is exactly what is needed when applying Theorem 1.2 or 1.3 to describe the set of homotopy classes of maps from the homotopy colimit of the BP's to $BG'\hat{_p}$; and hence (using Theorem 2.2) from BG to $BG'\hat{_p}$.

We are now ready to outline our general strategy for applying these techniques to construct and classify maps $BG \to BG'$, when G and G' are compact Lie groups and G' is connected. Of particular interest are the cases where G is connected or finite. As will be seen shortly, the condition that G' be connected is needed since we have to be able to apply Sullivan's arithmetic pullback square for localizations and completions of BG'.

Now fix G and G' as above. Choose maximal tori $T \subseteq G$ and $T' \subseteq G'$, and let $W = N(T)/T$ and $W' = N(T')/T'$ denote the Weyl groups. The construction and classification of maps $BG \to BG'$ is carried out in three steps:

Step 1 Admissible maps.

For the purposes of this survey, we define an admissible map to be a homomorphism $\phi : T \to T'$, for which there exists a homomorphism $\bar{\phi} : W \to W'$ such that ϕ is $\bar{\phi}$-equivariant. As noted below, this is more restrictive than the definition of Adams & Mahmud in [AM]. But the ones we consider are the only ones which can induce maps $BG \to BG'$ which are defined globally (rather than after a finite localization).

Theorem 3.3. (Adams & Mahmud [AM, Corollary 1.11]) *For any $f : BG \to BG'$, there exists an admissible map $\phi : T \to T'$ such that the following square commutes up to homotopy:*

$$\begin{array}{ccc} BT & \xrightarrow{B\phi} & BT' \\ \Big\downarrow{\scriptstyle \mathrm{incl}} & & \Big\downarrow{\scriptstyle \mathrm{incl}} \\ BG & \xrightarrow{f} & BG' \end{array} \tag{1}$$

If $\phi' : T \to T'$ is any other homomorphism for which this square commutes up to homotopy, then $\phi' = w \circ \phi$ for some element $w \in W'$. And conversely, if $\phi : T \to T'$ is any admissible map, then for some n there is a map $f : BG \to BG'[1/n]$ which makes (1) commute up to homotopy.

In fact, Adams and Mahmud showed only that the square in Theorem 3.3 commutes in rational cohomology. But by Notbohm's lemma (Corollary 3.2 above), this is equivalent to its being homotopy commutative.

Adams & Mahmud in [AM] actually define an admissible map to be a linear map $\tilde{\phi} : \tilde{T} \to \tilde{T}'$ between the universal covers, such that for some $n \geq 1$, $n \cdot \tilde{\phi}$ covers a homomorphism $\phi : T \to T'$ which is admissible in the sense defined above. Using this definition, their theorem says that square (1) above induces a one-to-one correspondence between

(i) homomorphisms $H^*(BG';\mathbb{Q}) \to H^*(BG;\mathbb{Q})$ which are induced by maps $f : BG \to BG'[1/n]$ (for some $n \geq 1$); and

(ii) W'-conjugacy classes of admissible maps ϕ. $\tilde{\phi} : \tilde{T} \to \tilde{T}'$.

We are now left with the problem: given an admissible map $\phi : T \to T'$, is it induced by some $f : BG \to BG'$? And if so, by how many homotopy classes of maps? Since we want to answer these by replacing BG by a p-local approximation, we must first consider extensions of ϕ to maps $BG \to BG'\hat{{}_p}$ for the individual primes p. The following proposition describes how this is done.

Proposition 3.4. *Fix an admissible map $\phi : T \to T'$, and let $[BG, BG']_\phi$, and $[BG, BG'\hat{{}_p}]_\phi$, be the sets of homotopy classes of maps whose restrictions to BT are* *homotopic to $B\phi$. Then*

$$[BG, BG']_\phi \cong \prod_{p\big||W|} [BG, BG'{}^{\hat{}}_p]_\phi.$$

The proof of Proposition 3.4 is based on Sullivan's arithmetic pullback square (applied to the simply connected space BG'), together with the facts that that $H^*(BG; \mathbb{Q})$ vanishes in odd dimensions, and that $BG'_\mathbb{Q}$ is a product of Eilenberg-Maclane spaces. The details are given in [JMO, Theorem 3.1]. The reason for taking the product only over primes dividing the order of W is that $[BG, BG'{}^{\hat{}}_p]_\phi$ always has order 1 when $p \nmid |W|$ (see Theorem 3.6 below).

Step 2 $\mathcal{R}_p$-invariant representations.

We now fix a prime p, and consider the problem of determining the set of homotopy classes of maps $BG \to BG'{}^{\hat{}}_p$ which extend a given admissible map $\phi : T \to T'$. Fix a maximal p-toral subgroup $N_p(T) \subseteq G$: i.e., a subgroup for which $N_p(T)/T$ is a p-Sylow subgroup of $N(T)/T$.

By an *$\mathcal{R}_p$-invariant representation* on G we mean an element

$$\rho \in \operatorname{Rep}(N_p(T), G') = \operatorname{Hom}(N_p(T), G')/\operatorname{Inn}(G')$$

with the property that the restrictions of ρ combine to form an element

$$\hat{\rho} = (\rho \mid P)_{G/P \in \mathcal{R}_p(G)} \in \varprojlim_{G/P \in \mathcal{R}_p(G)} \operatorname{Rep}(P, G').$$

Equivalently, $\rho : N_p(T) \to G'$ is $\mathcal{R}_p$-invariant if for any p-toral $P \subseteq G$ and any two homomorphisms $i_1, i_2 : P \rightarrowtail N_p(T)$ induced by inclusions and conjugation in G, $\rho \circ i_1$ is conjugate (in G') to $\rho \circ i_2$.

By Theorem 3.1, $[BP, BG'{}^{\hat{}}_p] \supseteq [BP, BG'] \cong \operatorname{Rep}(P, G')$ for all p-toral P. So for any $f : BG \to BG'{}^{\hat{}}_p$ which comes from a global map $BG \to BG'$, $f \mid BN_p(T) \simeq B\rho$ for some unique $\mathcal{R}_p$-invariant representation $\rho : N_p(T) \to G'$. The question now is: given an admissible map $\phi : T \to T'$, can it be extended to an $\mathcal{R}_p$-invariant representation, and if so to how many?

When G' is one of the matrix groups $U(n)$, $SU(n)$, $Sp(n)$, or $O(n)$, then $\mathcal{R}_p$-invariant representations are relatively easy to construct: character theory can be used to verify that the appropriate pairs of homomorphisms are conjugate. For example, assume $G = G' = U(n)$, fix $p \leq n$, and let $T \subseteq G$ be the group of diagonal matrices. For some k prime to $n!$, consider the admissible map $\phi_k : T \to T$ given by $\phi_k(t) = t^k$. Then $N(T)$ is the group of monomial matrices (one nonzero entry in every row and column). Let $\rho_k : N_p(T) \to G$, be the homomorphism defined by raising each entry in a matrix to the k-th power. Then for every $x \in N_p(T)$, $\mathrm{Tr}(\rho_k(x)) = \mathrm{Tr}(x^k)$; and using this relation one easily checks that ρ_k is $\mathcal{R}_p$-invariant. Later in this section, we will show how the ρ_k can be used to give a new construction of Sullivan's unstable Adams operations on $BU(n)$.

Several examples are given in [JMO2], where we construct $\mathcal{R}_p$-invariant representations (both to matrix groups and to the exceptional Lie group F_4), extending some of the admissible maps studied by Adams & Mahmud in [AM]. But we do still lack general techniques for doing this. For example, this is the missing step if we want to construct the unstable Adams operations for the exceptional Lie groups using these methods.

Step 3 Computation of higher limits.

An $\mathcal{R}_p$-invariant representation ρ determines a family of maps, compatible up to homotopy, from the $BP \simeq EG/P$ to $BG'\hat{_p}$. The obstructions to extending these to a map from $\underrightarrow{\mathrm{hocolim}}\,(EG/P)$ to $BG'\hat{_p}$—and hence from BG to $BG'\hat{_p}$—have already been described in Theorems 1.2 and 1.3 above.

It will be convenient to adopt a different notation for higher inverse limits. We will see in the next section that these groups may be thought of as cohomology groups of the underlying category with twisted coefficients. Accordingly, from now on, we write $H^*(\mathcal{C}; F)$ rather than $\underleftarrow{\lim}^*_{\mathcal{C}}(F)$ to denote the higher inverse limits of $F : \mathcal{C} \to \mathcal{A}b$.

For a given ρ, define functors

$$\Pi_1^\rho : \mathcal{R}_p(G) \to p\text{-}groups \qquad \text{and} \qquad \Pi_n^\rho : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}\text{-}mod \quad (n \geq 2)$$

by setting

$$\Pi_n^\rho(G/P) = \pi_n\left(\mathrm{map}(BP, BG'\hat{_p})_{B\rho|P}\right) \cong \begin{cases} \pi_1(BC_{G'}(\mathrm{Im}(\rho))) & \text{if } n = 1 \\ [\pi_n(BC_{G'}(\mathrm{Im}(\rho)))]\hat{_p} & \text{if } n \geq 2. \end{cases}$$

(Note that for any p-toral $P \subseteq G'$, $\pi_0(C_{G'}(P))$ is a p-group by [JMO, Proposition A.4].) Theorems 1.2 and 1.3 now take the following form:

Theorem 3.5. *For any $\mathcal{R}_p$-invariant representation ρ, $B\rho$ extends to a map $BG \to BG'\hat{_p}$ if the higher limits $H^{n+1}(\mathcal{R}_p(G); \Pi_n^\rho)$ vanish for all $n \geq 1$; and the extension is unique if $H^n(\mathcal{R}_p(G); \Pi_n^\rho) = 0$ for all $n \geq 1$. Furthermore, there is a spectral sequence*

$$E_2^{pq} = H^p(\mathcal{R}_p(G); \Pi_q^\rho) \Rightarrow \pi_{q-p}(\mathrm{map}(BG, BG')_\rho),$$

where $\mathrm{map}(-,-)_\rho$ *is the space of maps which extend $B\rho$.*

In Proposition 4.11 below, we will see that there is a number $k = k(G, p)$, such that $H^i(\mathcal{R}_p(G); F) = 0$ for all $i > k$ and any $F : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}$-*mod*. So the spectral sequence of Theorem 3.5 always converges strongly.

The last step when constructing maps $BG \to BG'$, or checking whether they are unique, is thus to compute the higher limits $H^*(\mathcal{R}_p(G); \Pi_*^\rho)$. A priori, one might expect the computation of these higher limits to be quite hard in general. However, as will be seen in the next section (Theorems 4.8 and 4.9), we have succeeded in developing some very powerful tools which are successful in making these computations in many cases.

As a first simple illustration of the application of the methods in this section, we show how they apply in the case where $p \nmid |W|$.

Theorem 3.6. [AM] *If G is connected and $p \nmid |W|$, then any admissible map $\phi : T \to T'$ lifts to a unique map $f : BG \to BG'^{\wedge}_p$.*

Proof. This is shown by Adams & Mahmud in [AM, Theorem 1.10]; but we note here how it follows from the theory just presented. When $p \nmid |W|$, then $N_p(T) = T$, and the only p-stubborn subgroups of G are the maximal tori (any p-stubborn subgroup $P \subseteq G$ is contained in a maximal torus, and $N(P)/P$ must be finite). In other words, $\mathcal{R}_p(G)$ is equivalent to the category with one object G/T, with $\mathrm{End}(G/T) \cong W$.

In particular, ϕ is automatically $\mathcal{R}_p$-invariant. Also, for each $i, j \geq 1$,

$$H^i(\mathcal{R}_p(G); \Pi_j^\phi) \cong H^i(W; \Pi_j^\phi(G/T)^{\wedge}_p) = 0$$

(again since $p \nmid |W|$). So ϕ extends to a unique map $BG \to BG'^{\wedge}_p$ by Theorems 3.5 and 2.2 above. $\square$

As a second example, set $G = G' = U(n)$, fix p and k such that $p|n!$ and $(k, n!) = 1$; and consider the $\mathcal{R}_p$-invariant representation $\rho_k : N_p(T) \to G$ defined above. Then for each $i \geq 1$, $\Pi_i^{\rho_k} \cong \Pi_i$, where Π_i is defined by setting $\Pi_i(G/P) = \pi_i(BC_G(P))^{\wedge}_p$ for G/P in $\mathcal{R}_p(G)$. Also, $H^j(\mathcal{R}_p(G); \Pi_i) = 0$ for all $i, j \geq 1$ by Lemma 5.3 below. Theorem 3.5 thus applies to show that ρ_k extends to a unique map $\psi^k : BG \to BG^{\wedge}_p$. Since $\psi^k|BT \simeq B\phi_k$, where $\phi_k : T \to T$ is the k-th power map, ψ^k is an unstable Adams operation of degree k on $BU(n)$ (see Definition 5.1 below).

As a last example, we use these procedures to classify maps $B\Gamma \to BSU(2)$ and $B\Gamma \to BSO(3)$ for any finite group Γ. Note that in these cases, Proposition 3.4 takes the form

$$[B\Gamma, BSU(2)] \cong \prod_{p \mid |\Gamma|} [B\Gamma, BSU(2)^{\wedge}_p] \quad \text{and} \quad [B\Gamma, BSO(3)] \cong \prod_{p \mid |\Gamma|} [B\Gamma, BSO(3)^{\wedge}_p].$$

Example 3.7. *Fix a prime p and a finite group Γ, and let $\Gamma_p \subseteq \Gamma$ be a Sylow p-subgroup. We consider maps $B\Gamma \to BSU(2)^{\wedge}_p$.*

(i) A homomorphism $\rho : \Gamma_p \to SU(2)$ is $\mathcal{R}_p$-invariant if and only if for any pair of elements $g, h \in \Gamma_p$ conjugate in Γ, $\rho(g)$ and $\rho(h)$ are conjugate in $SU(2)$.

(ii) Every $\mathcal{R}_p$-invariant representation $\rho : \Gamma_p \to SU(2)$ lifts to a map $f_\rho : B\Gamma \to BSU(2)\hat{}_p$ which is unique up to homotopy.

The above description of $[B\Gamma, BSU(2)]$ was shown by Mislin and Thomas [MT, Theorem 3.2], in the case when Γ has periodic cohomology. In the same paper [MT, Theorem 2.6], they also describe $[B\Gamma, BG]$ for any arbitrary compact connected Lie group G and any periodic group Γ satisfying the "2-normalizer condition".

In all of the examples described in [MT] or in Example 3.7 above, homotopy classes of maps $B\Gamma \to BG$ are detected by their restrictions to Sylow subgroups of Γ; i.e., by the sets $\mathrm{Rep}(\Gamma_p, G)$. When $G = SO(3)$, this is not always the case.

Example 3.8. *Fix a prime p and a finite group Γ, and let $\Gamma_p \subseteq \Gamma$ be a Sylow p-subgroup. We consider maps $B\Gamma \to BSO(3)\hat{}_p$.*

(i) A homomorphism $\rho : \Gamma_p \to SO(3)$ is $\mathcal{R}_p$-invariant if and only if for any pair of elements $g, h \in \Gamma_p$ conjugate in Γ, $\rho(g)$ and $\rho(h)$ are conjugate in $SO(3)$.

(ii) Every $\mathcal{R}_p$-invariant representation $\rho : \Gamma_p \to SO(3)$ lifts to a map $f_\rho : B\Gamma \to BSO(3)\hat{}_p$; and the lifting is unique up to homotopy if p is odd or if $\mathrm{Im}(\rho)$ is abelian. In all other cases, there are at most two homotopy classes of maps $B\Gamma \to BSO(3)\hat{}_p$ which lift ρ.

(iii) Assume that $p = 2$ and $\mathrm{Im}(\rho) \cong D(2^k)$ for $k \geq 3$. Let $H_1, H_2 \subseteq \mathrm{Im}(\rho)$ be representatives for the two conjugacy classes of subgroups $(\mathbb{Z}/2)^2 \subseteq D(2^k)$, and set $P_i = \rho^{-1}(H_i)$. Then ρ has two distinct liftings $B\Gamma \rightrightarrows BSO(3)\hat{}_2$ if and only if both maps

$$N(P_i)/P_i \xrightarrow{\text{conj}} \mathrm{Out}(P_i/\operatorname{Ker}(\rho)) \underset{\cong}{\xrightarrow{\rho}} \mathrm{Aut}(H_i) \cong \Sigma_3$$

are onto, and P_1 and P_2 are not conjugate in Γ.

Proofs. Point (i) in each case follows from the definition of $\mathcal{R}_p$-invariance; and the fact that when $G = SU(n)$ or $SO(2n+1)$, two homomorphisms $H \to G$ are conjugate in G if and only if they have the same character.

Points (ii) and (iii) follow upon showing that for each $n \geq 1$, $H^m(\mathcal{R}_p(\Gamma); \Pi_n^\rho) = 0$ for $m \geq 2$, and that $H^1(\mathcal{R}_p(\Gamma); \Pi_1^\rho)$ has order one or two as indicated. This is an easy consequence of Theorems 4.8 and 4.9 and Proposition 4.10 in the next section. □

Note in particular that when $p = 2$ or 3, then any *monomorphism* from Γ_p to $SO(3)$ or $SU(2)$ is $\mathcal{R}_p$-invariant. Also, when $\Gamma = GL_3(\mathbb{F}_2)$ and $\rho : \Gamma_2 \cong D(8) \rightarrowtail SO(3)$ is an injection, then point (iii) in Example 3.7 applies to show that there are two distinct homotopy classes of maps $B\Gamma \to BSO(3)\hat{}_2$ which extend ρ. Another easy consequence of these examples is that there are exactly 12 homotopy classes of maps from $BSL(2,\mathbb{F}_5)$ to $BSU(2)$ (compare with [Ad, Proposition 1.18]).

The reason the descriptions of $[B\Gamma, BSO(3)]$ and $[B\Gamma, BSU(2)]$ are so simple is in part because they are matrix rings (so conjugacy is easily determined), but mostly because their subgroups are well known and easily described. Presumably, similar results can be found (but with more complicated formulations) for maps to (for example) $BSU(3)$ or $BSp(2)$. However, it seems unlikely that any general result (or even conjecture) about $[BG, BG']$ can be formulated, neither for G and G' arbitrary (distinct) compact connected Lie groups, nor for G finite and G' connected.

4. Higher inverse limits

Let $\mathcal{C}$ be an arbitrary small category. We write $\mathcal{C}$-*mod* for the abelian category of contravariant functors $M : \mathcal{C} \to \mathcal{A}b$. The reason for this terminology is that when $\mathcal{C}$ is the category $\mathcal{O}_1(\Gamma)$ (recall that this category has a single object $\Gamma/1$ and $\mathrm{End}(\Gamma/1) \cong \Gamma$)) then $\mathcal{C}$-*mod* is just the usual category of $\mathbb{Z}[\Gamma]$-modules. This example will be used throughout this section to illustrate some of the abstract categorical notions via examples related to finite groups.

There is a functor $\varprojlim : \mathcal{C}\text{-}mod \to \mathcal{A}b$ which assigns to every M its inverse limit, i.e., the group of compatible families of elements $(x_c)_{c\in \mathrm{Ob}(\mathcal{C})}$. We want to study the derived functors $\varprojlim^i$ of the inverse limit (cf. [GZ, Appx. 2 §3], where the dual construction is described in detail). For example, for any $\mathbb{Z}[\Gamma]$-module M, regarded as a functor on $\mathcal{O}_1(\Gamma)$, $\varprojlim_{\mathcal{O}_1(\Gamma)}(M)$ can be identified with the fixed point subgroup M^Γ, and thus the higher derived functors are isomorphic to $H^*(\Gamma; M)$. In order to emphasize this analogy with group cohomology, we shall denote the functors $\varprojlim^i$ from now on by

$$H^i(\mathcal{C}; -) : \mathcal{C}\text{-}mod \to \mathcal{A}b.$$

The following proposition is just a special case of the usual long exact sequence induced by derived functors.

Proposition 4.1. *For any small category $\mathcal{C}$, and any short exact sequence of functors $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}$-mod, there exists a functorial long exact sequence*

$$\cdots \to H^i(\mathcal{C}; M') \to H^i(\mathcal{C}; M) \to H^i(\mathcal{C}; M'') \to H^{i+1}(\mathcal{C}; M) \to \dots.$$

Let $\mathbb{Z}$ denote the constant functor. One sees easily that for any M in $\mathcal{C}$-*mod*,

$$\varprojlim_{\mathcal{C}}(M) \cong \mathrm{Hom}_{\mathcal{C}\text{-}mod}(\mathbb{Z}, M).$$

Hence, instead of taking an injective resolution for M when computing its higher inverse limits, one can choose a single projective resolution P_* of $\mathbb{Z}$, and then $H^*(\mathcal{C};M) \cong H^*(\mathrm{Hom}_{\mathcal{C}\text{-}mod}(P_*,M))$ for all M. One way to do this is the following.

For any $c \in \mathrm{Ob}(\mathcal{C})$, let $A_c : \mathcal{C} \to \mathcal{A}b$ be the functor where for each $x \in \mathrm{Ob}(\mathcal{C})$, $A_c(x)$ is the free abelian group with basis $\mathrm{Mor}_{\mathcal{C}}(x,c)$. For any $f : x \to y$ in $\mathcal{C}$, $A_c(f) : A_c(y) \to A_c(x)$ is induced by the obvious map between bases. Then for any M in $\mathcal{C}$-*mod*,

$$\mathrm{Hom}_{\mathcal{C}\text{-}mod}(A_c, M) \cong M(c).$$

In particular, A_c is projective. One now checks that there is a projective resolution of $\mathbb{Z}$

$$\cdots \longrightarrow \bigoplus_{x_0\to x_1\to x_2} A_{x_0} \longrightarrow \bigoplus_{x_0\to x_1} A_{x_0} \longrightarrow \bigoplus_{x} A_x \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the boundary maps are alternating sums of maps induced by face maps in the nerve of $\mathcal{C}$. For example, when $\mathcal{C} = \mathcal{O}_1(\Gamma)$, then this is the usual bar resolution for Γ. This resolution leads to the following lemma, which is sometimes useful for proving results about higher limits.

Lemma 4.2. [BK, XI,6.2] *For any M in $\mathcal{C}$-mod, $H^*(\mathcal{C};M)$ is the cohomology of the cochain complex*

$$0 \longrightarrow \prod_{x} M(x) \longrightarrow \prod_{x_0\to x_1} M(x_0) \longrightarrow \prod_{x_0\to x_1\to x_2} M(x_0) \longrightarrow \dots,$$

whose boundary maps are alternating sums of homomorphisms induced by the face maps in the nerve of $\mathcal{C}$.

In order to describe one of the other properties of higher limits analogous to group cohomology, we first need to define the Kan extension of a functor. For any functor $F : \mathcal{C} \to \mathcal{D}$, the restriction functor $F^* : \mathcal{D}\text{-}mod \to \mathcal{C}\text{-}mod$ has a right adjoint

$$F_* : \mathcal{C}\text{-}mod \to \mathcal{D}\text{-}mod.$$

For any M in $\mathcal{C}$-*mod*, the functor F_*M is called the right Kan extension of M along F (cf. [HS, IX.5] or [GZ, Appx. 2, §3]). If F is an embedding onto a full subcategory, then $F^* \circ F_* = Id$ [HS, Proposition IX.5.2]; i.e., the Kan extension provides an ordinary extension from functors on $\mathcal{C}$ to functors on $\mathcal{D}$. If F has a right adjoint functor $G : \mathcal{D} \to \mathcal{C}$, then $F_* = G^*$. But in general, F_* does not have to be induced by any functor $\mathcal{D} \to \mathcal{C}$.

As a special case, consider a pair of groups $H \subseteq \Gamma$, and the induced functor $F : \mathcal{O}_1(H) \to \mathcal{O}_1(\Gamma)$. In this case, for an arbitrary H-module A, $F_*(A) \cong \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}\Gamma, A)$ is just the usual induced representation. The following result from [JM2] thus generalizes Shapiro's lemma for group cohomology.

Lemma 4.3. [JM2, Lemma 3.1] *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that $F_* : \mathcal{C}$-mod $\to \mathcal{D}$-mod preserves epimorphisms. Then there is a natural isomorphism*

$$H^*(\mathcal{C}; M) \cong H^*(\mathcal{D}; F_* M).$$

This lemma can be shown by adapting the usual proof of Shapiro's lemma; or alternatively using the Leray spectral sequence of the functor $F : \mathcal{C} \to \mathcal{D}$. Note that if $F_* = G^*$ for some functor $G : \mathcal{D} \to \mathcal{C}$, then F must preserve epimorphisms.

Of particular interest are the functors whose higher limits all vanish.

Definition 4.4. *A functor $M \in \mathcal{C}$-mod is called acyclic if $H^*(\mathcal{C}; M) = 0$ for all $i > 0$.*

Below, we will review various criteria obtained in [JM2] and [JMO] for acyclicity of functors. The starting point for these results is the following simple observation:

Lemma 4.5. *A functor M is acyclic if either of the following condition holds:*

(a) M splits off as a direct summand of a acyclic functor

(b) M admits a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that M_i/M_{i-1} is acyclic for any $i = 1, \ldots, n$.

We begin with a result based on Lemma 4.5(a). Note first that every functor on a category with final object is acyclic. This, together with Lemma 4.3, provides us with a large class of acyclic functors, provided that the category $\mathcal{C}$ satisfies some mild assumptions. The splitting needed to apply Lemma 4.5(a) is provided by an additonal structure on a functor: the Mackey structure (i.e., transfer maps satisfying the double coset formula). The idea of abstract Mackey functors, generalizing the relations among the restriction and induction maps for representation rings, was introduced by Dress (cf. [Dr, Section 1]) in the context of abstract induction theory. In [JM2], the first two authors define "proto-Mackey functors" (a slightly weaker notion than Dress', in particular not assuming the existence of an initial object in the category); and prove a general criterion for the vanishing of their higher limits [JM2, Corollary 5.16]. Here we quote only one sample application which is often useful.

Theorem 4.6. *Let p be a prime, and let $\mathcal{C}$ be either the category $\mathcal{O}_p(\Gamma)$ for a finite group Γ, or $\mathcal{A}_p^o(G)$ for a compact Lie group G. Then every proto-Mackey functor $\mathcal{C} \to \mathbb{Z}_{(p)}$-mod is acyclic.*

An important example is the functor $\Gamma/P \mapsto M^P$ on $\mathcal{O}_p(\Gamma)$, which is defined for any Γ-module M. This functor possesses an evident transfer map and is therefore acyclic by Theorem 4.6. (Note, however, that in this particular case the composite of transfer and restriction is multiplication by the index, so one can give an elementary proof of acyclicity—see for example Lemma 2.1 of [Mis]).

To apply Lemma 4.5(b) to the computation of higher limits over orbit categories, we consider the atomic functors: i.e., functors $F : \mathcal{C} \to \mathcal{A}b$ which vanish on all but possibly one single isomorphism class of objects. It is easy to see that every functor on an orbit category—or more generally on a category in which all endomorphisms are isomorphisms—admits filtrations whose quotient functors are atomic. Note that an atomic functor is determined by a single $\mathrm{End}(C)$-module for some object $C \in \mathrm{Ob}(\mathcal{C})$. The surprising thing is that in some interesting cases, the higher limits of an atomic functor can be described solely in terms of this module without refering to the whole category $\mathcal{C}$.

In order to make this explicit, we introduce a new invariant $\Lambda^*(\Gamma; M)$, for a finite group Γ and a $Z_{(p)}[\Gamma]$-module M.

Definition 4.7. *For any prime p, any finite group Γ, and any $\mathbb{Z}_{(p)}[\Gamma]$-module M, let $F_M : \mathcal{O}_p(\Gamma) \to \mathcal{A}b$ be the atomic functor concentrated on the free orbit $\Gamma/1$ with $F_M(\Gamma/1) = M$; and set*

$$\Lambda^*(\Gamma; M) = H^*(\mathcal{O}_p(\Gamma); F_M).$$

Note that $\Lambda^*(\Gamma; M)$ depends implicitly on the prime p, as well as on Γ and M. Also, by [JMO, Corollary 1.8], $H^*(\mathcal{O}_p(\Gamma); F) \cong H^*(\mathcal{R}_p(\Gamma); F)$ for any finite group Γ and any F in $\mathcal{O}_p(\Gamma)$-*mod* (so $\Lambda^*(\Gamma; M)$ could just as easily be defined as the higher limits of a functor over $\mathcal{R}_p(\Gamma)$).

We now concentrate on functors in $\mathcal{R}_p(\Gamma)$-*mod*, where Γ is an arbitrary compact Lie group. The change of categories lemma (Lemma 4.3) implies easily:

Theorem 4.8. [JMO, Lemma 5.4] *For any atomic functor $F : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}$-mod concentrated on the orbit G/P,*

$$H^*(\mathcal{R}_p(G); F) \cong \Lambda^*(N(P)/P; F(G/P)).$$

This method of filtering the higher limits $H^*(\mathcal{R}_p(G); -)$ looks quite convenient, but it is not yet clear why it should be very useful for concrete calculations. The reason is that for many such computations, most of the groups $\Lambda^*(\Gamma; M)$ which occur are zero. This is due mainly to the following vanishing result, shown in [JMO, Propositions 5.5 and 6.1].

Theorem 4.9. *Fix a prime p, a finite group Γ, and a $\mathbb{Z}_{(p)}[\Gamma]$-module M. Then $\Lambda^*(\Gamma; M) = 0$ (for all $* \geq 0$) if $p \big| |Ker[\Gamma \to \mathrm{Aut}(M)]|$, or if Γ contains a nontrivial normal p-subgroup.*

The following theorem describes some of the other basic properties of the functors Λ^*, and gives some feel for how these groups behave:

Proposition 4.10. *Fix a prime p, a finite group Γ, and a $\mathbb{Z}_{(p)}[\Gamma]$-module M.*

(a) If $p \nmid |\Gamma|$, then $\Lambda^0(\Gamma; M) \cong M^\Gamma$, and $\Lambda^i(\Gamma; M) = 0$ for $i \geq 1$. If $p \big| |\Gamma|$, then $\Lambda^0(\Gamma; M) = 0$.

(b) $\Lambda^*(\Gamma; M) \cong \Lambda^*(\Gamma/H; M)$ if $H \lhd \Gamma$ is the kernel of the Γ-action on M and $p \nmid |\Gamma|$.

(c) Let $\Gamma_p \subseteq \Gamma$ be a Sylow p-subgroup, and assume $|\Gamma_p| = p$. Then $\Lambda^1(\Gamma; M) \cong M^{N(\Gamma_p)}/M^\Gamma$, and $\Lambda^i(\Gamma; M) = 0$ for $i \neq 1$.

Proposition 4.10 is shown in [JMO]: the first two parts in Proposition 6.1, and the third in Proposition 6.2.

The functors Λ^* enable us to show the following finiteness result which holds for an arbitrary functor $F : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}$-*mod*:

Proposition 4.11. *For any G and p, there is some $k = k(G,p)$ such that for any functor $F : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}$-mod, $H^*(\mathcal{R}_p(G); F) = 0$ for $i > k$. If G is finite and its Sylow p-subgroup has order p^m, then we can take $k = m$.*

By [JMO, Proposition 1.6], any $\mathcal{R}_p(G)$ contains only finitely many isomorphism classes of objects (i.e., G contains finitely many conjugacy classes of p-stubborn subgroups). So upon filtering F by atomic functors, Proposition 4.11 is reduced to proving the corresponding result for the $\Lambda^*(\Gamma; M)$, for finite groups Γ. And this is shown via induction on the order of Γ, and using the acyclicity (Theorem 4.6) of the functor F'_M in $\mathcal{O}_p(\Gamma)$-*mod* defined by $F'_M(\Gamma/P) = M^P$.

One important consequence of Proposition 4.11 is that the spectral sequences for spaces of maps $BG \to X\hat{_p}$ (Theorems 1.3 and 3.5) always converge strongly.

5. Self maps of BG

In Section 3, we discussed our general strategy for constructing and classifying maps between classifying spaces. Those procedures are very effective in many concrete situations. But it seems unlikely that there is any completely general description of the sets $[BG, BG']$ for arbitrary pairs G, G' of compact connected Lie groups.

One case where there *are* general results of this type is that of self maps: either arbitrary self maps of BG when G is connected and simple, or $\mathbb{Q}$-equivalences $BG \to BG$ when G is any compact connected Lie group. It is these results which are the subject of this section.

We start by defining the unstable Adams operations on BG. As mentioned earlier, it was Sullivan's construction of unstable Adams operations on $BU(n)$ which gave the first examples of maps between classifying spaces not induced by homomorphisms.

Let $U = \bigcup_{n=1}^{\infty} U(n)$ be the infinite dimensional unitary group, and let $T_\infty \subseteq U$ be the subgroup of all diagonal matrices. The Adams operation $\psi^k : K(X) \to K(X)$ determines a (homotopy class of) maps $BU \to BU$, also denoted ψ^k; and $\psi^k | BT_\infty \simeq B\phi_k$ where $\phi_k : T_\infty \to T_\infty \subseteq U$ is the k-th power homomorphism. In particular, the induced homomorphism $H^{2i}(\psi^k; \mathbb{Q})$ is multiplication by k^i for each i. This motivates the following definition.

Definition-Proposition 5.1. *Let G be any compact connected Lie group. A self-map $f : BG \to BG$ is called an unstable Adams operation of degree k if any of the following equivalent conditions hold:*

(i) For each $i \geq 0$, $H^{2i}(f; \mathbb{Q})$ is multiplication by k^i.

(ii) For any maximal torus $T \subseteq G$, the following square commutes up to homotopy:

$$\begin{array}{ccc} BT & \xrightarrow{\text{incl}} & BG \\ {\scriptstyle B\phi_k}\downarrow & & {\scriptstyle f}\downarrow \\ BT & \xrightarrow{\text{incl}} & BG \end{array} \qquad (\phi_k(t) = t^k) \tag{1}$$

(iii) For some embedding (equivalently, all embeddings) $G \hookrightarrow U$, the following square commutes up to homotopy:

$$\begin{array}{ccc} BG & \longrightarrow & BU \\ {\scriptstyle f}\downarrow & & {\scriptstyle \psi^k}\downarrow \\ BG & \longrightarrow & BU \end{array} \tag{2}$$

Note that by Notbohm's lemma (Corollary 3.2 above), square (1) commutes up to homotopy if and only if it commutes in rational cohomology. Similarly, square (2) commutes up to homotopy if and only if it commutes in rational cohomology: this follows from arguments involving the Chern character.

Unstable Adams operations were first constructed by Sullivan [Su], for $G = U(n)$ or $SU(n)$, and any k prime to $n!$. His idea was to use étale homotopy to relate the profinite completion $BU(n)\hat{}$ to the structure of $BU(n)$ as a variety over $\mathbb{C}$ without topological structure—and then consider the action of the Galois group $\mathrm{Aut}(\mathbb{C})$ on it. This gave unstable Adams operations on the p-completions $BU(n)\hat{_p}$; and they were then combined to get self maps of $BU(n)$, using Sullivan's arithmetic pullback square [Su, §3.4]. This construction was later extended by Wilkerson [Wi] to the case where G is an arbitrary compact connected Lie group and k is prime to the order of its Weyl group.

Another approach to constructing unstable Adams operations was given by Friedlander in [Fr2], who showed that BG can be approximated by classifying spaces of

Lie groups over fields of finite characteristic. For example, when $k = p$ is a prime, then $BU(n)^\wedge_q \simeq BGL_n(\bar{\mathbb{F}}_p)^\wedge_q$ for all primes $q \neq p$ [Fr1]. The Frobenius automorphism of $\bar{\mathbb{F}}_p$ then induces an unstable Adams operation $(\psi^p)^\wedge_q$ of degree p on $BU(n)^\wedge_q$.

Neither of these constructions gave any indication as to whether the unstable Adams operations are unique up to homotopy. The following theorem answers this question, and is one of the main results in [JMO].

Theorem 5.2. [JMO, Theorem 4.3] *Let G be any compact connected Lie group. Then for each k, there is up to homotopy at most one unstable Adams operation $\psi^k : BG \to BG$ of degree k.*

For $G = SU(2)$, this result was proved by Mislin [Mis].

The proof of Theorem 5.2 provides another illustration of how the program outlined in Section 3 can be applied. To keep things short, we assume $k \neq 0$ (the case $k = 0$ is in fact simpler, but must be considered separately).

Fix a maximal torus $T \subseteq G$, and let $W = N(T)/T$ be the Weyl group. We must show that the admissible map $\phi_k : T \to T$, where $\phi_k(t) = t^k$, has at most one lifting to a map $BG \to BG$. By Proposition 3.4, it suffices to show that for each $p \big| |W|$, there is at most one lifting of ϕ_k to a map $BG \to BG^\wedge_p$.

Fix p, and let $W_p = N_p(T)/T$ be a Sylow p-subgroup of $W = N(T)/T$. Then $N_p(T)$ is a maximal p-toral subgroup of G. The first step is to show that ϕ_k extends to at most one $\mathcal{R}_p$-invariant representation (up to conjugacy in G). If we fix one $\mathcal{R}_p$-invariant representation $\rho_k : N_p(T) \to G$ and compare the other representations to it, we obtain a one-to-one correspondence between the set of all extensions of ϕ_k to representations $N_p(T) \to G$ and the group $H^1(W_p; T)$. Under this correspondence, the $\mathcal{R}_p$-invariant representations all lie in the image of $H^1(W; T)_{(p)} \hookrightarrow H^1(W_p; T)$. Also, $H^1(W; T)$ is a 2-group, and is detected by restriction to the reflections in W. Using this, it is straightforward to check that any other $\mathcal{R}_p$-invariant representation must represent the trivial element in $H^1(W; T)$, and hence is conjugate to ρ_k. For the details, see the proof in [JMO, Proposition 3.5].

It remains to show that $B\rho_k : BN_p(T) \to BG^\wedge_p$ extends to at most one map $BG \to BG^\wedge_p$. By Theorem 3.5 above, this means showing that the higher limits $H^i(\mathcal{R}_p(G); \Pi_i^{\rho_k})$ vanish for all $i \geq 1$. One first checks that $\Pi_i^{\rho_k} \cong \Pi_i$ as functors $\mathcal{R}_p(G) \to \mathbb{Z}_{(p)}$-*mod*, where $\Pi_i(G/P) \cong \pi_i(BC_G(P))^\wedge_p$. The latter homotopy group can actually be determined explicitly, since when P is p-stubborn, the centralizer $C_G(P)$ is equal to the center $Z(P)$ (see [JMO, Lemma 1.5]), and is in particular the product of a torus and a finite abelian group. Note, for example, that $\Pi_i = 0$ for $i \geq 3$, and that $\Pi_1(G/P)$ is abelian for all P.

The techniques described in Section 4 are now used to show that $H^m(\mathcal{R}_p(G); \Pi_n) = 0$ for all $m, n \geq 1$ (see [JMO, Sections 5 & 6], where this is shown for simply connected G). To illustrate these computations, we sketch the proof when $G = U(n)$ (and the same proof works when $G = SU(n)$ or $Sp(n)$).

Lemma 5.3. *Fix any prime p, and set $G = U(n)$. Then for any $i \geq 1$,*

$$H^j(\mathcal{R}_p(G); \Pi_i) \cong \begin{cases} 0 & \text{if } j > 0 \\ \pi_i(BZ(G))\hat{_p} & \text{if } j = 0. \end{cases}$$

Proof. The proof is based on results and notation from Section 4. Recall in particular Theorem 4.8: if F is a p-local functor on $\mathcal{R}_p(G)$, and vanishes except on one orbit type G/P, then $H^*(\mathcal{R}_p(G); F) \cong \Lambda^*(NP/P; F(G/P))$.

Assume first that $P \subseteq G$ is a p-stubborn subgroup such that

$$\Lambda^*(N(P)/P; \Pi_i(G/P)) \neq 0.$$

We want to show that P contains a maximal torus. By Theorem 4.9, the kernel of the action of $N(P)/P$ on $\Pi_i(G/P) \cong \pi_i(BZ(P))\hat{_p}$ must have order prime to p. In particular, $p \nmid |[N(P) \cap C_G(Z(P))]/P|$; and so P is a maximal p-toral subgroup of $C_G(Z(P))$ (see [JMO, Lemma A.2]). But $Z(P)$ is abelian, and hence (since $G = U(n)$) is contained in a maximal torus. So $C_G(Z(P))$, and hence P, must contain maximal tori.

Now let $\hat{\Pi}_i \subseteq \Pi_i : \mathcal{R}_p(G) \to \mathbb{Z}_{(p)}$-*mod* be the subfunctor defined by setting $\hat{\Pi}_i(G/P) = \Pi_i(G/P)$ if P contains a maximal torus, and $\hat{\Pi}_i(G/P) = 0$ otherwise. By what was shown in the last paragraph,

$$\Lambda^*(NP/P; (\Pi_i/\hat{\Pi}_i)(G/P)) = 0$$

for each G/P in $\mathcal{R}_p(G)$. So $H^*(\mathcal{R}_p(G); \Pi_i/\hat{\Pi}_i) = 0$ by Lemma 4.5 and Theorem 4.8; and $H^*(\mathcal{R}_p(G); \Pi_i) \cong H^*(\mathcal{R}_p(G); \hat{\Pi}_i)$ by Proposition 4.1.

Now fix a maximal torus T, and let $W = N(T)/T$ be the Weyl group. Let $\alpha_i : \mathcal{R}_p(W) \to \mathbb{Z}_{(p)}$-*mod* be the functor

$$\alpha_i(W/Q) = \Pi_i(G/P) \cong \pi_i(BZ(P)) \cong \pi_{i-1}(T^Q). \quad (Q = P/T \subseteq W)$$

It is not hard to see that

$$H^*(\mathcal{R}_p(G); \hat{\Pi}_i) \cong H^*(\mathcal{R}_p(W); \alpha_i)$$

(regard $\mathcal{R}_p(W)$ as a subcategory of $\mathcal{R}_p(G)$). But α_i is a Mackey functor in the sense of [JM2], and so its higher limits vanish by Theorem 4.6 above. More precisely, by [JMO, Proposition 5.2],

$$H^j(\mathcal{R}_p(W); \alpha_i) \cong \begin{cases} 0 & \text{if } j > 0 \\ \pi_{i-1}(T_W) & \text{if } j = 0. \end{cases}$$

Since $T^W = Z(G)$ ($G = U(n)$), this completes the proof. $\square$

We are now ready to describe the set of homotopy classes of self maps of BG when G is connected and simple. Note that for such G, $\mathrm{Rep}(G,G) \cong \{0\} \amalg \mathrm{Out}(G)$, where 0 denotes the trivial homomorphism and $\mathrm{Out}(G)$ the outer automorphism group. By the smash product of two monoids M_1, M_2 with zero element is meant the quotient monoid

$$(M_1 \times M_2)/\langle (x_1, 0) = (0,0) = (0, x_2) : x_i \in M_i \rangle.$$

Theorem 5.4. *Let G be any compact connected simple Lie group with maximal torus T and Weyl group W. Then there is a bijection*

$$\beta : (\{0\} \amalg \mathrm{Out}(G)) \wedge \{k \geq 0 : k = 0 \text{ or } (k, |W|) = 1\} \xrightarrow{\cong} [BG, BG]$$

of monoids with zero element, which sends (α, k) to $\psi^k \circ B\alpha$ (for any unstable Adams operation ψ^k of degree k). In particular, for any $f, f' : BG \to BG$, the following are equivalent:

(1) f and f' are homotopic

(2) $f|BT \simeq f'|BT : BT \to BG$

(3) $H^*(f; \mathbb{Q}) = H^*(f'; \mathbb{Q})$.

Theorem 5.4 was proven in several stages. Hubbuck [Hub1], [Hub2] and Mahmud (see [AM]) showed, using methods involving the Steenrod algebra, that each $f : BG \to BG$ is the composite of an unstable Adams operation and a map induced by an automorphism of G. Later, Ishiguro [Is] showed that unstable Adams operations of type ψ^k exist only for $k = 0$ or k prime to the order of the Weyl group of G. A shorter proof of these results, using Notbohm's theorem about p-toral groups (Theorem 3.1 above) is given in [JMO, Theorem 3.4].

As mentioned before, the existence of unstable Adams operations $\psi^k : BG \to BG$, whenever $(k, |W|) = 1$, was shown by Sullivan [Su], Wilkerson [Wi] and Friedlander [Fr2]. And the last step in the proof of Theorem 5.4 was the result in [JMO] about uniqueness of unstable Adams operations, stated in Theorem 5.2 above.

The case of self maps of BG for an arbitrary compact connected Lie group G is much more complicated. In [JMO, Example 7.1], we construct two maps $f, f' : BSU(3) \to BSO(8)$ which induce the same map on rational cohomology but are not homotopic. In particular, this shows that homotopy classes of self maps of BG are not detected by rational cohomology when $G = SU(3) \times SO(8)$ (i.e., conditions (1) and (3) in Theorem 5.4 are not equivalent in this case). However, if one restricts attention to $\mathbb{Q}$-equivalences $BG \to BG$ for arbitrary connected G, then one does get a strong result, analogous to Theorem 5.4. One way of formulating it is the following.

Theorem 5.5. [JMO3] *Let G be any compact connected Lie group, and let $T{\subseteq}G$ be a maximal torus. Then two $\mathbb{Q}$-equivalences $f, f' : BG{\to}BG$ are homotopic if and only if they have the same effect in rational cohomology, if and only if $f|BT{\simeq}f'|BT$, if and only if f and f' are extensions of the same surjective admissible map $\phi : T{\twoheadrightarrow}T$.*

The set of $\mathbb{Q}$-equivalences is described explicitly as follows. For each prime $p\big||W|$, let $G\langle p\rangle \subseteq G$ be the subgroup generated by the simple summands of G whose Weyl group has order divisible by p. For each $n \geq 1$, let $H_n \lhd G$ be the product of all simple components (normal subgroups) of G isomorphic to $SO(2n{+}1)$. Then a surjective admissible map $\phi : T{\twoheadrightarrow}T$ can be extended to a $\mathbb{Q}$-equivalence $BG{\to}BG$ if and only if the following two conditions hold:

(a) $p \nmid |Ker(\phi) \cap G\langle p\rangle|$ for all $p\big||W|$; and

(b) $\phi(H_n \cap T) = H_n \cap T$ for all $n \geq 1$.

In other words, there is a one-to-one correspondence between $\mathbb{Q}$-equivalences $f : BG{\to}BG$ and admissible surjections $\phi : T{\twoheadrightarrow}T$ which satisfy conditions (a) and (b) above. There is also a version of Theorem 5.5 dealing with self equivalences of $BG^\wedge_p$ (also shown in [JMO3]).

The following example shows why condition (b) is needed in Theorem 5.5. Fix $n \geq 1$, and set $G = SO(2n{+}1){\times}Sp(n)$. The standard maximal tori $T_1{\subseteq}SO(2n{+}1)$ and $T_2{\subseteq}Sp(n)$ can be identified in a way such that the map ϕ which switches them is an admissible automorphism of the maximal torus $T_1{\times}T_2 = T{\subseteq}G$. But ϕ cannot extend to a self map of BG, since $BSO(2n{+}1)$ and $BSp(n)$ have distinct homotopy types at the prime 2.

6. Homotopical uniqueness of classifying spaces

In the last section, we saw that very complete results can be obtained about self maps of classifying spaces of compact connected simple Lie groups. This is not their only unusual property. It turns out that in many cases, the p-completed homotopy types of classifying spaces of compact connected Lie groups are fully determined by their $\mathbb{F}_p$-cohomology as algebras over the Steenrod algebra A_p. This is of course a very special phenomenon, even among other classifying spaces: for example, $B\mathbb{Z}/p^2$ and $B\mathbb{Z}/p^3$ have isomorphic cohomology in this sense, and are p-complete, but they are not homotopy equivalent.

Proving the uniqueness of BG in this sense means realizing an isomorphism $H^*(X;\mathbb{F}_p){\cong}H^*(BG;\mathbb{F}_p)$ of A_p-algebras by a map $BG{\to}X^\wedge_p$. The starting point for doing this is a theorem of Lannes (Theorem 6.1 below), which says that when V is an elementary abelian p-group and X satisfies certain mild conditions, then any A_p-algebra map $H^*(X;\mathbb{F}_p){\to}H^*(BV;\mathbb{F}_p)$ can be realized by a map $BV{\to}X^\wedge_p$ (unique up to homotopy).

Let $\mathcal{U}$ denote the category of unstable modules over the Steenrod algebra A_p, and let $\mathcal{K}$ be the category of unstable A_p-algebras. For example, for any space X, $H^*(X;\mathbb{F}_p)$ is in both $\mathcal{K}$ and $\mathcal{U}$. For each elementary abelian p-group V, Lannes and Zarati constructed a functor $T_V : \mathcal{K} \to \mathcal{K}$ which is left adjoint to the tensor product functor $-\otimes H^*(BV;\mathbb{F}_p)$. In fact, T_V can be regarded as a functor either from $\mathcal{K}$ to itself or from $\mathcal{U}$ to itself; and as a functor $\mathcal{U} \to \mathcal{U}$ it preserves exact sequences and tensor products. For more details, see either [La1] or [La2].

For any space X, the evaluation map $\mathrm{map}(BV,X)\times BV\to X$ induces a map of cohomology

$$H^*(X;\mathbb{F}_p) \to H^*(\mathrm{map}(BV,X);\mathbb{F}_p) \otimes H^*(BV;\mathbb{F}_p);$$

and this is adjoint to a homomorphism

$$T_V(H^*(X;\mathbb{F}_p)) \to H^*(\mathrm{map}(BV,X);\mathbb{F}_p). \tag{1}$$

(Indeed, the existence of this homomorphism was the reason for introducing the functor T_V in the first place.) Also, for any algebra K in $\mathcal{K}$,

$$(T_V K)^0 \cong \mathrm{map}[\mathrm{Hom}_{\mathcal{K}}(K, H^*(BV)), \mathbb{F}_p].$$

The following theorem thus describes cases in which (1) is an isomorphism.

Theorem 6.1. (Lannes [La2, Théorèmes 0.4–0.6]) *Let V be an elementary abelian p-group, and let X be a p-complete space such that $H^*(X;\mathbb{F}_p)$ is finite in each dimension. Then the following hold:*

(i) $[BV,X] \cong \mathrm{Hom}_{\mathcal{K}}(H^*(X;\mathbb{F}_p), H^*(BV;\mathbb{F}_p))$.

(ii) Assume that either $T_V(H^*(X;\mathbb{F}_p))$ vanishes in dimension 1; or that there is some p-complete space Z and a map $Z\to\mathrm{map}(BV,X)$ such that the induced map $T_V(H^*(X;\mathbb{F}_p))\to H^*(Z;\mathbb{F}_p)$ is an isomorphism. Then the map (1) above is an isomorphism; i.e.,

$$H^*(\mathrm{map}(BV,X);\mathbb{F}_p) \cong T_V(H^*(X;\mathbb{F}_p)).$$

Lannes' theorem (part (i), at least) is needed to prove the theorem of Dwyer & Zabrodsky and Notbohm (Theorem 3.1 above), which plays such an important role when describing maps between classifying spaces. Lannes' theorem has thus appeared indirectly in previous sections (via the use of Theorem 3.1), but in this one and the next it will be used explicitly.

Theorem 6.1 says in particular that if X and Y are p-complete spaces such that $H^*(X;\mathbb{F}_p) \cong H^*(Y;\mathbb{F}_p)$ as A_p-algebras, then $\mathrm{map}(BV,X)$ and $\mathrm{map}(BV,Y)$ also have the same cohomology (componentwise) as A_p-algebras. When $X = BG$ for some compact connected Lie group G, then this (with an appropriate choice of V) can often be used to construct a "maximal torus" $BT{\to}Y$ analogous to the maximal torus in G. Examples of this procedure will be given in the sketches of the proofs of Theorems 6.3 and 6.5 below.

We now turn to uniqueness results for classifying spaces. The ones discussed here are all of the form: for certain G and p, a p-complete space X is homotopy equivalent to $BG^{\wedge}_p$ if its $\mathbb{F}_p$-cohomology is isomorphic to that of BG (as algebras over the Steenrod algebra A_p). The first result of this type was for the group $G = SU(2)$:

Theorem 6.2. (Dwyer, Miller, & Wilkerson [DMW1]) *Let p be any prime, and let X be any p-complete space such that $H^*(X;\mathbb{F}_p){\cong}H^*(BSU(2);\mathbb{F}_p)$ as A_p-algebras. Then $X{\simeq}BSU(2)^{\wedge}_p$.*

For $p = 2$, Theorem 6.2 is proved using the pushout decomposition of $BSO(3)$ described in diagram (2) of Section 2. First, a 2-complete space Y is constructed, together with a fibration $K(\mathbb{Z}/2,1){\to}X{\to}Y$, such that $H^*(Y;\mathbb{F}_2) \cong H^*(BSO(3);\mathbb{F}_2)$. Then, Y is included in a homotopy commutative square

$$\begin{array}{ccc} BD(8) & \longrightarrow & BO(2) \\ \Big\downarrow & & \Big\downarrow \\ B\Sigma_4 & \longrightarrow & Y \end{array} \tag{2}$$

(and this is of course the hard part). Square (2) induces a map $BSO(3)^{\wedge}_2 \to Y$, which is shown to be an $\mathbb{F}_2$-homology equivalence and hence a homotopy equivalence. And this map is then lifted to a homotopy equivalence $BSU(2)^{\wedge}_2{\simeq}X$.

If X is a space such that $H^*(X;\mathbb{F}_p){\cong}H^*(BSU(2);\mathbb{F}_p)$ (as A_p-algebras) for all primes p, then Theorem 6.2 says only that X has the same genus as $BSU(2)$; i.e., $X^{\wedge}_p{\simeq}BSU(2)^{\wedge}_p$ for all p. Results of Rector [Re] show that the genus of $BSU(2)$ contains uncountably many distinct homotopy types.

The next theorem is a special case of a general existence and uniqueness theorem proven in [DMW2]. That full theorem will be stated in Theorem 7.1 below.

Theorem 6.3. (Dwyer, Miller, & Wilkerson [DMW2]) *Let G be a compact connected Lie group, and let p be any prime which does not divide the order of the Weyl group of G. Then for any p-complete space X such that $H^*(X;\mathbb{F}_p){\cong}H^*(BG;\mathbb{F}_p)$ as A_p-algebras, $X{\simeq}BG^{\wedge}_p$.*

Theorem 6.3 is proven by showing that any p-complete X for which

$$H^*(X;\mathbb{F}_p) \cong H^*(BG;\mathbb{F}_p) \cong H^*(BT;\mathbb{F}_p)^W$$

is homotopy equivalent to the Borel construction of W acting on $BT^{\wedge}_p$ (and $BG^{\wedge}_p{\simeq}EW \times_W BT^{\wedge}_p$ when $p \nmid |W|$: see the discussion following Theorem 2.2 above).

Let $V \subseteq T$ be the subgroup of elements of order p. Lannes' theorem (Theorem 6.1) is used to construct a map $i : BV \to X$ which induces the appropriate map in cohomology, and then to show that $\mathrm{map}(BV, X)_i \simeq BT\hat{_p}$. This gives a "maximal torus" for X; i.e., a map $BT\hat{_p} \to X$ which induces the usual map $H^*(BG; \mathbb{F}_p) \rightarrowtail H^*(BT; \mathbb{F}_p)$ in cohomology. The rest of the proof involves showing that i can be extended to a map $EW \times_W BT\hat{_p} \to X$ defined on the Borel construction.

The most general uniqueness results so far are given in recent work of Notbohm. It is not yet clear how generally his methods apply, but it seems quite possible that they could lead to a proof of the following conjecture.

Conjecture 6.4. (Notbohm) *Let G be a connected compact Lie group, and let p be any prime such that $H^*(BG; \mathbb{Z})$ is p-torsion free. Then for any p-complete space X such that $H^*(X; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p)$ as A_p-algebras, $X \simeq BG\hat{_p}$.*

In fact, no counterexamples to Conjecture 6.4 are known, even when $H^*(BG; \mathbb{Z})$ is not p-torsion free.

The following theorem describes some of the cases of the conjecture which have been proven by now. The proof has not yet appeared as a preprint, but Notbohm has been helpful enough to send us the details.

Theorem 6.5. (Notbohm) *Conjecture 6.4 holds when p is an odd prime and G is one of the groups $U(n)$, $SU(n)$, $Sp(n)$, or $SO(n)$.*

Notbohm has also proven the conjecture in a number of other cases, including many of those involving exceptional simple Lie groups.

In order to give a feeling for the techniques used to prove Theorem 6.5, we sketch the proof in the case when p is odd and $G \cong U(n)$. The proofs in the other cases differ only in occasional details. The same proof can also be used to prove Theorem 6.3 (but it is much simpler in that case).

Fix a p-complete space X, and let $\varphi : H^*(X; \mathbb{F}_p) \xrightarrow{\cong} H^*(BG; \mathbb{F}_p)$ be an isomorphism of A_p-algebras. Let $T \subseteq G$ be the standard maximal torus of diagonal matrices, and let $W \cong \Sigma_n$ be the Weyl group. The goal is to construct an $\mathbb{F}_p$-cohomology equivalence $BG \to X$ realizing φ. This is done using the approximation of BG via classifying spaces of p-stubborn subgroups (Theorem 2.2).

The first step is to construct a map $f_T : BT \to X$ such that $f_T \circ Bw \simeq f_T$ for each $w \in W \subseteq \mathrm{Aut}(G)$, and such that the following "big triangle" commutes:

$$\begin{tikzcd} & H^*(BT; \mathbb{F}_p) & \\ H^*(X; \mathbb{F}_p) \arrow[ur, "f_T^*"] \arrow[rr, "\varphi", "\cong"'] & & H^*(BG; \mathbb{F}_p) \arrow[ul, "(\mathrm{incl})^*"'] \end{tikzcd}$$

To do this, let $V \cong (\mathbb{Z}/p)^n$ be the p-torsion subgroup in T. Let $i \in [BV, X]$ be the component corresponding to the inclusion $BV \hookrightarrow BG$, under the identification

$$\begin{aligned}[BV, X] &\cong \mathrm{Hom}_{\mathcal{K}}\left(H^*(X;\mathbb{F}_p), H^*(BV;\mathbb{F}_p)\right) \\ &\cong \mathrm{Hom}_{\mathcal{K}}\left(H^*(BG;\mathbb{F}_p), H^*(BV;\mathbb{F}_p)\right) \cong [BV, BG]\end{aligned}$$

of Theorem 6.1(i). With proper choice of basepoint $x \in BV$, the evaluation map

$$e_x : \mathrm{map}(BV, X)_i \to X$$

is W-equivariant (with the trivial action on X). Also,

$$\begin{aligned} H^*(\mathrm{map}(BV,X)_i;\mathbb{F}_p) &\cong H^*(\mathrm{map}(BV,BG)_{\mathrm{incl}};\mathbb{F}_p) && \text{(Theorem 6.2(ii))} \\ &\cong H^*(BC_G(V);\mathbb{F}_p) \cong H^*(BT;\mathbb{F}_p). && \text{(Theorem 3.1)}\end{aligned}$$

This isomorphism between the $\mathbb{F}_p$-cohomology rings of $\mathrm{map}(BV, X)_i$ and BT is equivariant with respect to the W-actions on the two spaces. Notbohm then shows that $H^2(\mathrm{map}(BV,X)_i;\hat{\mathbb{Z}}_p) \cong H^2(BT;\hat{\mathbb{Z}}_p)$ as $\hat{\mathbb{Z}}_p[W]$-modules; and the isomorphism induces a homotopy equivariant homology equivalence $[\mathrm{map}(BV,X)_i]\hat{_p} \to BT\hat{_p}$. Since X is p-complete, e_x factors through a map $f_T : BT \to X$ with the required properties.

The next step is to extend f_T to a map $f_{NT} : BN(T) \to X$, and show that the "little triangle"

$$\begin{array}{ccc} & H^*(BN(T);\mathbb{F}_p) & \\ f_{NT}^* \nearrow & & \nwarrow (\mathrm{incl})^* \\ H^*(X;\mathbb{F}_p) & \underset{\cong}{\xrightarrow{\varphi}} & H^*(BG;\mathbb{F}_p) \end{array}$$

commutes. Here, $f_{NT} : BN(T) \to X$ is constructed by obstruction theory, by regarding $BN(T)$ as the homotopy colimit of $EN(T)/T \simeq BT$ over the category $\mathcal{O}_1(W)$ of the free orbit $W/1$. By Theorem 1.2 above, the obstructions to doing this lie in the higher limits

$$H^{i+1}(W;\pi_i(\mathrm{map}(BT,X)_{f_T})\hat{_p}) \cong H^{i+1}(W;\pi_i(BT)\hat{_p}) \cong H^{i+1}(\Sigma_n;(\hat{\mathbb{Z}}_p)^n) = 0.$$

The commutativity of the little triangle then follows upon showing that $H^*(BU(n);\mathbb{F}_p) \to H^*(BT;\mathbb{F}_p)$ has a unique lifting to $H^*(BN(T);\mathbb{F}_p)$ (as maps of A_p-algebras).

Now, for each p-stubborn subgroup $P \subseteq N(T)$, set $f_P = f_{NT}|BP$. The third step is to show that the f_P are compatible up to homotopy; i.e., that they define an element in the inverse limit $\underset{\longleftarrow \mathcal{R}_p(G)}{\lim} [BP, X]$. This is the trickiest part of the proof. It requires a choice of conjugacy class representatives for the p-stubborn subgroups which have "sufficiently large" intersection with T, and a lemma which compares the mapping spaces $\mathrm{map}(BA,X)$ and $\mathrm{map}(BA,BT)$ for any abelian p-toral group A.

The last step is to show that the obstructions to extending the f_P to a map

$$BG \xleftarrow{\simeq} \underset{G/P\in\mathcal{R}_p(G)}{\underrightarrow{\mathrm{hocolim}}} (EG/P) \xrightarrow{f} X$$

all vanish. By Theorem 1.2, they lie in the higher limits

$$H^{i+1}(\mathcal{R}_p(G);\alpha_i), \qquad \text{where} \qquad \alpha_i(G/P) = \pi_i(\mathrm{map}(BP,X)_{f_P}).$$

One first shows that $\mathrm{map}(BP,X)$ and $\mathrm{map}(BP,BG)$ have the same homotopy (at least in the relevant components); so that $\alpha_i(G/P) \cong \pi_i(BC_G(P))\hat{_p}$ for each p-stubborn $P\subseteq G$. Then $\alpha_i \cong \Pi_i$ in the notation of [JMO, §§4-5], and $H^j(\mathcal{R}_p(G);\Pi_i) = 0$ for all $i,j \geq 1$ by Lemma 5.3 above.

7. Realizations of polynomial algebras

For any compact connected Lie group G, $H^*(BG;\mathbb{F}_p)$ is a polynomial algebra over $\mathbb{F}_p$ for almost all (and in many cases all) primes p. It is thus natural to ask which other finitely generated polynomial algebras over $\mathbb{F}_p$ can be realized as the cohomology algebras of spaces.

Recall that if G is a compact connected Lie group with maximal torus T and Weyl group W, and p is prime to the order of W, then $H^*(BG;\mathbb{F}_p)\cong H^*(BT;\mathbb{F}_p)^W$, and $BG\hat{_p}$ has the homotopy type of the Borel construction $EW\times_W BT\hat{_p}$. In other words, $BG\hat{_p}$ can be constructed in this case as a homotopy colimit involving only the space $BT\hat{_p}$. This procedure was extended by Clark & Ewing [CEw], to realize many other polynomial algebras as cohomology algebras of spaces. Later, Adams & Wilkerson [AW] proved that the methods of Clark & Ewing apply in the following generality:

Theorem 7.1. [AW], [DMW2] *Fix a prime p, and let K^* be any A_p-algebra which is polynomial on generators of degrees prime to p. Then there exists a simply connected p-complete space X, unique up to homotopy, such that $H^*(X;\mathbb{F}_p)\cong K^*$.*

The existence part of Theorem 7.1 is due to Adams & Wilkerson [AW]. They prove that (for p odd) any such A_p-algebra K^* has the form $K^*\cong H^*(BT;\mathbb{F}_p)^W$ for some finite group W, some torus T, and some action of W on $H^*(BT;\mathbb{F}_p)$. The results of Clark & Ewing [CEw] are then used to realize the W-action as an action on $BT\hat{_p}$; and K^* is the cohomology of the Borel construction $EW\times_W BT\hat{_p}$. In fact, Adams & Wilkerson also gave necessary and sufficient conditions for any polynomial algebra with A_p-algebra structure to be the fixed point set of some finite group acting on $H^*(BT;\mathbb{F}_p)$.

The uniqueness part of Theorem 7.1 is shown in [DMW2, Theorem 1.1]. The proof is essentially the same as that sketched in Section 6 (Theorem 6.3) for the special case when $K^*\cong H^*(BG;\mathbb{F}_p)$ for some G.

Just as in the case of maps between classifying spaces and the uniqueness problem, the Borel construction must be replaced by more general homotopy colimits if one wants further results. The first examples of "exotic" spaces with polynomial $\mathbb{F}_p$-cohomology, where p does divide the degrees of some generators, were given by Zabrodsky [Za] (who used an *ad hoc* construction) and Aguade [Ag] (who used homotopy colimits to recover Zabrodsky's examples and create new ones). Here, "exotic" means that they are not the classifying spaces of compact Lie groups. The following theorem describes the general setup used when realizing these spaces.

Theorem 7.2. (Aguade [Ag]) *Fix a prime p, a pair $H\subseteq G$ of finite groups, and a finite $\mathbb{F}_p[G]$-module V. Assume that $H^*(G;M)\cong H^*(H;M)$ (induced by the restriction map) for any $\mathbb{F}_p[G]$-module M. Set $P^*=S^*(V)$, the graded symmetric algebra, where $V=P^2$. Assume we are given a G-space X_0 for which $H^*(X_0;\mathbb{F}_p)\cong P^*$ (as $\mathbb{F}_p[G]$-algebras), a space X_1 with $H^*(X_1;\mathbb{F}_p)\cong (P^*)^H$, and a map $f: X_0/H\to H_1$ such that $H^*(f;\mathbb{F}_p)$ is the inclusion $(P^*)^H\hookrightarrow P^*$. Then there is a space X such that $H^*(X;\mathbb{F}_p)\cong (P^*)^G$ as algebras over the Steenrod algebra.*

Aguade applied Theorem 7.2 to realize seven different algebras as cohomology algebras of spaces. For example, the theorem can be applied to the following triples (G,p,V), where V has the obvious structure as an $\mathbb{F}_p[G]$-module:

$$(GL_2(\mathbb{F}_3),3,(\mathbb{F}_3)^2),\quad (W(E_7),5,(\mathbb{F}_5)^7),\quad (W(E_7),7,(\mathbb{F}_7)^7),\quad (W(E_8),7,(\mathbb{F}_7)^8).$$

Here, $W(E_i)$ denotes the Weyl group of E_i.

To prove Theorem 7.2, Aguade considered the category $I=I(G,H)$, defined for any pair $H\subseteq G$, with two objects 0 and 1 and such that $\mathrm{End}(0)\cong G$, $\mathrm{End}(1)=1$, $\mathrm{Mor}(0,1)\cong G/H$, and $\mathrm{Mor}(1,0)=\emptyset$. The spaces X_0 and X_1 in the hypotheses of the theorem thus define a functor $F: I\to \mathcal{T}op$; and one easily sees that $H^0(I;H^*(F(-);\mathbb{F}_p))\cong P^G$. The assumption $H^*(G;M)\cong H^*(H;M)$ is used when showing that all higher inverse limits of the functor $H^*(F(-);\mathbb{F}_p)$ vanish, and hence (using Theorem 1.1 above) that $H^*(\underrightarrow{\mathrm{hocolim}}\,(F);\mathbb{F}_p)\cong P^G$.

Aguade's examples were all constructed at odd primes. The first example of such a space at the prime 2 was constructed recently by Dwyer & Wilkerson.

Theorem 7.3. (Dwyer & Wilkerson [DW2]) *There exists a 2-complete space $BD(4)$ whose $\mathbb{F}_2$-cohomology is the ring of rank 4 Dickson invariants over $\mathbb{F}_2$. In other* *words, as algebras over the Steenrod algebra,*

$$H^*(BD(4);\mathbb{F}_2) \cong H^*(B(\mathbb{Z}/2)^4;\mathbb{F}_2)^{GL_4(\mathbb{F}_2)}.$$

The space $BD(4)$ is also constructed as the homotopy colimit of simpler spaces, but this time over a much more complicated category: one which was motivated by the general decompositions in Theorems 2.4 and 2.5 above. Theorem 2.5 suggests that if $BD(4)$ exists, then it should be the homotopy colimit of a diagram over the category of "injective" maps $BV \to BD(4)$; or (using Lannes' theorem) the category whose objects are A_2-algebra homomorphisms

$$H^*(B(\mathbb{Z}/2)^4;\mathbb{F}_2)^{GL_4(\mathbb{F}_2)} \to H^*(BV;\mathbb{F}_2)$$

where V is an elementary abelian 2-group and $H^*(BV)$ is a finitely generated module over the Dickson algebra. This category is equivalent to the category $\mathcal{A}$, whose objects are the four vector spaces $A_i = (\mathbb{F}_2)^i$ for $1 \le i \le 4$, and whose morphisms are the monomorphisms. A functor $F : \mathcal{A}^o \to \mathcal{T}op_h$ is then constructed, where $\mathcal{T}op_h$ is the homotopy category, and where $F(A_i) = BC_{\mathrm{Spin}(7)}(A_i)$ for an appropriate choice of embeddings $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \subseteq \mathrm{Spin}(7)$. For example, $F(A_1) = \mathrm{Spin}(7)$, and $F(A_4) = A_4 \cong (\mathbb{Z}/2)^4$. This step — defining F and checking that it is a well defined functor to $\mathcal{T}op_h$ — is the hardest part of the construction of $BD(4)$.

Once F has been constructed, one wants to define $BD(4) = \underrightarrow{\mathrm{hocolim}}_{\mathcal{A}}(F)$, and then use the spectral sequence of Theorem 1.1 to show that $H^*(BD(4);\mathbb{F}_2)$ is the rank 4 Dickson algebra. The calculations in this last step are similar to those used to prove Theorem 2.5. The problem is that homotopy colimits are defined only for functors defined to $\mathcal{T}op$; not to the homotopy category. So it is also necessary to study the obstructions to lifting F to a functor $\tilde{F} : \mathcal{A} \to \mathcal{T}op$. It is this problem which will be discussed throughout the rest of the section.

Let $F : \mathcal{C} \to \mathcal{T}op_h$ be an arbitrary functor defined on a discrete category $\mathcal{C}$. By a homotopy lifting of F is meant a functor $\tilde{F} : \mathcal{C} \to \mathcal{T}op$, together with a fixed homotopy equivalence $\alpha(x) : \tilde{F}(x) \xrightarrow{\simeq} F(x)$ for all $x \in \mathrm{Ob}(\mathcal{C})$, such that for each $x \to y$ in $\mathcal{C}$ the obvious square commutes up to homotopy.

The obstructions to constructing a lifting have been studied by Dwyer & Kan: first for the general lifting problem in [DK1], and then for a certain specialized case in [DK2]. The problem of constructing a homotopy lifting of F is in fact equivalent to the problem of constructing the "homotopy colimit" of F in some appropriate sense. If F does lift to a functor $\tilde{F} : \mathcal{C} \to \mathcal{T}op$, then $\underrightarrow{\mathrm{hocolim}}(\tilde{F})$ is a homotopy colimit for F. Conversely, once $p_F : \underrightarrow{\mathrm{hocolim}}(F) \to \mathcal{C}$ has been constructed with the right properties, then $\tilde{F}$ can be defined by letting $\tilde{F}(x)$ (for $x \in \mathrm{Ob}(\mathcal{C})$) be the pullback in the following square:

$$\begin{tikzcd}
\tilde{F}(x) \arrow[r] \arrow[d] & \underrightarrow{\mathrm{hocolim}}_{\mathcal{C}}\,(F) \arrow[d, "p_F"] \\
B(\mathcal{C}{\downarrow}x) \arrow[r, "B\Phi_x"] & B\mathcal{C}
\end{tikzcd}$$

Here, $\mathcal{C}{\downarrow}x$ is the category over x ($\mathrm{Ob}(\mathcal{C}{\downarrow}x) = \{(y{\to}x) \in \mathrm{Mor}(\mathcal{C})\}$), and $\Phi_x : \mathcal{C}{\downarrow}x \to \mathcal{C}$ is the obvious functor.

A functor $F : \mathcal{C} \to \mathcal{T}op_h$ is called *centric* if for each $f : x{\to}y$ in $\mathcal{C}$, the map

$$\mathrm{Aut}(F(x))_1 := \mathrm{map}(F(x), F(x))_{\mathrm{Id}} \xrightarrow{F(f)\circ -} \mathrm{map}(F(x), F(y))_{F(f)}$$

is a homotopy equivalence. This is a particularly convenient case to work with, because the obstructions to lifting F are actually higher inverse limits of functors on $\mathcal{C}$. And many of the diagrams constructed when working with classifying spaces, including the decomposition of BG via p-stubborn subgroups (Theorem 2.2), and the functor $F : \mathcal{A} \to \mathcal{T}op_h$ constructed by Dwyer & Wilkerson and described above, are centric.

The next theorem describes the obstructions to diagram lifting in the special case where the homotopy functor is centric.

Theorem 7.4. (Dwyer & Kan [DK2]) *Let $F : \mathcal{C} \to \mathcal{T}op_h$ be a centric functor from a (small) discrete category $\mathcal{C}$ to the homotopy category. For each $i \geq 1$, let $\alpha_i : \mathcal{C} \to \mathcal{A}b$ denote the contravariant functor $\alpha_i(x) = \pi_i(\mathrm{Aut}(F(x))_1)$. Then the obstructions to the existence of a homotopy lifting of F to $\tilde{F} : \mathcal{C} \to \mathcal{T}op$ lie in the higher inverse limits $\underleftarrow{\lim}^{i+2}(\alpha_i)$, and the obstructions to the uniqueness of such a lifting lie in the higher limits $\underleftarrow{\lim}^{i+1}(\alpha_i)$.*

As suggested above, the most direct way to "see" how the obstructions in Theorem 7.4 arise is to try directly to construct an appropriate space $\underrightarrow{\mathrm{hocolim}}\,(F)$. More precisely, we want to construct an identification space

$$\underrightarrow{\mathrm{hocolim}}(F) = \Big(\coprod_{n\geq 0} \coprod_{x_0\to\cdots\to x_n} F(x_0) \times \Delta^n \Big) \Big/ \sim,$$

where each face map between the $x_0{\to}\cdots{\to}x_n$ gives rise to an identification between the corresponding $F(x_0){\times}\Delta^n$. The "1-skeleton" $\underrightarrow{\mathrm{hocolim}}^{(1)}(F)$ is easily constructed, by taking the disjoint union of the $F(x)$ (for all $x{\in}\mathrm{Ob}(\mathcal{C})$), and then attaching one copy of $F(x) \times \Delta^1$ for each morphism $x{\to}y$. The 2-skeleton is then obtained by attaching a copy of $F(x) \times \Delta^2$ for each $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{C}$, using the fact that $F(gf) \simeq F(g) \circ F(f)$ (F being a functor to the homotopy category).

The first obstructions to this procedure occur when constructing the 3-skeleton. For each sequence $x_0 \to x_1 \to x_2 \to x_3$, $F(x_0) \times \Delta^3$ can be attached to the 2-skeleton, except for $F(x_0) \times D^2$, where $D^2 \subseteq \partial\Delta^3$ is a small disk near the last vertex.

The obstruction to completing the attachment lies in

$$\pi_1\left(\mathrm{map}(F(x_0), F(x_3))_{F(x_0 \to x_3)}\right) \cong \pi_1\left(\mathrm{Aut}(F(x_0))_1\right) = \alpha_1(x_0).$$

Thus, for each 3-simplex $x_0 \to x_1 \to x_2 \to x_3$ in $B\mathcal{C}$, we get an obstruction in $\alpha_1(x_0)$. Using Lemma 4.2 above, these obstructions can be combined to give an element in $\underset{\longleftarrow \mathcal{C}}{\lim}{}^3(\alpha_1)$. This is the first obstruction to constructing $\underset{\longrightarrow}{\mathrm{hocolim}}\,(F)$; and the higher obstructions are obtained in a similar way.

References

[Ad] J. F. Adams, Maps between classifying spaces, II, Inventiones math. 49 (1978), 1–65

[AM] J. F. Adams & Z. Mahmud, Maps between classifying spaces, Inventiones math. 35 (1976), 1–41

[AM2] J. F. Adams & Z. Mahmud, Maps between classifying spaces, III, Topological topics, Cambridge Univ. Press (1983), 136–153

[AW] J. F. Adams & C. Wilkerson, Finite H-spaces and algebras over the Steenrod algebra, Annals of Math. 111 (1980), 95–143

[Ag] J. Aguade, Constructing modular classifying spaces, Israel Jour. Math. 66 (1989), 23–40

[Bo] A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts, Ann. Math. 57 (1953), 115–207

[Bo] A. Bousfield, Homotopy spectral sequences and obstructions, Israel J. Math. 66 (1989), 54–104

[BK] A. Bousfield & D. Kan, Homotopy limits, completions and localizations, Lecture Notes in Math. 304, Springer-Verlag (1972)

[Br] G. Bredon, Sheaf theory, McGraw-Hill (1967)

[Ca] G. Carlsson, Equivariant stable homotopy and Sullivan's conjecture, Inventiones math. (to appear)

[CE] H. Cartan & S. Eilenberg, Homological algebra, Princeton Univ. Press (1965)

[CEw] A. Clark & J. Ewing, The realization of polynomial algebras as cohomology rings, Pacific J. Math. 50 (1974), 425–434

[Dr] A. Dress, Induction and structure theorems for orthogonal representations of finite groups, Annals of Math. 102 (1975), 291–325

[DK1] W. Dwyer & D. Kan, Realizing diagrams in the homotopy category by means of diagrams of simplicial sets, Proc. Amer. Math. Soc. 91 (1984), 456–460

[DK2] W. Dwyer & D. Kan, Centric maps and realization of diagrams in the homotopy category (preprint)

[DMN] W. Dwyer, H. Miller, & J. Neisendorfer, Fiberwise completion and unstable Adams spectral sequences, Israel J. Math. 66 (1989), 160–178

[DMW1] W. Dwyer, H. Miller, & C. Wilkerson, The homotopic uniqueness of BS^3, Algebraic topology, Barcelona, 1976, Lecture Notes in Math. 1298, Springer-Verlag (1987), 90–105

[DMW2] W. Dwyer, H. Miller, & C. Wilkerson, Homotopical uniqueness of classifying spaces (preprint)

[DM] W. Dwyer & G. Mislin, On the homotopy type of the components of $map_*(BS^3, BS^3)$, Algebraic topology, Barcelona, 1986, Lecture Notes in Math. 1298, Springer-Verlag (1987), 82-89

[DW1] W. Dwyer & C. Wilkerson, A cohomology decomposition theorem (preprint)

[DW2] W. Dwyer & C. Wilkerson, A new finite loop space at the prime two (preprint)

[DZ] W. Dwyer & A. Zabrodsky, Maps between classifying spaces, Algebraic topology, Barcelona, 1986, Lecture Notes in Math. 1298, Springer-Verlag (1987), 106–119

[Fe] M. Feshbach, The Segal conjecture for compact Lie groups, Topology 26 (1987), 1–20

[Fr1] E. Friedlander, Unstable K-theories of the algebraic closure of a finite field, Comment. Math. Helv. 50 (1975), 145–154

[Fr2] E. Friedlander, Exceptional isogenies and the classifying spaces of simple Lie groups, Annals of Math. 101 (1975), 510–520

[FM1] E. Friedlander & G. Mislin, Cohomology of classifying spaces of complex Lie groups and related discrete groups, Comment. Math. Helv. 59 (1984), 347–361

[FM2] E. Friedlander & G. Mislin, Locally finite approximations of Lie groups. I, Inventiones math. 83 (1986), 425–436

[FM3] E. Friedlander & G. Mislin, Locally finite approximations of Lie groups. II, Math. Proc. Camb. Phil. Soc. 100 (1986), 505–517

[GZ] P. Gabriel & M. Zisman, Calculus of fractions and homotopy theory, Springer-Verlag (1967)

[HS] P. Hilton & U. Stammbach, A course in homological algebra, Springer-Verlag (1971)

[Ho] M. Hopkins, Thesis, Oxford Univ. (1984)

[Hub1] J. Hubbuck, Homotopy representations of Lie groups, New developments in topology, London Math. Soc. Lecture Notes 11, Cambridge Univ. Press (1974), 33–41

[Hub2] J. Hubbuck, Mapping degrees for classifying spaces I, Quarterly Journal of Math. Oxford Ser. (2) 25, (1974), 113–133

[Hur] W. Hurewicz, Beiträge zur Topologie der Deformationen IV: Asphärische Räume, Proc. Akad. Wetensch. Amsterdam 39 (1936), 215–224

[Is] K. Ishiguro, Unstable Adams operations on classifying spaces, Math. Proc. Camb. Phil. Soc. 102 (1987), 71–75

[JM1] S. Jackowski & J. McClure, Homotopy approximations for classifying spaces of compact Lie groups, Proceedings of Arcata conference in algebraic topology, Lecture Notes in Math. 1370, Springer-Verlag (1989), 221–234

[JM2] S. Jackowski & J. McClure, Homotopy decomposition of classifying spaces via elementary abelian subgroups, Topology (to appear)

[JMO] S. Jackowski, J. McClure, & B. Oliver, Homotopy classification of self-maps of BG via G-actions, Annals of Math. (to appear)

[JMO2] S. Jackowski, J. McClure, & B. Oliver, Maps between classifying spaces revisited (in preparation)

[JMO3] S. Jackowski, J. McClure, & B. Oliver, Self homotopy equivalences of BG (in preparation)

[La1] J. Lannes, Sur la cohomologie modulo p des p-groupes abéliens élémentaires, Homotopy theory, Proceedings of the Durham symposium 1985, Cambridge Univ. Press (1987), 97–116

[La2] J. Lannes, Sur les espaces fonctionnels dont la source est la classifiant d'un p-groupe abélien élémentaire (preprint)

[Mil] H. Miller, The Sullivan conjecture on maps from classifying spaces, Ann. Math. 120 (1984), 39–87

[Mln] J. Milnor, On axiomatic homology theory, Pac. J. Math. 12 (1962), 337–341

[Mis] G. Mislin, The homotopy classification of self-maps of infinite quaternionic projective space, Quarterly J. Math. Oxford 38 (1987), 245–257

[MT] G. Mislin & C. Thomas, On the homotopy set $[B\pi, BG]$ with π finite and G a compact connected Lie group, Quarterly J. Math. Oxford 40 (1989), 65–78

[No] D. Notbohm, Abbildungen zwischen klassifizierenden Räume, Dissertation, Göttingen (1988)

[Ol1] R. Oliver, Smooth compact Lie group actions on disks, Math. Z. 149 (1976), 79–96

[Ol2] R. Oliver, A proof of the Conner conjecture, Ann. Math. 103 (1976), 637–644

[Pu] L. Puig, Structure locale dans les groups finis, Bull. Soc. Math. France, Mémoire 47 (1976)

[Q1] D. Quillen, The spectrum of an equivariant cohomology ring I, Ann. Math. 94 (1971), 549–572

[Q2] D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, Ann. Math. 96 (1972), 552–586

[Re] D. Rector, Loop structure on the homotopy type of S^{3l}, Symposium on algebraic topology, Lecture Notes in Math. 249, Springer-Verlag (1971), 99–105

[Se] G. Segal, Classifying spaces and spectral sequences, Publ. Math. I.H.E.S. 34 (1968), 105–112

[Sl] J. Słomińska, Homotopy colimits on E-I-categories, Algebraic topology, Poznan, 1989, Lecture Notes in Math., Springer-Verlag (to appear)

[Su] D. Sullivan, Geometric topology, Part I: Localization, periodicity and Galois symmetry, Mimeographed notes, M.I.T. (1970)

[Wi] C. W. Wilkerson, Self-maps of classifying spaces, Localization in group theory and homotopy theory, Lecture Notes in Math. 418, Springer-Verlag (1974), 150–157

[Wo1] Z. Wojtkowiak, On maps from holim F to Z, Algebraic topology, Barcelona, 1986, Lecture Notes in Math. 1298, Springer-Verlag (1987), 227–236

[Za] A. Zabrodsky, On the realization of invariant subgroups of π_*X, Trans. Amer. Math. Soc. 285 (1984), 467–496