FUSION SYSTEMS REALIZING CERTAIN TODD MODULES

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Abstract. We study a certain family of simple fusion systems over finite 3-groups, ones that involve Todd modules of the Mathieu groups $2M_{12}$, M_{11}, and $A_6 = O^2(M_{10})$ over $\mathbb{F}_3$, and show that they are all isomorphic to the 3-fusion systems of almost simple groups. As one consequence, we give new 3-local characterizations of Conway's sporadic simple groups.

Fix a prime p. A fusion system over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are injective homomorphisms between the subgroups satisfying certain axioms first formulated by Puig [Pu], and modeled on the Sylow theorems for finite groups. The motivating example is the fusion system of a finite group G with $S \in \mathrm{Syl}_p(G)$, whose morphisms are those homomorphisms between subgroups of S induced by conjugation in G.

The general theme in this paper is to study fusion systems over finite p-groups S that contain an abelian subgroup $A \trianglelefteq S$ such that $A \ntrianglelefteq \mathcal{F}$ and $C_S(A) = A$. In such situations, we let $\varGamma = \mathrm{Aut}_{\mathcal{F}}(A)$ be its automizer, try to understand what restrictions the existence of such a fusion system imposes on the pair $(A, O^{p'}(\varGamma))$, and also look for tools to describe all fusion systems that "realize" a given pair $(A, O^{p'}(\varGamma))$ for A an abelian p-group and $\varGamma \leq \mathrm{Aut}(A)$.

This paper is centered around one family of examples: those where $p = 3$, where $O^{3'}(\varGamma) \cong 2M_{12}$, M_{11}, or $A_6 = O^{3'}(M_{10})$, and where A is elementary abelian of rank 6, 5, or 4, respectively. But we hope that the tools we use to handle these cases will also be useful in many other situations. Our main results can be summarized as follows:

Theorem A. *Let $\mathcal{F}$ be a saturated fusion system over a finite 3-group S with an elementary abelian subgroup $A \leq S$ such that $C_S(A) = A$, and such that either*

(i) $\mathrm{rk}(A) = 6$ *and* $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cong 2M_{12}$; *or*

(ii) $\mathrm{rk}(A) = 5$ *and* $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cong M_{11}$; *or*

(iii) $\mathrm{rk}(A) = 4$ *and* $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cong A_6$.

Assume also that $A \ntrianglelefteq \mathcal{F}$. Then $A \trianglelefteq S$, S splits over A, and $O^{3'}(\mathcal{F})$ is simple and isomorphic to the 3-fusion system of Co_1 in case (i)*, to that of Suz, Ly, or Co_3 in case* (ii)*, or to that of $U_4(3)$, $U_6(2)$, McL, or Co_2 in case* (iii)*.*

Theorem A is proven below as Theorem 4.16 (case (i)) and Theorem 5.23 (cases (ii) and (iii)). As one consequence of these results, we give new 3-local characterizations of the three Conway groups as well as of McL and $U_6(2)$ (Theorems 6.1, 6.2, and 6.3).

All three cases of Theorem A have already been shown in earlier papers using very different methods. In [vB, Theorem A], Martin van Beek determined (among other results) all fusion

2020 *Mathematics Subject Classification.* Primary 20D20. Secondary 20C20, 20D05, 20E45.

Key words and phrases. finite groups, Sylow subgroups, fusion, finite simple groups, modular representations.

B. Oliver is partially supported by UMR 7539 of the CNRS. Part of this work was carried out at the Isaac Newton Institute for Mathematical Sciences during the programme GRA2, supported by EPSRC grant nr. EP/K032208/1.

systems $\mathcal{F}$ over a Sylow 3-subgroup of Co_1 with $O_3(\mathcal{F}) = 1$. In [BFM], Baccanelli, Franchi, and Mainardis listed all saturated fusion systems $\mathcal{F}$ with $O_3(\mathcal{F}) = 1$ over a Sylow 3-subgroup of the *split* extension $E_{81} \rtimes A_6$, and this includes the four systems that appear in case (iii) of the above theorem. In [PSm], Parker and Semeraro develop computer algorithms that they use to list, among other things, all saturated fusion systems $\mathcal{F}$ over 3-groups of order at most 3^7 with $O_3(\mathcal{F}) = 1$ and $O^3(\mathcal{F}) = \mathcal{F}$. However, our goals are different from those in the earlier papers, in that we want to develop tools which can be used in other situations within the framework of the general problem described above, and are using these Todd modules as test cases.

The proof of Theorem A is straightforward, following a program that also seems to work in many other cases. Set $Z = Z(S)$. We first show that $\mathcal{F} = \langle C_{\mathcal{F}}(Z), N_{\mathcal{F}}(A)\rangle$. We then construct a special subgroup $Q \trianglelefteq S$ of exponent 3 with $Z(Q) = [Q, Q] = Z$ (of order 3 or 9) and $Q/Z(Q) \cong E_{81}$, and show that Q is normal in $C_{\mathcal{F}}(Z)$. This is the hardest part of the proof, especially when $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cong 2M_{12}$. Finally, we determine the different possibilities for $O^{3'}(\mathrm{Out}_{\mathcal{F}}(Q))$, and show that this group together with $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(A))$ determines $O^{3'}(\mathcal{F})$ up to isomorphism.

Theorem A involves just one special case of the following general problem. Given a prime p, a finite group $\Gamma = O^{p'}(\Gamma)$, and a finite $\mathbb{F}_p\Gamma$-module M (or more generally, a finite $\mathbb{Z}/p^k\Gamma$-module for some $k > 1$), we say that a saturated fusion system $\mathcal{F}$ over a finite p-group S "realizes" (Γ, M) if there is an abelian subgroup $A \le S$ such that $C_S(A) = A$, $A \ntrianglelefteq \mathcal{F}$, and $(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A)), A) \cong (\Gamma, M)$. We want to know whether a given module can be realized in this sense, and if so, list all of the distinct saturated fusion systems that realize it.

In the papers [O1], [COS], and [OR1], we studied this question under the additional assumption that $|\Gamma|$ be a multiple of p but not of p^2, and the answer in that case was already quite complicated. In this more general setting, all we can hope to do for now is to look at a few more cases, and try to develop some tools that can be used in greater generality. For example, in a second paper [O3] still in preparation, we give some criteria for the nonrealizability of certain $\mathbb{F}_p\Gamma$-modules. As one application of those results, when $\Gamma \cong M_{11}$, M_{12}, or $2M_{12}$, we show that up to extensions by trivial modules, the only $\mathbb{F}_p\Gamma$-modules that can be realized in the above sense are the Todd modules of M_{11} and $2M_{12}$ and their duals (when $p = 3$), and the simple 10-dimensional $\mathbb{F}_{11}[2M_{12}]$-modules.

As pointed out by the referee, Theorem A in this paper is closely related to the list of amalgams by Papadopoulos in [Pp]. It seems quite possible that the results in this paper can be used to strengthen or generalize the main theorem in [Pp], but if so, that will have to wait for a separate (short) paper.

General definitions and properties involving saturated fusion systems are surveyed in Section 1, while the more technical results needed to carry out the programme described above are listed in Section 2. In Section 3, we set up some notation for working with Todd modules for $2M_{12}$ and M_{11}; notation which we hope might also be useful in other contexts. Case (i) of Theorem A is proven in Section 4, and the remaining cases in Section 5. The 3-local characterizations of the Conway groups and some others are given in Section 6. We finish with two appendices: one containing a few general group theoretic results, and another more specifically focused on groups with strongly p-embedded subgroups.

Notation and terminology: Most of our notation for working with groups is fairly standard. When $P \le G$ and $x \in N_G(P)$, we let $c_x^P \in \mathrm{Aut}(P)$ denote conjugation by x on the left: $c_x^P(g) = {}^x g = xgx^{-1}$ (though the direction of conjugation very rarely matters). Our commutators have the form $[x, y] = xyx^{-1}y^{-1}$. If G is a group and $\alpha \in \mathrm{Aut}(G)$, then $[\alpha] \in \mathrm{Out}(P)$ denotes its class modulo $\mathrm{Inn}(G)$. If $\varphi \in \mathrm{Hom}(G, H)$ is a homomorphism, Q is normal in both G and H, and $\varphi(Q) = Q$, then $\varphi/Q \in \mathrm{Hom}(G/Q, H/Q)$ denotes the induced map between quotients. Also, $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of a finite group G, $\mathscr{S}(G)$ is the set of all subgroups of G, and $Z_2(G)$ is the second term in its upper central series ($Z_2(G)/Z(G) = Z(G/Z(G))$).

Other notation used here includes:

- E_{p^m} is always an elementary abelian p-group of rank m;
- p^{a+b} denotes a special p-group P with $Z(P) = [P, P] \cong E_{p^a}$ and $P/Z(P) \cong E_{p^b}$;
- p_+^{1+2m} (when p is odd) is an extraspecial p-group of order p^{1+2m} and exponent p;
- $A \circ B$ is a central product of groups A and B;
- $A \rtimes B$ and $A.B$ are a semidirect product and an arbitrary extension of A by B;
- $UT_n(q)$ is the group of upper triangular $(n \times n)$-matrices over $\mathbb{F}_q$ with 1's on the diagonal; and
- $\Gamma L_n(q)$ and $P\Gamma L_n(q)$ denote the extensions of $GL_n(q)$ and $PGL_n(q)$ by their field automorphisms.

Also, $2M_{12}$, $2A_n$, and $2\Sigma_n$ ($n = 4, 5, 6$) denote nonsplit central extensions of C_2 by the groups M_{12}, A_n, and Σ_n, respectively.

Thanks: The author would especially like to thank the referee for the many helpful suggestions, including some several involving potential connections with other papers. He would also like to thank the Isaac Newton Institute for its hospitality while the paper was being revised.

1. Background

We begin with a survey of the basic definitions and terminology involving fusion systems that will be needed here, such as normalizer fusion systems, the Alperin-Goldschmidt fusion theorem for fusion systems, and the model theorem. Most of these definitions and results are originally due to Puig [Pu].

1.1. Basic definitions and terminology.

A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ are such that

- $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all $P, Q \le S$; and
- every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

For this to be very useful, more conditions are needed.

Definition 1.1. Let $\mathcal{F}$ be a fusion system over a finite p-group S.

(a) Two subgroups $P, P' \le S$ are *$\mathcal{F}$-conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P, P') \ne \varnothing$, and two elements $x, y \in S$ are $\mathcal{F}$-conjugate if there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle x \rangle, \langle y \rangle)$ such that $\varphi(x) = y$. The $\mathcal{F}$-conjugacy classes of $P \le S$ and $x \in S$ are denoted $P^{\mathcal{F}}$ and $x^{\mathcal{F}}$, respectively.

(b) A subgroup $P \le S$ is *fully normalized in $\mathcal{F}$ (fully centralized in $\mathcal{F}$)* if $|N_S(P)| \ge |N_S(Q)|$ ($|C_S(P)| \ge |C_S(Q)|$) for each $Q \in P^{\mathcal{F}}$.

(c) The fusion system $\mathcal{F}$ is *saturated* if it satisfies the following two conditions:

- (Sylow axiom) For each subgroup $P \le S$ fully normalized in $\mathcal{F}$, P is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
- (extension axiom) For each isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$ in $\mathcal{F}$ such that Q is fully centralized in $\mathcal{F}$, φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ where
$$N_\varphi = \{g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(Q)\}.$$

In the following lemma, we describe another important property of fully normalized subgroups.

Lemma 1.2 ([AKO, Lemma I.2.6(c)]). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then for each $P \le S$ and each $Q \in P^{\mathcal{F}} \cap \mathcal{F}^f$, there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), S)$ such that $\psi(P) = Q$.*

We next recall a few more classes of subgroups in a fusion system. As usual, for a fixed prime p, a proper subgroup H of a finite group G is *strongly p-embedded* if $p \mid |H|$, and $p \nmid |H \cap {}^xH|$ for each $x \in G \smallsetminus H$.

Definition 1.3. Let $\mathcal{F}$ be a fusion system over a finite p-group S. For $P \le S$,

- P is *$\mathcal{F}$-centric* if $C_S(Q) \le Q$ for each $Q \in P^{\mathcal{F}}$;
- P is *$\mathcal{F}$-essential* if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and the group $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ contains a strongly p-embedded subgroup;
- P is *weakly closed in $\mathcal{F}$* if $P^{\mathcal{F}} = \{P\}$;
- P is *strongly closed in $\mathcal{F}$* if for each $x \in P$, $x^{\mathcal{F}} \subseteq P$; and
- P is *normal in $\mathcal{F}$* ($P \trianglelefteq \mathcal{F}$) if each morphism in $\mathcal{F}$ extends to a morphism that sends P to itself. Let $O_p(\mathcal{F}) \trianglelefteq \mathcal{F}$ be the largest subgroup of S normal in $\mathcal{F}$.
- P is *central in $\mathcal{F}$* if each morphism in $\mathcal{F}$ extends to a morphism that sends P to itself via the identity. Let $Z(\mathcal{F}) \trianglelefteq \mathcal{F}$ be the largest subgroup of S central in $\mathcal{F}$.

Clearly, if P is weakly closed in $\mathcal{F}$, then it must be normal in S.

It follows immediately from the definitions that if P_1 and P_2 are both normal in $\mathcal{F}$, then so is P_1P_2. So $O_p(\mathcal{F})$ is defined, and a similar argument applies to show that $Z(\mathcal{F})$ is defined.

The following notation is useful when referring to some of these classes of subgroups.

Notation 1.4. *For each fusion system $\mathcal{F}$ over a finite p-group S, define*

- $\mathcal{F}^f = \{P \le S \,|\, P \text{ is fully normalized in } \mathcal{F}\}$;
- $\mathcal{F}^c = \{P \le S \,|\, P \text{ is } \mathcal{F}\text{-centric}\}$ *and* $\mathcal{F}^{cf} = \mathcal{F}^c \cap \mathcal{F}^f$; *and*
- $\mathbf{E}_{\mathcal{F}} = \{P \le S \,|\, P \text{ is } \mathcal{F}\text{-essential}\}$.

1.2. The Alperin-Goldschmidt fusion theorem for fusion systems.

The following is one version of the Alperin-Goldschmidt fusion theorem for fusion systems. This theorem is our main motivation for defining $\mathcal{F}$-essential subgroups here.

Theorem 1.5 ([AKO, Theorem I.3.6]). *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then each morphism in $\mathcal{F}$ is a composite of restrictions of automorphisms $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ for $R \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$.*

Equivalently, Theorem 1.5 says that $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(P) \,|\, P \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \rangle$. Here, whenever $\mathcal{F}$ is a fusion system over S, and $\mathscr{X}$ is a set of fusion subsystems and morphisms in $\mathcal{F}$, we let $\langle \mathscr{X} \rangle$ denote the smallest fusion system over S that contains $\mathscr{X}$. Since an intersection of fusion subsystems over S is always a fusion system over S (not necessarily saturated, of course), the subsystem $\langle \mathscr{X} \rangle$ is well defined.

In fact, up to $\mathcal{F}$-conjugacy, the essential subgroups form the smallest possible set of subgroups that generate $\mathcal{F}$.

Proposition 1.6. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathscr{T}$ be a set of subgroups of S such that $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(P) \,|\, P \in \mathscr{T} \rangle$. Then each $\mathcal{F}$-essential subgroup $R < S$ is $\mathcal{F}$-conjugate to a member of $\mathscr{T}$.*

Proof. Fix $R \in \mathcal{F}^f$ such that $R < S$ and $R^{\mathcal{F}} \cap \mathscr{T} = \varnothing$, and set

$$\mathrm{Aut}^0_{\mathcal{F}}(R) = \bigl\langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(R) \,\big|\, \alpha = \overline{\alpha}|_R, \text{ some } \overline{\alpha} \in \mathrm{Hom}_{\mathcal{F}}(P,S) \text{ where } R < P \leq S \bigr\rangle.$$

We will prove that $\mathrm{Aut}^0_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{F}}(R)$. It will then follow that R is not $\mathcal{F}$-essential (see [AKO, Proposition I.3.3(b)]), thus proving the proposition.

Fix $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$. By assumption, there are isomorphisms

$$R = R_0 \xrightarrow[\cong]{\alpha_1} R_1 \xrightarrow[\cong]{\alpha_2} R_2 \xrightarrow[\cong]{\alpha_3} \cdots \xrightarrow[\cong]{\alpha_k} R_k = R$$

such that $\alpha = \alpha_k \circ \cdots \circ \alpha_1$, together with automorphisms $\beta_i \in \mathrm{Aut}_{\mathcal{F}}(P_i)$ for $1 \leq i \leq k$ such that $\langle R_{i-1}, R_i \rangle \leq P_i \in \mathscr{T}$ and $\alpha_i = \beta_i|_{R_{i-1}}$.

By Lemma 1.2 and since $R \in \mathcal{F}^f$, for each $0 \leq i \leq k$, there is $\chi_i \in \mathrm{Hom}_{\mathcal{F}}(N_S(R_i), N_S(R))$ such that $\chi_i(R_i) = R$, where we take $\chi_0 = \chi_k = \mathrm{Id}_{N_S(R)}$. For each $1 \leq i \leq k$, set

$$\widehat{R}_{i-1} = N_{P_i}(R_{i-1}) \quad \text{and} \quad \widehat{\alpha}_i = (\chi_i) \circ (\beta_i|_{\widehat{R}_{i-1}}) \circ (\chi_{i-1}^{-1}|_{\chi_{i-1}(\widehat{R}_{i-1})}) \in \mathrm{Hom}_{\mathcal{F}}(\widehat{R}_{i-1}, S).$$

Then $\widehat{\alpha}_i|_R = (\chi_i|_{R_i}) \circ \alpha_i \circ (\chi_{i-1}^{-1}|_{R_{i-1}}) \in \mathrm{Aut}_{\mathcal{F}}(R)$ for each i.

For each i, $P_i > R_{i-1}$ since $P_i \in \mathscr{T}$ while $R_{i-1} \in R^{\mathcal{F}}$ and $R^{\mathcal{F}} \cap \mathscr{T} = \varnothing$. Hence $\widehat{R}_{i-1} > R$ for each $1 \leq i \leq k$. By construction, $\alpha = (\widehat{\alpha}_k|_R) \circ \cdots \circ (\widehat{\alpha}_1|_R)$, and so $\alpha \in \mathrm{Aut}^0_{\mathcal{F}}(R)$. Since $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ was arbitrary, this proves that $\mathrm{Aut}^0_{\mathcal{F}}(R) = \mathrm{Aut}_{\mathcal{F}}(R)$, as claimed. □

The next two lemmas give different conditions for a subgroup to be normal in a fusion system. Both are consequences of Theorem 1.5.

Lemma 1.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. A subgroup $Q \leq S$ is normal in $\mathcal{F}$ if and only if it is weakly closed and contained in all $\mathcal{F}$-essential subgroups.*

Proof. This is essentially the equivalence (a ⇔ c) in [AKO, Proposition I.4.5]. □

In general, strongly closed subgroups in a saturated fusion system need not be normal. The next lemma describes one case where this does happen.

Lemma 1.8 ([AKO, Corollary I.4.7(a)])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. If $A \trianglelefteq S$ is an abelian subgroup that is strongly closed in $\mathcal{F}$, then $A \trianglelefteq \mathcal{F}$.*

1.3. Normalizer fusion subsystems and models.

If $\mathcal{F}$ is a fusion system over a finite p-group S, then a *fusion subsystem* $\mathcal{E} \le \mathcal{F}$ over a subgroup $T \le S$ is a subcategory $\mathcal{E}$ whose objects are the subgroups of T, such that $\mathcal{E}$ is itself a fusion system over T. For example, the full subcategory of $\mathcal{F}$ with objects the subgroups of T is a fusion subsystem of $\mathcal{F}$. If we want our fusion subsystems to be saturated, then, of course, the problem of constructing them is more subtle.

One case where this is straightforward is the construction of normalizers and centralizers of subgroups in a fusion system.

Definition 1.9. Let $\mathcal{F}$ be a fusion system over a finite p-group S. For each $Q \le S$, we define fusion subsystems $C_\mathcal{F}(Q) \le N_\mathcal{F}(Q) \le \mathcal{F}$ over $C_S(Q) \le N_S(Q)$ by setting

$$\begin{aligned}\mathrm{Hom}_{C_\mathcal{F}(Q)}(P,R) &= \bigl\{\varphi|_P \bigm| \varphi \in \mathrm{Hom}_\mathcal{F}(PQ,RQ),\ \varphi(P) \le R,\ \varphi|_Q = \mathrm{Id}_Q\bigr\}\\ \mathrm{Hom}_{N_\mathcal{F}(Q)}(P,R) &= \bigl\{\varphi|_P \bigm| \varphi \in \mathrm{Hom}_\mathcal{F}(PQ,RQ),\ \varphi(P) \le R,\ \varphi(Q) = Q\bigr\}.\end{aligned}$$

It follows immediately from the definitions that a subgroup $Q \le S$ is normal or central in $\mathcal{F}$ if and only if $N_\mathcal{F}(Q) = \mathcal{F}$ or $C_\mathcal{F}(Q) = \mathcal{F}$, respectively.

Theorem 1.10 ([AKO, Theorem I.5.5])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and fix $Q \le S$. Then $C_\mathcal{F}(Q)$ is saturated if Q is fully centralized in $\mathcal{F}$, and $N_\mathcal{F}(Q)$ is saturated if Q is fully normalized in $\mathcal{F}$.*

We next look at models for constrained fusion systems, and in particular, for normalizer fusion subsystems of centric subgroups.

Definition 1.11. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.

(a) The fusion system $\mathcal{F}$ is *constrained* if there is a subgroup $Q \le S$ that is normal in $\mathcal{F}$ and $\mathcal{F}$-centric; equivalently, if $O_p(\mathcal{F}) \in \mathcal{F}^c$.

(b) A *model* for a constrained fusion system $\mathcal{F}$ over S is a finite group M with $S \in \mathrm{Syl}_p(M)$, such that $S \in \mathrm{Syl}_p(M)$, $\mathcal{F}_S(M) = \mathcal{F}$, and $C_M(O_p(M)) \le O_p(M)$.

By the *model theorem* (see [AKO, Theorem III.5.10]), every constrained fusion system has a model, unique up to isomorphism. We will need this only in the following situation.

Proposition 1.12. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then for each $Q \in \mathcal{F}^{cf}$, the normalizer fusion subsystem $N_\mathcal{F}(Q)$ is constrained, and hence has a model: a finite group M with $N_S(Q) \in \mathrm{Syl}_p(M)$ such that $Q \trianglelefteq M$, $C_M(Q) \le Q$, and $\mathcal{F}_{N_S(Q)}(M) = N_\mathcal{F}(Q)$. Furthermore, M is unique in the following sense: if M^* is another model for $N_\mathcal{F}(Q)$, also with $Q \trianglelefteq M^*$ and $N_S(Q) \in \mathrm{Syl}_p(M^*)$, then $M \cong M^*$ via an isomorphism that restricts to the identity on $N_S(Q)$.*

Proof. The subsystem $N_\mathcal{F}(Q)$ is constrained since the subgroup Q is normal and $N_\mathcal{F}(Q)$-centric. So by the model theorem [AKO, Theorem III.5.10], it has a model, and any two models for $N_\mathcal{F}(Q)$ are isomorphic via an isomorphism that is the identity on $N_S(Q)$. □

1.4. Subsystems of index prime to p.

We next turn to fusion subsystems of index prime to p. By analogy with groups, this really corresponds to subgroups of a finite group G that contain $O^{p'}(G)$ (but are not necessarily normal).

Definition 1.13. Let $\mathcal{F}$ be a fusion system over a finite p-group S. A fusion subsystem $\mathcal{E} \le \mathcal{F}$ has *index prime to* p if $\mathcal{E}$ is also a fusion system over S, and $\mathrm{Aut}_{\mathcal{E}}(P) \ge O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for each $P \le S$.

There is clearly always a smallest fusion subsystem of $\mathcal{F}$ of index prime to p: the subsystem $O^{p'}_*(\mathcal{F})$ over S generated by the automorphism groups $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$. The corresponding result for saturated fusion subsystems is more subtle.

Theorem 1.14. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then there is a (unique) smallest saturated fusion subsystem $O^{p'}(\mathcal{F}) \le \mathcal{F}$ of index prime to p. This has the property that for each $P \le S$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, there are morphisms $\varphi_0 \in \mathrm{Hom}_{O^{p'}(\mathcal{F})}(P,S)$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\varphi = \alpha \circ \varphi_0$.*

Proof. See [AKO, Theorem I.7.7] or [BCGLO, Theorem 5.4] for the existence and uniqueness of $O^{p'}(\mathcal{F})$. The last statement follows from Lemma 3.4(c) in [BCGLO], or since the map $\theta\colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \varGamma_{p'}(\mathcal{F})$ sends $\mathrm{Aut}_{\mathcal{F}}(S)$ surjectively. □

In fact, the theorems in [AKO] and in [BCGLO] cited above both describe the subsystem $O^{p'}(\mathcal{F})$ in more precise detail.

Proposition 1.15. *For each saturated fusion system $\mathcal{F}$ over a finite p-group S, we have $O^{p'}(\mathcal{F})^c = \mathcal{F}^c$, $O^{p'}(\mathcal{F})^f = \mathcal{F}^f$, and $\mathbf{E}_{O^{p'}(\mathcal{F})} = \mathbf{E}_{\mathcal{F}}$.*

Proof. By Theorem 1.14, if $P \le S$ and $Q \in P^{\mathcal{F}}$, then there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha(Q) \in P^{O^{p'}(\mathcal{F})}$. From this, it follows immediately that $O^{p'}(\mathcal{F})$ and $\mathcal{F}$ have the same centric subgroups, and the same fully normalized subgroups. To see that they have the same essential subgroups, it remains to check that $\mathrm{Out}_{O^{p'}(\mathcal{F})}(P)$ has a strongly p-embedded subgroup if and only if $\mathrm{Out}_{\mathcal{F}}(P)$ does, and this is shown in Lemma B.1. □

We also need the following result, which gives a more precise description of $O^{p'}(\mathcal{F})$, but under very restrictive conditions on $\mathcal{F}$.

Proposition 1.16. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, such that*

(i) *$\mathbf{E}_{\mathcal{F}} \ne \varnothing$ and each member of $\mathbf{E}_{\mathcal{F}}$ is weakly closed in $\mathcal{F}$, and*

(ii) *no intersection of two distinct members of $\mathbf{E}_{\mathcal{F}}$ is $\mathcal{F}$-centric.*

Then

(a) $\mathrm{Aut}_{O^{p'}(N_{\mathcal{F}}(R))}(P) = \bigl\{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \bigm| \alpha|_R \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R))\bigr\}$ *for each $R \in \mathbf{E}_{\mathcal{F}}$ and each $R \le P \le S$; and*

(b) $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S) = \langle \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}}(R))}(S) \,|\, R \in \mathbf{E}_{\mathcal{F}}\rangle$.

Proof. For each $R \in \mathbf{E}_{\mathcal{F}}$, set $\mathcal{E}_R = O^{p'}(N_{\mathcal{F}}(R))$.

(a) Fix $R \in \mathbf{E}_{\mathcal{F}}$, and let H be a model for $N_{\mathcal{F}}(R)$ (see Proposition 1.12). Then $O^{p'}(H)$ is a model for $\mathcal{E}_R$, and an extension of R by $O^{p'}(H/R) \cong O^{p'}(\mathrm{Out}_{\mathcal{F}}(R))$. Hence

$$\mathrm{Aut}_{\mathcal{E}_R}(R) = \mathrm{Aut}_{O^{p'}(H)}(R) = O^{p'}(\mathrm{Aut}_H(R)) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)).$$

Let P be such that $R \le P \le S$. Then $\alpha \in \mathrm{Aut}_{\mathcal{E}_R}(P)$ implies $\alpha|_R \in \mathrm{Aut}_{\mathcal{E}_R}(R) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R))$. Conversely, if $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ is such that $\alpha|_R \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)) = \mathrm{Aut}_{\mathcal{E}_R}(R)$, then by the extension axiom and since $\alpha|_R$ normalizes $\mathrm{Aut}_P(R)$, there is $\beta \in \mathrm{Aut}_{\mathcal{E}_R}(P)$ such that $\beta|_R = \alpha|_R$. So by [AKO, Lemma I.5.6] and since $R \in \mathcal{F}^c$, there is $x \in Z(R)$ such that $\alpha = \beta \circ c_x$, and hence $\alpha \in \mathrm{Aut}_{\mathcal{E}_R}(P)$.

(b) Set

$$\mathcal{F}_0 = \langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)) \,|\, R \in \mathbf{E}_{\mathcal{F}} \rangle \qquad \text{and} \qquad O^{p'}_*(\mathcal{F}) = \langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \,|\, P \le S \rangle$$

as (not necessarily saturated) fusion systems over S. Thus $O^{p'}_*(\mathcal{F})$ is the minimal fusion subsystem in $\mathcal{F}$ of index prime to p. For $P \in \mathcal{F}^c$, since P is contained in at most one member of $\mathbf{E}_{\mathcal{F}}$ by (ii), the sets $\mathrm{Hom}_{\mathcal{F}}(P,S)$ and $\mathrm{Hom}_{\mathcal{F}_0}(P,S)$ and groups $\mathrm{Aut}_{\mathcal{F}}(P)$ and $\mathrm{Aut}_{\mathcal{F}_0}(P)$ are described as follows:

$\mathcal{E}$	$\mathrm{Hom}_{\mathcal{E}}(P,S)$	$\mathrm{Aut}_{\mathcal{E}}(P)$
$\mathcal{F}$	$\{\alpha\|_P \,\|\, \alpha \in \mathrm{Aut}_{\mathcal{F}}(R)\}$	$\{\alpha\|_P \,\|\, \alpha \in \mathrm{Aut}_{\mathcal{F}}(R),\ \alpha(P) = P\}$
$\mathcal{F}_0$	$\{\alpha\|_P \,\|\, \alpha \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R))\}$	$\{\alpha\|_P \,\|\, \alpha \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)),\ \alpha(P) = P\}$

TABLE 1.17. In each case, either R is the unique member of $\mathbf{E}_{\mathcal{F}}$ such that $P \le R$, or $R = S$ if there is no such member.

In particular, this shows that $\mathrm{Aut}_{\mathcal{F}_0}(P)$ is normal of index prime to p in $\mathrm{Aut}_{\mathcal{F}}(P)$ for each $P \in \mathcal{F}^c$, and hence by [AKO, Lemma I.7.6(a)] that $\mathcal{F}_0$ has index prime to p in $\mathcal{F}$. Thus $\mathcal{F}_0 = O^{p'}_*(\mathcal{F})$ (the inclusion $\mathcal{F}_0 \le O^{p'}_*(\mathcal{F})$ is immediate from the definitions). So

$$\begin{aligned}
\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S) &= \big\langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha|_P \in \mathrm{Hom}_{O^{p'}_*(\mathcal{F})}(P,S),\ \text{some } P \in \mathcal{F}^c \big\rangle \\
&= \big\langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha|_P \in \mathrm{Hom}_{\mathcal{F}_0}(P,S) \text{ some } P \in \mathcal{F}^c \big\rangle \\
&= \big\langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \exists\, P \in \mathcal{F}^c,\ P \le R \in \mathbf{E}_{\mathcal{F}} \cup \{S\},\ \beta \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)),\ \text{s.t. } \alpha|_P = \beta|_P \big\rangle \\
&= \big\langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha|_R \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)) \text{ some } R \in \mathbf{E}_{\mathcal{F}} \cup \{S\} \big\rangle \\
&= \langle \mathrm{Aut}_{\mathcal{E}_R}(S) \,|\, R \in \mathbf{E}_{\mathcal{F}} \rangle \ :
\end{aligned}$$

the first equality by [AKO, Theorem I.7.7], the second since $\mathcal{F}_0 = O^{p'}_*(\mathcal{F})$, the third by Table 1.17, the fourth since $\alpha|_P = \beta|_P$ implies $\alpha|_R = \beta \circ c_x$ for some $x \in Z(P)$ (see [AKO, Lemma I.5.6]), and the last by (a) (applied with $P = S$). □

One can also show that $O^{p'}(\mathcal{F}) = \langle O^{p'}(N_{\mathcal{F}}(R)) \,|\, R \in \mathbf{E}_{\mathcal{F}} \rangle$ under the hypotheses of Proposition 1.16. However, that will not be needed here.

1.5. **Quotient fusion systems.**

Quotient fusion systems of $\mathcal{F}$ over S are formed by dividing out by a subgroup of S, not by a fusion subsystem of $\mathcal{F}$.

Definition 1.18. Let $\mathcal{F}$ be a fusion system, and assume $Q \le S$ is strongly closed in $\mathcal{F}$. In particular, $Q \trianglelefteq S$. Let $\mathcal{F}/Q$ be the fusion system over S/Q where for each $P, R \le S$ containing Q, we set

$$\mathrm{Hom}_{\mathcal{F}/Q}(P/Q, R/Q) = \big\{ \varphi/Q \in \mathrm{Hom}(P/Q, R/Q) \,\big|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q),\ (\varphi/Q)(gQ) = \varphi(g)Q\ \forall\, g \in P \big\}.$$

We refer to [Cr, Proposition II.5.11] for the proof that $\mathcal{F}/Q$ is saturated whenever $\mathcal{F}$ is. In fact, the definition and saturation of $\mathcal{F}/Q$ hold whenever Q is weakly closed in $\mathcal{F}$. This is not surprising, since we are looking only at morphisms in $\mathcal{F}$ between subgroups containing Q, so that $\mathcal{F}/Q = N_{\mathcal{F}}(Q)/Q$.

If Q is strongly closed in $\mathcal{F}$, then every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R)$, for arbitrary $P,Q \le S$, induces a (unique) morphism $\overline{\varphi} \in \mathrm{Hom}(PQ/Q, RQ/Q)$. (Just note that $\varphi(P\cap Q) \le R\cap Q$.) A much deeper theorem states that each such morphism $\overline{\varphi}$ also lies in $\mathcal{F}/Q$. We refer to [AKO, Theorem II.5.12] and [Cr, Theorem II.5.14] for proofs of this result first shown by Puig. In this paper, however, we work with $\mathcal{F}/Q$ only in the special case where $Q \trianglelefteq \mathcal{F}$, in which case this property is automatic.

We will need the following lemma, comparing essential subgroups in $\mathcal{F}$ and in $\mathcal{F}/Z$ when Z is central in $\mathcal{F}$.

Lemma 1.19. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and fix $Z \le Z(\mathcal{F})$. Then for each $R \le S$, $R \in \mathbf{E}_{\mathcal{F}}$ if and only if $R \ge Z$ and $R/Z \in \mathbf{E}_{\mathcal{F}/Z}$.*

Proof. If $R \in \mathbf{E}_{\mathcal{F}}$, then $R \in \mathcal{F}^c$, and hence $R \ge Z(S) \ge Z$. So from now on, we always assume that $R \ge Z$. We will show that the following hold for each $R \le S$ containing Z:

(a) $R \in \mathcal{F}^f$ if and only if $R/Z \in (\mathcal{F}/Z)^f$;

(b) the natural map $\Psi\colon \mathrm{Out}_{\mathcal{F}}(R) \longrightarrow \mathrm{Out}_{\mathcal{F}/Z}(R/Z)$ is surjective and its kernel is a p-group; and

(c) $R/Z \in (\mathcal{F}/Z)^c$ if and only if $R \in \mathcal{F}^c$ and Ψ is an isomorphism.

It follows immediately from (a), (b), and (c) and Definition 1.3 that $R \in \mathbf{E}_{\mathcal{F}}$ if $R/Z \in \mathbf{E}_{\mathcal{F}/Z}$. Conversely, if $R \in \mathbf{E}_{\mathcal{F}}$, then $O_p(\mathrm{Out}_{\mathcal{F}}(R)) = 1$ since $\mathrm{Out}_{\mathcal{F}}(R)$ has a strongly p-embedded subgroup (see [AKO, Proposition A.7(c)]), so Ψ is an isomorphism, and $R/Z \in \mathbf{E}_{\mathcal{F}/Z}$ by (a), (b), and (c) again.

Point (a) is clear, since $(R/Z)^{\mathcal{F}/Z} = \{P/Z \,|\, P \in R^{\mathcal{F}}\}$, and $N_{S/Z}(P/Z) = N_S(P)/Z$ whenever $Z \le P \le S$.

The natural map $\Psi\colon \mathrm{Aut}_{\mathcal{F}}(R) \longrightarrow \mathrm{Aut}_{\mathcal{F}/Z}(R/Z)$ is surjective by definition of $\mathcal{F}/Z$. If $[\alpha] \in \mathrm{Ker}(\Psi)$, where $[\alpha]$ is the class of $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$, then for some $x \in R$, αc_x^R induces the identity on R/Z and (since $Z \le Z(\mathcal{F})$) the identity on Z, and hence has p-power order by Lemma B.5. So $\mathrm{Ker}(\Psi)$ is a p-group, proving (b).

By (a), it suffices to prove (c) when $R \in \mathcal{F}^f$ and $R/Z \in (\mathcal{F}/Z)^f$. Assume $R/Z \in (\mathcal{F}/Z)^c$. Then $C_S(R)/Z \le C_{S/Z}(R/Z) \le R/Z$, so $R \in \mathcal{F}^c$. For each $[\alpha] \in \mathrm{Ker}(\Psi)$, the class of $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$, we have $[\alpha] \in O_p(\mathrm{Out}_{\mathcal{F}}(R)) \le \mathrm{Out}_S(R)$, so $\alpha = c_x^R$ for some $x \in N_S(R)$ such that $c_x^R \in \mathrm{Aut}(R)$ induces an inner automorphism on R/Z. Hence $xZ \in (R/Z)C_{S/Z}(R/Z)$, so $xZ \in R/Z$ since $R/Z \in (\mathcal{F}/Z)^c$, and $x \in R$. Thus $\alpha \in \mathrm{Inn}(R)$, and Ψ is an isomorphism in this case.

Conversely, assume $R \in \mathcal{F}^c$ and Ψ is an isomorphism, and let $y \in N_S(R)$ be such that $yZ \in C_{S/Z}(R/Z)$. Then $[y,R] \le Z$, so $[c_y^R] \in \mathrm{Ker}(\Psi) = 1$. So $c_y^R \in \mathrm{Inn}(R)$, and $y \in RC_S(R) = R$ since R is $\mathcal{F}$-centric. This shows that $C_{S/Z}(R/Z) \le R/Z$ and hence $R/Z \in (\mathcal{F}/Z)^c$, finishing the proof of (c). □

If $\mathcal{F}$ is a saturated fusion system over S and $P \le Q \le S$, then $P \trianglelefteq \mathcal{F}$ and $Q \trianglelefteq \mathcal{F}$ implies $Q/P \trianglelefteq \mathcal{F}/P$: this follows easily from the definitions. However, $P \trianglelefteq \mathcal{F}$ and $Q/P \trianglelefteq \mathcal{F}/P$ need not imply that $Q \trianglelefteq \mathcal{F}$, as is seen by the following example. Let p be any prime, set $G = C_p \wr \varSigma_p$ (wreath product), fix $S \in \mathrm{Syl}_p(G)$ (so $S \cong C_p \wr C_p$), and set $\mathcal{F} = \mathcal{F}_S(G)$. Set $P = O_p(G) \cong E_{p^p}$. Then $P \trianglelefteq \mathcal{F}$ and $S/P \trianglelefteq \mathcal{F}/P$, but S is not normal in $\mathcal{F}$.

In the following lemma, we give two conditions under which $P \trianglelefteq \mathcal{F}$ and $Q/P \trianglelefteq \mathcal{F}/P$ does imply that $Q \trianglelefteq \mathcal{F}$.

Lemma 1.20. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $P \leq Q \leq S$ be such that $P \trianglelefteq \mathcal{F}$ and $Q/P \trianglelefteq \mathcal{F}/P$. If Q is abelian, or if $P \leq Z(\mathcal{F})$, then $Q \trianglelefteq \mathcal{F}$.*

Proof. Since Q/P is normal, it is strongly closed in $\mathcal{F}/P$, and hence Q is strongly closed in $\mathcal{F}$. So if Q is abelian, then it is normal by Lemma 1.8. If $P \leq Z(\mathcal{F})$, then Q is contained in all $\mathcal{F}$-essential subgroups by Lemma 1.19 and since Q/P is contained in all $\mathcal{F}/P$-essential subgroups (Lemma 1.7), and so $Q \trianglelefteq \mathcal{F}$ by Lemma 1.7 again. □

2. General lemmas

As noted in the introduction, in our general setting, we want to analyze a saturated fusion system $\mathcal{F}$ over a finite p-group S with an abelian subgroup $A \leq S$ and $\mathit{\Gamma} = \mathrm{Aut}_{\mathcal{F}}(A)$, where the group A and the action of $O^{p'}(\mathit{\Gamma})$ are given. In this section, we give some of the tools that will be used in Sections 4 and 5 to do this.

In practice, we don't get very far without knowing that the subgroup A is normal in S and weakly closed in $\mathcal{F}$, and this should perhaps be included in our general assumptions. But in many cases, it follows easily from the weaker assumptions on A and $O^{p'}(\mathit{\Gamma})$.

Lemma 2.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $A \leq S$ be such that no member of $A^{\mathcal{F}} \smallsetminus \{A\}$ is contained in $N_S(A)$. Then A is weakly closed in $\mathcal{F}$.*

Proof. Assume otherwise: then $S > N_S(A)$, and hence $N_S(N_S(A)) > N_S(A)$. Choose $x \in N_S(N_S(A)) \smallsetminus N_S(A)$. Then ${}^{x}A \neq A$, contradicting the assumption that A not be S-conjugate to any other subgroup of $N_S(A)$. □

The importance of A being weakly closed in our general situation is illustrated by the following lemma.

Lemma 2.2. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $A \trianglelefteq S$ is an abelian subgroup that is weakly closed in $\mathcal{F}$.*

(a) *If $R \in \mathcal{F}^f$, and $R \in Q^{\mathcal{F}}$ for some $Q \leq A$, then $R \leq A$.*

(b) *For each $P, Q \leq A$, $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_{N_{\mathcal{F}}(A)}(P,Q)$. Hence each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ extends to some $\overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(A)$.*

(c) *No element of $C_S(A) \smallsetminus A$ is $\mathcal{F}$-conjugate to any element of A.*

Proof. **(a)** Assume $Q \leq A$ and $R \leq S$ are $\mathcal{F}$-conjugate and $R \in \mathcal{F}^f$. By the extension axiom, each $\psi \in \mathrm{Iso}_{\mathcal{F}}(Q,R)$ extends to some $\overline{\psi} \in \mathrm{Hom}_{\mathcal{F}}(C_S(Q),S)$. Then $C_S(Q) \geq A$ since A is abelian, $\overline{\psi}(A) = A$ since A is weakly closed in $\mathcal{F}$, and so $R = \overline{\psi}(Q) \leq A$.

(b) Assume $P, Q \leq A$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, and choose $R \in P^{\mathcal{F}}$ that is fully centralized in $\mathcal{F}$. Then $R \leq A$ by (a), and there is $\psi \in \mathrm{Iso}_{\mathcal{F}}(\varphi(P),R)$. By the extension axiom again, ψ extends to $\widehat{\psi} \in \mathrm{Hom}_{\mathcal{F}}(A,S)$ and $\psi\varphi$ extends to $\widehat{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(A,S)$, and $\widehat{\psi}(A) = A = \widehat{\varphi}(A)$ since A is weakly closed. Then $\widehat{\psi}^{-1}\widehat{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(A)$, and $(\widehat{\psi}^{-1}\widehat{\varphi})|_P = \psi^{-1}(\psi\varphi) = \varphi$.

(c) Assume $x \in C_S(A) \smallsetminus A$ is $\mathcal{F}$-conjugate to $y \in A$. By (a), we can arrange that $\langle y \rangle \in \mathcal{F}^f$, so by Lemma 1.2, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(\langle x \rangle),S)$ such that $\varphi(x) = y$. But $A \leq N_S(\langle x \rangle)$, $\varphi(A) = A$ since A is weakly closed, and this is impossible since $\varphi(x) \in A$ and $x \notin A$. So no element in $C_S(A) \smallsetminus A$ is $\mathcal{F}$-conjugate to any element of A. □

In many of the cases we want to consider, the assumptions we choose on A and on $\varGamma$ imply that $O^{p'}(\mathcal{F})$ is simple (see, e.g., [AKO, Definition I.6.1]). For example, if $\mathcal{F}$ is a saturated fusion system over S, and $A \trianglelefteq S$ is such that $C_S(A) = A$, and we set $\varGamma = \mathrm{Aut}_{\mathcal{F}}(A)$ and $\varGamma_0 = O^{p'}(\varGamma)$, and assume also that $\varOmega_1(A)$ is a simple $\mathbb{F}_p\boldsymbol{\varGamma}$-module and $\varGamma_0/O_{p'}(\varGamma_0)$ is a simple group (and $\varGamma_0 \ncong C_p$), then either $A \trianglelefteq \mathcal{F}$ or the fusion system $O^{p'}(\mathcal{F})$ is simple. However, this will not be needed, and before proving it here, we would first have to define normal fusion subsystems.

2.1. Proving that $\mathcal{F} = \langle N_{\mathcal{F}}(A), C_{\mathcal{F}}(Z)\rangle$.

When analyzing fusion systems in our setting, we first check whether $\mathcal{F} = \langle N_{\mathcal{F}}(A), C_{\mathcal{F}}(Z)\rangle$ for some choice of $Z \le Z(S)$. The following lemma will be our tool for doing this.

Proposition 2.3. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, let $A \trianglelefteq S$ be an abelian subgroup that is weakly closed in $\mathcal{F}$, and fix $1 \ne Z \le Z(S) \cap A$. Then either $\mathcal{F} = \langle C_{\mathcal{F}}(Z), N_{\mathcal{F}}(A)\rangle$, or there are $R \in \mathbf{E}_{\mathcal{F}}$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that α is not a morphism in $\langle C_{\mathcal{F}}(Z), N_{\mathcal{F}}(A)\rangle$, and such that $\alpha(Z) \nleq A$, $\alpha(Z) \in N_{\mathcal{F}}(A)^f$, and $R = C_S(\alpha(Z)) = N_S(\alpha(Z))$.*

Proof. Set $\mathcal{F}_0 = \langle C_{\mathcal{F}}(Z), N_{\mathcal{F}}(A)\rangle$: the smallest fusion system over S (not necessarily saturated) that contains both $C_{\mathcal{F}}(Z)$ and $N_{\mathcal{F}}(A)$. We first claim that

$$N_{\mathcal{F}}(Z) \le \langle C_{\mathcal{F}}(Z), \mathrm{Aut}_{\mathcal{F}}(S)\rangle \le \mathcal{F}_0. \tag{2.4}$$

The second inclusion is clear: $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{N_{\mathcal{F}}(A)}(S)$ since A is weakly closed in $\mathcal{F}$ by assumption. If $\varphi \in \mathrm{Hom}_{N_{\mathcal{F}}(Z)}(P,Q)$, where $P,Q \ge Z$, then since $S = C_S(Z)$, $\varphi|_Z \in \mathrm{Aut}_{\mathcal{F}}(Z)$ extends to some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ by the extension axiom, and $\varphi = \alpha \circ (\alpha^{-1}\varphi)$ where $\alpha^{-1}\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(Z)}(P,S)$. This proves the first inclusion in (2.4).

By Lemma 1.2 and since $Z \le Z(S)$ is fully normalized in $\mathcal{F}$, for each $X \in Z^{\mathcal{F}}$, there is $\psi_X \in \mathrm{Hom}_{\mathcal{F}}(N_S(X), S)$ such that $\psi_X(X) = Z$. Set

$$\mathcal{Z} = \{X \in Z^{\mathcal{F}} \mid \psi_X \in \mathrm{Mor}(\mathcal{F}_0)\}.$$

If $\psi' \in \mathrm{Hom}_{\mathcal{F}}(N_S(X), S)$ is another morphism such that $\psi'(X) = Z$, then $\psi' \circ \psi_X^{-1} \in \mathrm{Mor}(N_{\mathcal{F}}(Z))$, and hence $\psi' \in \mathrm{Mor}(\mathcal{F}_0)$ if and only if $\psi_X \in \mathrm{Mor}(\mathcal{F}_0)$ by (2.4). So $\mathcal{Z}$ is independent of the choices of the ψ_X.

If $X \in Z^{\mathcal{F}}$ and $X \le A$, then $A \le N_S(X)$ and $\psi_X(A) = A$, so $\psi_X \in \mathrm{Mor}(\mathcal{F}_0)$. Thus

$$X \in Z^{\mathcal{F}} \text{ and } X \le A \quad\Longrightarrow\quad X \in \mathcal{Z}. \tag{2.5}$$

If $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ is such that $P \ge Z$ and $X = \varphi(Z) \in \mathcal{Z}$, then $\varphi(P) \le C_S(X)$ since $P \le S = C_S(Z)$, so $\psi_X \circ \varphi$ is defined and in $N_{\mathcal{F}}(Z) \le \mathcal{F}_0$, and hence $\varphi = (\psi_X|_{\varphi(P)})^{-1} \circ (\psi_X \circ \varphi)$ is also in $\mathcal{F}_0$. Thus

$$\text{for each } \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S) \text{ with } Z \le P \le S, \quad \varphi(Z) \in \mathcal{Z} \implies \varphi \in \mathrm{Mor}(\mathcal{F}_0). \tag{2.6}$$

Assume $\mathcal{F} > \mathcal{F}_0$. By Theorem 1.5 (the Alperin-Goldschmidt fusion theorem), there are $R \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$ and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\alpha \notin \mathrm{Mor}(\mathcal{F}_0)$. Since $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}_0}(S)$ by (2.4), we have $R \in \mathbf{E}_{\mathcal{F}}$. Choose such R and α with $|R|$ maximal. Since R is $\mathcal{F}$-centric, we have $R \ge Z(S) \ge Z$. Set $X = \alpha(Z)$; then $X \notin \mathcal{Z}$ by (2.6), and hence $X \nleq A$ by (2.5). Also, $R \le C_S(X) \le N_S(X)$ since $R \le C_S(Z) = S$.

For each $Y \in Z^{\mathcal{F}} \smallsetminus \mathcal{Z}$, we have $\psi_Y \notin \mathrm{Mor}(\mathcal{F}_0)$ by definition of $\mathcal{Z}$. Hence ψ_Y is a composite of restrictions of automorphisms of members of $\mathbf{E}_{\mathcal{F}} \cup \{S\}$ of order at least $|N_S(Y)|$, and at least one of these automorphisms is not in $\mathcal{F}_0$. So by the maximality assumption on R,

$|R| \geq |N_S(Y)|$ for all $Y \in Z^{\mathcal{F}} \smallsetminus \mathcal{Z}$, and in particular, for all $Y \in X^{N_{\mathcal{F}}(A)}$. Since $R \leq N_S(X)$, this shows that X is fully normalized in $N_{\mathcal{F}}(A)$, and also that $R = C_S(X) = N_S(X)$. $\square$

Note in particular the following special case of Proposition 2.3.

Corollary 2.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, let $A \trianglelefteq S$ be an abelian subgroup that is weakly closed in $\mathcal{F}$, and fix $1 \neq Z \leq Z(S) \cap A$. Assume that $A \trianglelefteq C_{\mathcal{F}}(Z)$ but $A \ntrianglelefteq \mathcal{F}$. Then there are $R \in \mathbf{E}_{\mathcal{F}}$ and $\alpha \in \operatorname{Aut}_{\mathcal{F}}(R)$ such that $\alpha(Z) \nleq A$, $\alpha(Z) \in N_{\mathcal{F}}(A)^f$, and $R = C_S(\alpha(Z)) = N_S(\alpha(Z))$.*

Proof. By assumption, $C_{\mathcal{F}}(Z) \leq N_{\mathcal{F}}(A) < \mathcal{F}$. So $\langle C_{\mathcal{F}}(Z), N_{\mathcal{F}}(A)\rangle \neq \mathcal{F}$, and the result follows from Proposition 2.3. $\square$

2.2. Normality of subgroups.

The results in this subsection will be useful when showing that certain subgroups, especially abelian subgroups, are strongly closed or normal in a fusion system.

Lemma 2.8. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $Q \trianglelefteq S$ be a normal subgroup that is not weakly closed in $\mathcal{F}$. Then there are $P \in Q^{\mathcal{F}} \smallsetminus \{Q\}$, $R \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, and $\alpha \in \operatorname{Aut}_{\mathcal{F}}(R)$ such that $R \geq Q$, $P = \alpha(Q)$, $R = N_S(P)$, $P \in N_{\mathcal{F}}(Q)^f$, and $|R| \geq |N_S(U)|$ for all $U \in Q^{\mathcal{F}} \smallsetminus \{Q\}$.*

Proof. Let $\mathscr{W}$ be the set of pairs (R, α) where $R \in \mathbf{E}_{\mathcal{F}} \cup \{S\}$, $R \geq Q$, $\alpha \in \operatorname{Aut}_{\mathcal{F}}(R)$, and $\alpha(Q) \neq Q$. Since Q is not weakly closed in $\mathcal{F}$, there is $\varphi \in \operatorname{Hom}_{\mathcal{F}}(Q, S)$ such that $\varphi(Q) \neq Q$, and hence $\mathscr{W} \neq \varnothing$ by the Alperin-Goldschmidt fusion theorem (Theorem 1.5).

Choose $(R, \alpha) \in \mathscr{W}$ such that $|R|$ is maximal. By Lemma 1.2, for each $U \in Q^{\mathcal{F}} \smallsetminus \{Q\}$, there is a morphism $\varphi \in \operatorname{Hom}_{\mathcal{F}}(N_S(U), S)$ such that $\varphi(U) = Q$. By Theorem 1.5 again, there is $(R_1, \alpha_1) \in \mathscr{W}$ such that $|R_1| \geq |N_S(U)|$, and $|R| \geq |R_1|$ by the maximality of $|R|$. Thus $|R| \geq |N_S(U)|$ for each $U \in Q^{\mathcal{F}} \smallsetminus \{Q\}$.

Now set $P = \alpha(Q)$. Then $P \trianglelefteq R$ since $Q \trianglelefteq R$, so $R \leq N_S(P)$, with equality since we just saw $|R| \geq |N_S(P)|$. Also, $P \in N_{\mathcal{F}}(Q)^f$ since $|R| \geq |N_S(U)|$ for each $U \in Q^{\mathcal{F}} \smallsetminus \{Q\} \supseteq P^{N_{\mathcal{F}}(Q)}$. $\square$

The following is a more technical result that will be needed when proving that $Q/Z \trianglelefteq C_{\mathcal{F}}(Z)/Z$ in case (i) of Theorem A.

Proposition 2.9. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $A \trianglelefteq S$ be an abelian subgroup that is weakly closed in $\mathcal{F}$ but not normal. Let $1 = A_0 < A_1 < \cdots < A_m = A$ be such that $[S, A_i] \leq A_{i-1}$ for each $1 \leq i \leq m$. Set $\mathcal{E}_0 = \mathcal{F}$, and for each $1 \leq i \leq m$, set $\overline{A}_i = A_i/A_{i-1}$ and $\mathcal{E}_i = C_{\mathcal{E}_{i-1}}(\overline{A}_i)/\overline{A}_i$, regarded as a fusion system over S/A_i. (Note that $\overline{A}_i \leq Z(S/A_{i-1})$.) Then there are $0 \leq \ell \leq m-2$, $R \leq S$, and $\alpha \in \operatorname{Aut}_{\mathcal{F}}(R)$, such that*

- *$R \geq A_{\ell+1}$, $[\alpha, A_i] \leq A_{i-1}$ for $1 \leq i \leq \ell$, and $X \overset{\text{def}}{=} \alpha(A_{\ell+1}) \nleq A$;*
- *$R = N_S(X)$, $R/A_\ell = C_{S/A_\ell}(X/A_\ell)$, and $X/A_\ell \in N_{\mathcal{E}_\ell}(A/A_\ell)^f$; and*
- *$R/A_\ell \in \mathbf{E}_{\mathcal{E}_\ell}$.*

Proof. The fusion systems $\mathcal{E}_i$ are all saturated by Theorem 1.10 and [Cr, Proposition II.5.11], applied iteratively. Also, A/A_{m-1} is weakly closed in $\mathcal{E}_{m-1}$ since A is weakly closed in $\mathcal{F}$. All $\mathcal{E}_{m-1}$-essential subgroups contain $Z(S/A_{m-1}) \geq A/A_{m-1}$ since they are centric, so $A/A_{m-1} \trianglelefteq \mathcal{E}_{m-1}$ by Lemma 1.7. Since $A \ntrianglelefteq \mathcal{E}_0 = \mathcal{F}$ by assumption, there is $0 \leq \ell \leq m-2$ such that $A/A_\ell \ntrianglelefteq \mathcal{E}_\ell$ and $A/A_{\ell+1} \trianglelefteq \mathcal{E}_{\ell+1}$.

We now apply Corollary 2.7, with A/A_ℓ, $A_{\ell+1}/A_\ell$, and $\mathcal{E}_\ell$ in the role of A, Z, and $\mathcal{F}$. Here, $A_{\ell+1}/A_\ell \le Z(S/A_\ell)$ since $[A_{\ell+1}, S] \le A_\ell$, while $A/A_\ell \ntrianglelefteq \mathcal{E}_\ell$ by assumption. Since A/A_ℓ is abelian, it is normal in $C_{\mathcal{E}_\ell}(\overline{A}_{\ell+1})$ by Lemma 1.20 and since $A/A_{\ell+1} \trianglelefteq \mathcal{E}_{\ell+1} = C_{\mathcal{E}_\ell}(\overline{A}_{\ell+1})/\overline{A}_{\ell+1}$. So by Corollary 2.7, there are $R \le S$ containing A_ℓ, and $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{E}_\ell}(R/A_\ell)$, such that $R/A_\ell = C_{S/A_\ell}(\overline{\alpha}(\overline{A}_{\ell+1})) \in \mathbf{E}_{\mathcal{E}_\ell}$, and

$$X/A_\ell \overset{\text{def}}{=} \overline{\alpha}(\overline{A}_{\ell+1}) \nleq A/A_\ell, \quad R/A_\ell = N_{S/A_\ell}(X/A_\ell), \quad \text{and} \quad X/A_\ell \in N_{\mathcal{E}_\ell}(A/A_\ell)^f. \tag{2.10}$$

Also, $R/A_\ell \ge Z(S/A_\ell) \ge \overline{A}_{\ell+1}$ since R/A_ℓ is $\mathcal{E}_\ell$-centric, so $R \ge A_{\ell+1}$.

Set $\alpha_\ell = \overline{\alpha}$, and choose $\alpha_i \in \mathrm{Aut}_{C_{\mathcal{E}_i}(\overline{A}_{i+1})}(R/A_i) \le \mathrm{Aut}_{\mathcal{E}_i}(R/A_i)$ for decreasing indices $i = \ell-1, \ell-2, \dots, 0$ so that $\alpha_i/\overline{A}_{i+1} = \alpha_{i+1}$ for each $i < \ell$. Set $\alpha = \alpha_0 \in \mathrm{Aut}_{\mathcal{F}}(R)$; then $[\alpha, A_i] \le A_{i-1}$ for each i by by definition of the $\mathcal{E}_i$, and $X = \alpha(A_{\ell+1}) \nleq A$ since $X/A_\ell = \overline{\alpha}(\overline{A}_{\ell+1}) \nleq A/A_\ell$. The other claims listed in the proposition follow easily from (2.10). $\square$

2.3. **Equalities between fusion systems.**

We finish the section with two sets of conditions for showing that two fusion systems over the same p-group are equal. Proposition 2.11 will be applied to the fusion systems encountered in Section 4, and Proposition 2.13 to those in Section 5.

Proposition 2.11. *Let $\mathcal{F}_1 \ge \mathcal{E} \le \mathcal{F}_2$ be saturated fusion systems over a finite p-group S. Assume that $Q \trianglelefteq S$ is centric and normal in all three, and that $\mathrm{Aut}_{\mathcal{F}_1}(Q) = \mathrm{Aut}_{\mathcal{F}_2}(Q)$. Assume also that the homomorphism*

$$H^1(\mathrm{Out}_{\mathcal{F}_1}(Q); Z(Q)) \longrightarrow H^1(\mathrm{Out}_{\mathcal{E}}(Q); Z(Q))$$

induced by restriction is surjective. Then $\mathcal{F}_1 = \mathcal{F}_2$.

Proof. Let $M_1 \ge H \le M_2$ be models for $\mathcal{F}_1 \ge \mathcal{E} \le \mathcal{F}_2$ (Definition 1.11), where $S \le H$ is a Sylow p-subgroup of all three. Thus M_1 and M_2 are both extensions of Q by $\mathrm{Out}_{\mathcal{F}_1}(Q) = \mathrm{Out}_{\mathcal{F}_2}(Q)$, and the difference of the two extensions (up to isomorphism) is represented by an element $\chi \in H^2(\mathrm{Out}_{\mathcal{F}_1}(Q); Z(Q))$ (see [McL, Theorem IV.8.8]). Also, χ vanishes after restriction to $H^2(\mathrm{Out}_{\mathcal{E}}(Q); Z(Q))$ since M_1 and M_2 both contain H, so $\chi = 0$ since $\mathrm{Out}_{\mathcal{E}}(Q)$ has index prime to p in $\mathrm{Out}_{\mathcal{F}_1}(Q)$. Thus there is an isomorphism $\psi\colon M_1 \longrightarrow M_2$ such that $\psi|_Q = \mathrm{Id}_Q$. Note that ψ also induces the identity on H/Q and on S/Q since they inject into $\mathrm{Aut}(Q)$, but need not induce the identity on S.

Set $\psi_0 = \psi|_H \in \mathrm{Aut}(H)$. Consider the commutative diagram

$$\begin{tikzcd}
H^1(M_1/Q; Z(Q)) \arrow[r, "\eta_1", "\cong"'] \arrow[d, "\rho_1"'] & C_{\mathrm{Aut}(M_1)}(Q)/\mathrm{Aut}_{Z(Q)}(M_1) \arrow[d, "\rho_2"'] \\
H^1(H/Q; Z(Q)) \arrow[r, "\eta_2", "\cong"'] & C_{\mathrm{Aut}(H)}(Q)/\mathrm{Aut}_{Z(Q)}(H)
\end{tikzcd}$$

where η_1, η_2 are defined as in [OV, Lemma 1.2]. Since ρ_1 is surjective by assumption, ρ_2 is also surjective. So there is $\alpha \in \mathrm{Aut}(M_1)$ such that $\alpha|_H = \psi_0 c_z|_H$ for some $z \in Z(Q)$, and upon replacing α by αc_z^{-1}, we can arrange that $\alpha|_H = \psi_0$.

Now set $\varphi = \psi\alpha^{-1}\colon M_1 \xrightarrow{\cong} M_2$. Then $\varphi|_H = \psi_0\psi_0^{-1} = \mathrm{Id}_H$, and in particular, $\varphi|_S = \mathrm{Id}_S$. Since M_1 and M_2 are models for $\mathcal{F}_1$ and $\mathcal{F}_2$, we conclude that $\mathcal{F}_1 = \mathcal{F}_2$. $\square$

The other criterion we give for two fusion systems to be equal applies only to fusion systems satisfying some very restrictive hypotheses, which are stated separately for easier reference.

Hypotheses 2.12. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Assume $A, Q \trianglelefteq S$ are such that*

(i) $\mathbf{E}_{\mathcal{F}} = \{A, Q\}$;

(ii) *A is abelian, $S = AQ$, and $C_S(A \cap Q) = A$; and*

(iii) $p \nmid \bigl|N_{\mathrm{Aut}(A)}(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A)))/O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A))\bigr|$.

Note that $\mathcal{F} = N_{\mathcal{F}}(R)$ if $\mathbf{E}_{\mathcal{F}} = \{R\}$ has order 1, while $\mathcal{F} = N_{\mathcal{F}}(S)$ if $\mathbf{E}_{\mathcal{F}} = \varnothing$. So the next proposition still holds if we assume $\mathbf{E}_{\mathcal{F}} \subseteq \{A, Q\}$ instead of assuming equality. However, since the extra cases that would be added are rather trivial and will not be encountered in this paper, we decided to use the more restrictive version.

Proposition 2.13. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two saturated fusion systems over the same finite p-group S, and let $A, Q \trianglelefteq S$ be normal subgroups with respect to which Hypotheses 2.12 hold for $\mathcal{F}_1$ and for $\mathcal{F}_2$. Assume also that $O^{p'}(N_{\mathcal{F}_1}(A)) = O^{p'}(N_{\mathcal{F}_2}(A))$ and $O^{p'}(\mathrm{Aut}_{\mathcal{F}_1}(Q)) = O^{p'}(\mathrm{Aut}_{\mathcal{F}_2}(Q))$. Then $O^{p'}(\mathcal{F}_1) = O^{p'}(\mathcal{F}_2)$.*

Proof. If Hypotheses 2.12 hold for $\mathcal{F}_i$ ($i = 1, 2$), then they also hold for $O^{p'}(\mathcal{F}_i)$ (note in particular that $\mathbf{E}_{O^{p'}(\mathcal{F}_i)} = \mathbf{E}_{\mathcal{F}_i}$ by Proposition 1.15). So it suffices to prove the proposition when $\mathcal{F}_i = O^{p'}(\mathcal{F}_i)$ for $i = 1, 2$.

Since $S = AQ$ where A and Q are both properly contained in S, we have $Q \ngeq A$ and $A \ngeq Q$. Note that Q is nonabelian, since otherwise $C_S(A \cap Q) = S$, contradicting 2.12(ii). Also, A and Q are weakly closed in $\mathcal{F}_i$ for $i = 1, 2$, since otherwise, there would be $\alpha \in \mathrm{Aut}_{\mathcal{F}_i}(S)$ with $\alpha(A) \neq A$ or $\alpha(Q) \neq Q$, which is impossible since α permutes the members of $\mathbf{E}_{\mathcal{F}_i}$.

Set

$$\Theta = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_2}(S) \rangle \leq \mathrm{Aut}(S).$$

Fix $R \in \{A, Q\}$. Each element of Θ normalizes R since R is weakly closed in $\mathcal{F}_1$ and in $\mathcal{F}_2$. For each $\alpha \in \Theta$ such that $\alpha|_R = \mathrm{Id}_R$, α also induces the identity on S/R since $C_S(R) \leq R$ (since $R \in \mathbf{E}_{\mathcal{F}_i}$ by 2.12(i)), and hence α has p-power order. Thus

$$\bigl\{\alpha \in \Theta \bigm| \alpha|_R = \mathrm{Id}_R\bigr\} \leq O_p(\Theta) \quad (\text{for } R \in \{A, Q\}): \tag{2.14}$$

this subgroup is normal in Θ since all elements in Θ normalize R.

By points (i) and (ii) in Hypotheses 2.12 and since A and Q are weakly closed, the conclusions of Lemma 1.16 hold for $\mathcal{F}_1$ and $\mathcal{F}_2$. (Note that $Q \cap A \notin \mathcal{F}^c$ since it is strictly contained in the abelian group A.) By Lemma 1.16(b) and since $O^{p'}(\mathcal{F}_i) = \mathcal{F}_i$ for $i = 1, 2$ by assumption,

$$\mathrm{Aut}_{\mathcal{F}_i}(S) = \bigl\langle \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(A))}(S), \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(Q))}(S) \bigr\rangle \tag{2.15}$$

for $i = 1, 2$.

Again fix $R \in \{A, Q\}$. If $\alpha \in \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_1}(R))}(S)$, then $\alpha|_R \in O^{p'}(\mathrm{Aut}_{\mathcal{F}_1}(R)) = O^{p'}(\mathrm{Aut}_{\mathcal{F}_2}(R))$ by Lemma 1.16(a), so $\alpha|_R = \beta|_R$ for some $\beta \in \mathrm{Aut}_{\mathcal{F}_2}(S)$ by the extension axiom and since $\alpha|_R$ is normalized by $\mathrm{Aut}_S(R)$. By Lemma 1.16(a) again, $\beta \in \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_2}(R))}(S)$. Also, $\alpha^{-1}\beta \in O_p(\Theta)$ by (2.14) and since $\alpha|_R = \beta|_R$. Upon repeating this argument with the roles of $\mathcal{F}_1$ and $\mathcal{F}_2$ exchanged, we have shown that

$$\mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_1}(R))}(S) O_p(\Theta) = \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_2}(R))}(S) O_p(\Theta).$$

Together with (2.15), this implies that

$$\mathrm{Aut}_{\mathcal{F}_1}(S)O_p(\Theta) = \mathrm{Aut}_{\mathcal{F}_2}(S)O_p(\Theta). \tag{2.16}$$

For $R \in \{A, Q\}$, set

$$\varGamma^{(R)} = O^{p'}(\mathrm{Aut}_{\mathcal{F}_1}(R)) = O^{p'}(\mathrm{Aut}_{\mathcal{F}_2}(R)),$$

where the last two groups are equal by assumption. Then for $i = 1, 2$,

$$\mathrm{Aut}_{\mathcal{F}_i}(R) = \varGamma^{(R)} \cdot \{\alpha|_R \,|\, \alpha \in \mathrm{Aut}_{\mathcal{F}_i}(S)\} \tag{2.17}$$

by the Frattini argument and the extension axiom (and since $R \trianglelefteq S$).

Set $\Theta^{(A)} = \langle \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_2}(A)\rangle$. Then $\varGamma^{(A)} \trianglelefteq \Theta^{(A)}$ since it is normal in each $\mathrm{Aut}_{\mathcal{F}_i}(A)$. Since $N_{\mathrm{Aut}(A)}(\varGamma^{(A)})/\varGamma^{(A)}$ has order prime to p by 2.12(iii), we have $O^{p'}(\Theta^{(A)}) = O^{p'}(\varGamma^{(A)}) = \varGamma^{(A)}$. By (2.17), for each $\alpha \in \mathrm{Aut}_{\mathcal{F}_1}(A)$, there are $\alpha_0 \in \varGamma^{(A)}$ and $\widehat{\alpha} \in \mathrm{Aut}_{\mathcal{F}_1}(S)$ such that $\alpha = \alpha_0(\widehat{\alpha}|_A)$. By (2.16), there is $\widehat{\beta} \in \mathrm{Aut}_{\mathcal{F}_2}(S)$ such that $\widehat{\alpha}^{-1}\widehat{\beta} \in O_p(\Theta)$. Set $\beta = \alpha_0(\widehat{\beta}|_A) \in \mathrm{Aut}_{\mathcal{F}_2}(A)$. Then $\alpha^{-1}\beta = (\widehat{\alpha}^{-1}\widehat{\beta})|_A$ has p-power order, hence lies in $O^{p'}(\Theta^{(A)}) = \varGamma^{(A)}$, and we have shown that $\mathrm{Aut}_{\mathcal{F}_1}(A) \le \mathrm{Aut}_{\mathcal{F}_2}(A)$. A similar argument proves the opposite inclusion, and thus

$$\mathrm{Aut}_{\mathcal{F}_1}(A) = \mathrm{Aut}_{\mathcal{F}_2}(A). \tag{2.18}$$

For $i = 1, 2$,

$$\begin{aligned}
\mathrm{Aut}_{\mathcal{F}_i}(Q) &= \varGamma^{(Q)} \cdot \big\{\alpha|_Q \,\big|\, \alpha \in \mathrm{Aut}_{\mathcal{F}_i}(S)\big\} \\
&= \varGamma^{(Q)} \cdot \big\langle \big\{\alpha|_Q \,\big|\, \alpha \in \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(Q))}(S)\big\}\,,\, \big\{\alpha|_Q \,\big|\, \alpha \in \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(A))}(S)\big\}\big\rangle \\
&\le \varGamma^{(Q)} \cdot \big\langle \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(Q))}(Q)\,,\, \big\{\alpha|_Q \,\big|\, \alpha \in \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(A))}(S)\big\}\big\rangle \\
&= \varGamma^{(Q)} \cdot \big\{\alpha|_Q \,\big|\, \alpha \in \mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(A))}(S)\big\} :
\end{aligned}$$

the first equality by (2.17), the second by (2.15), and the last since $\mathrm{Aut}_{O^{p'}(N_{\mathcal{F}_i}(Q))}(Q) = \varGamma^{(Q)}$ by Lemma 1.16(a). The opposite inclusion is clear, so

$$\mathrm{Aut}_{\mathcal{F}_1}(Q) = \mathrm{Aut}_{\mathcal{F}_2}(Q) \tag{2.19}$$

since $O^{p'}(N_{\mathcal{F}_1}(A)) = O^{p'}(N_{\mathcal{F}_2}(A))$ by assumption.

For $R \in \{A, Q\}$, consider the homomorphism

$$\Theta = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_2}(S)\rangle \xrightarrow{\ \Psi_R\ } N_{\mathrm{Aut}_{\mathcal{F}_1}(R)}(\mathrm{Aut}_S(R)) = N_{\mathrm{Aut}_{\mathcal{F}_2}(R)}(\mathrm{Aut}_S(R)),$$

where $\mathrm{Aut}_{\mathcal{F}_1}(R) = \mathrm{Aut}_{\mathcal{F}_2}(R)$ by (2.18) or (2.19), and where Ψ_R is induced by restriction to R and is surjective by the extension axiom. Hence Ψ_R sends $O_p(\Theta)$ into the group $O_p(N_{\mathrm{Aut}_{\mathcal{F}_i}(R)}(\mathrm{Aut}_S(R))) = \mathrm{Aut}_S(R)$. So for each $\beta \in O_p(\Theta)$, there are $g, h \in S$ such that $\beta|_A = c_h^A$ and $\beta|_Q = c_g^Q$. Then $\beta(c_g^S)^{-1}$ is the identity on Q and conjugation by hg^{-1} after restriction to A, so $hg^{-1} \in C_S(Q \cap A) = A$ by 2.12(ii), and $\beta(c_g^S)^{-1}|_A = \mathrm{Id}$. Since $S = AQ$ by 2.12(ii), this shows that $\beta = c_g^S$, and hence that $O_p(\Theta) = \mathrm{Inn}(S)$. So $\mathrm{Aut}_{\mathcal{F}_1}(S) = \mathrm{Aut}_{\mathcal{F}_2}(S)$ by (2.16). Since $\mathbf{E}_{\mathcal{F}_i} = \{A, Q\}$ by 2.12(i), this together with (2.18) and (2.19) (and Theorem 1.5) shows that

$$\mathcal{F}_1 = \langle \mathrm{Aut}_{\mathcal{F}_1}(S), \mathrm{Aut}_{\mathcal{F}_1}(A), \mathrm{Aut}_{\mathcal{F}_1}(Q)\rangle = \langle \mathrm{Aut}_{\mathcal{F}_2}(S), \mathrm{Aut}_{\mathcal{F}_2}(A), \mathrm{Aut}_{\mathcal{F}_2}(Q)\rangle = \mathcal{F}_2. \qquad \square$$

3. Todd modules in characteristic 3

We describe here the notation we use in Sections 4 and 5 to make computations involving Todd modules: first the Todd module for $2M_{12}$, and afterwards those for M_{11} and $A_6 \cong O^2(M_{10})$.

3.1. The ternary Golay code and the group $2M_{12}$.

We first set up notation for handling the ternary Golay code $\mathscr{G}$ and its automorphism group $2M_{12}$. Our notation is based on that used by Griess in [Gr, Chapter 7] to describe the ternary Golay code. We begin by fixing some very general notation for describing n-tuples of elements in a field.

Notation 3.1. *For a finite set $X = \{1, 2, \dots, n\}$ and a field K, we regard K^X as the vector space of maps $X \longrightarrow K$, and let $\{e_i \,|\, i \in X\}$ be its canonical basis:*

$$\{e_i \,|\, i \in X\} \subseteq K^X \qquad\qquad \text{where } e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } i, j \in X.$$

We also set $e_J = \sum_{j \in J} e_j$ for $J \subseteq X$. Let

$$\mathrm{Perm}_X(K) \leq \mathrm{Mon}_X(K) \leq \mathrm{Aut}(K^X)$$

be the subgroups of permutation automorphisms and monomial automorphisms, respectively: automorphisms that permute the basis $\{e_i\}$ or the subspaces $\{Ke_i\}$, respectively. Thus if $|X| = n$, then $\mathrm{Perm}_X(K) \cong \Sigma_n$ and $\mathrm{Mon}_X(K) \cong K^\times \wr \Sigma_n$. Let

$$\pi = \pi_{X,K} \colon \mathrm{Mon}_X(K) \longrightarrow \mathrm{Perm}_X(K)$$

be the canonical projection that sends a monomial automorphism to the corresponding permutation automorphism; thus $\mathrm{Ker}(\pi_{X,K})$ is the group of automorphisms that send each Ke_i to itself.

Now set $I = \{1, 2, 3, 4\}$, and regard $\mathbb{F}_3^I$ as the space of 4-tuples of elements of $\mathbb{F}_3$ as well as that of functions $I \longrightarrow \mathbb{F}_3$. Let $\mathscr{T} \subseteq \mathbb{F}_3^I$ be the *tetracode subgroup*:

$$\begin{aligned} \mathscr{T} &= \{(a, b, b + a, b + 2a) \,|\, a, b \in \mathbb{F}_3\} \\ &= \bigl\{\xi \in \mathbb{F}_3^I \,\big|\, \xi(3) = \xi(1) + \xi(2),\ \xi(4) = \xi(1) + \xi(3)\bigr\}. \end{aligned} \tag{3.2}$$

Thus $\mathscr{T}$ is a 2-dimensional subspace of $\mathbb{F}_3^I$. By [Gr, Lemma 7.3],

$$\mathrm{Aut}(\mathscr{T}) \overset{\text{def}}{=} \bigl\{\alpha \in \mathrm{Mon}_I(\mathbb{F}_3) \,\big|\, \alpha(\mathscr{T}) = \mathscr{T}\bigr\} \cong GL_2(3) \cong 2\Sigma_4. \tag{3.3}$$

More precisely, each linear automorphism of $\mathscr{T}$ extends to a unique monomial automorphism of $\mathbb{F}_3^I$; and each permutation of I lifts to a monomial automorphism of $\mathbb{F}_3^I$, unique up to sign, that acts on $\mathscr{T}$.

Set $\Delta = \mathbb{F}_3 \times I$, so that $\mathbb{F}_3^\Delta$ is a 12-dimensional vector space over $\mathbb{F}_3$. Define $C_1, C_2, C_3, C_4 \in \mathbb{F}_3^\Delta$ by setting

$$C_i = e_{(0,i)} + e_{(1,i)} + e_{(2,i)} \qquad \text{for } i \in I,$$

and set $\mathscr{C} = \{C_i \,|\, i \in I\}$. Thus $e_\Delta = \sum_{i \in I} C_i$. Define

$$\mathfrak{Gr} \colon \mathbb{F}_3^I \longrightarrow \mathbb{F}_3^\Delta \qquad \text{by setting} \qquad \mathfrak{Gr}(\xi) = \sum\nolimits_{i \in I} e_{(\xi(i), i)}$$

(the "graph" of ξ). Thus for each $(c, i) \in \Delta$, $\mathfrak{Gr}(\xi)(c, i) = 1$ if $c = \xi(i)$ and is zero otherwise. Finally, define $\mathscr{G} < \overline{\mathscr{G}} < \mathbb{F}_3^\Delta$ by setting

$$\overline{\mathscr{G}} = \bigl\langle \mathscr{C} \cup \mathfrak{Gr}(\mathscr{T}) \bigr\rangle \qquad \text{and} \qquad \mathscr{G} = \bigl\langle C_i + \mathfrak{Gr}(\xi) \,\big|\, i \in I, \xi \in \mathscr{T} \bigr\rangle. \tag{3.4}$$

Finally, for $i,j\in I$ and $\xi\in\mathscr{T}$, we define

$$C_{ij}=C_i-C_j\in\mathscr{G}\qquad\text{and}\qquad\mathfrak{gr}_\xi=\mathfrak{Gr}(\xi)-\mathfrak{Gr}(0)\in\mathscr{G}.$$

The C_i are clearly linearly independent in $\overline{\mathscr{G}}$. The relations

$$\mathfrak{Gr}(\xi)+\mathfrak{Gr}(\eta)+\mathfrak{Gr}(\theta)=\sum_{\substack{i\in I\\ \xi(i)\neq\eta(i)}}C_i\quad\text{for all }\xi,\eta,\theta\in\mathscr{T}\text{ such that }\xi+\eta+\theta=0\tag{3.5}$$

among the C_i and $\mathfrak{Gr}(\xi)$ are easily checked. So for any $\mathbb{F}_3$-basis $\{\xi_1,\xi_2\}$ of $\mathscr{T}$,

$$\begin{aligned}\overline{\mathscr{G}}&=\bigl\langle C_1,C_2,C_3,C_4,\mathfrak{Gr}(0),\mathfrak{Gr}(\xi_1),\mathfrak{Gr}(\xi_2)\bigr\rangle\\ \mathscr{G}&=\bigl\langle C_{12},C_{13},C_{14},\mathfrak{gr}_{\xi_1},\mathfrak{gr}_{\xi_2},C_1+\mathfrak{Gr}(0)\bigr\rangle.\end{aligned}$$

These elements in each of these two sets are independent in $\mathbb{F}_3^\Delta$, and hence form bases for $\overline{\mathscr{G}}$ and $\mathscr{G}$, respectively. So $\dim(\overline{\mathscr{G}})=7$ and $\dim(\mathscr{G})=6$.

The subspace $\mathscr{G}$ is the ternary Golay code. We refer to [Gr, Lemmas 7.8 & 7.9] for more details and more properties. Note in particular that $\mathscr{G}=\mathscr{G}^\perp$ under the standard inner product on $\mathbb{F}_3^\Delta$ (i.e., that for which the standard basis $\{e_{(c,i)}\,|\,(c,i)\in\Delta\}$ is orthonormal).

We next look at automorphisms of $\mathscr{G}$.

Notation 3.6. (a) *Set* $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}=\{\xi\in\mathrm{Mon}_\Delta(\mathbb{F}_3)\,|\,\xi(\mathscr{G})=\mathscr{G}\}$.

(b) *For* $\eta\in\mathbb{F}_3^I$, *let* $\mathfrak{tr}_\eta\in\mathrm{Perm}_\Delta(\mathbb{F}_3)$ *be the translation that sends* $e_{(c,i)}$ *to* $e_{(c+\eta(i),i)}$. *Thus for* $\xi\in\mathbb{F}_3^\Delta$, *we have* $\mathfrak{tr}_\eta(\xi)(c,i)=\xi(c-\eta(i),i)$.

(c) *Fix* $\alpha\in\mathrm{Mon}_I(\mathbb{F}_3)$, *and let* $\varepsilon_i\in\mathbb{F}_3^\times$ $(i\in I)$ *and* $\sigma\in\Sigma_I$ *be such that* $\alpha(e_i)=\varepsilon_ie_{\sigma(i)}$ *for all* i. *Let* $\boldsymbol{\tau}(\alpha)\in\mathrm{Perm}_\Delta(\mathbb{F}_3)$ *be the automorphism that sends* $e_{(c,i)}$ *to* $e_{(\varepsilon_ic,\sigma(i))}$. *Thus for* $\xi\in\mathbb{F}_3^\Delta$, *we have* $(\boldsymbol{\tau}(\alpha)(\xi))(c,i)=\xi(\varepsilon_{\sigma^{-1}(i)}c,\sigma^{-1}(i))$.

(d) *Define*

$$\boldsymbol{N_0}=\mathfrak{tr}_{\mathscr{T}}\rtimes\boldsymbol{\tau}(Aut(\mathscr{T}))=\bigl\langle\mathfrak{tr}_\eta,\boldsymbol{\tau}(\alpha)\,\big|\,\eta\in\mathscr{T},\alpha\in\mathrm{Aut}(\mathscr{T})\bigr\rangle\le\widehat{\boldsymbol{M}}_{\boldsymbol{12}},$$

and set $\boldsymbol{N}=\boldsymbol{N_0}\times\{\pm\mathrm{Id}\}\le\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$.

By [Gr, Proposition 7.29], $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}\cong 2M_{12}$.

Note the following relations, for $\eta,\theta\in\mathbb{F}_3^I$, $i\in I$, and $\alpha\in\mathrm{Mon}_I(\mathbb{F}_3)$:

$$\begin{aligned}\mathfrak{tr}_\eta(C_i)&=C_i &\qquad \boldsymbol{\tau}(\alpha)(C_i)&=C_{\pi(\alpha)(i)}\\ \mathfrak{tr}_\eta(\mathfrak{Gr}(\theta))&=\mathfrak{Gr}(\theta+\eta) &\qquad \boldsymbol{\tau}(\alpha)(\mathfrak{Gr}(\theta))&=\mathfrak{Gr}(\alpha(\theta)).\end{aligned}$$

To see the last equality, note that for $\alpha\in\mathrm{Mon}_I(\mathbb{F}_3)$ with $\varepsilon_i\in\mathbb{F}_3^\times$ and $\sigma\in\Sigma_I$ as above, and for $\theta=\sum_{i\in I}\theta(i)e_i$ in $\mathbb{F}_3^I$, we have

$$\boldsymbol{\tau}(\alpha)(\mathfrak{Gr}(\theta))=\sum_{i\in I}\boldsymbol{\tau}(\alpha)(e_{(\theta(i),i)})=\sum_{i\in I}e_{(\varepsilon_i\theta(i),\sigma(i))}=\mathfrak{Gr}(\theta')$$

where $\theta'=\sum_{i\in I}\varepsilon_i\theta(i)e_{\sigma(i)}=\alpha(\theta)$. In particular, these formulas show that the action of $\boldsymbol{N_0}$ on $\mathbb{F}_3^\Delta$ sends $\overline{\mathscr{G}}$ and $\mathscr{G}$ to themselves.

Lemma 3.7. *We have* $\boldsymbol{N}=N_{\widehat{\boldsymbol{M}}_{\boldsymbol{12}}}(\mathfrak{tr}_{\mathscr{T}})$, *and this is a maximal subgroup of* $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$.

Proof. By construction, $\boldsymbol{N}\le N_{\widehat{\boldsymbol{M}}_{\boldsymbol{12}}}(\mathfrak{tr}_{\mathscr{T}})$. Conversely, by [Gr, Theorem 7.20], $\boldsymbol{N}$ is the subgroup of all elements of $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ whose action on Δ permutes the columns $\mathbb{F}_3\times\{i\}$, and hence contains the normalizer of $\mathfrak{tr}_{\mathscr{T}}$.

For the maximality of $\boldsymbol{N} \le \widehat{\boldsymbol{M}}_{\mathbf{12}}$ (or of $\boldsymbol{N}/\{\pm\mathrm{Id}\} \cong E_9 \rtimes GL_2(3)$ in $\widehat{\boldsymbol{M}}_{\mathbf{12}}/\{\pm\mathrm{Id}\} \cong M_{12}$), see [Co, p. 235] or [A3, p. 8]. Note that if we regard M_{12} as a group of permutations of 12 points, then $\boldsymbol{N}/\{\pm\mathrm{Id}\} \cong M_9 \rtimes \Sigma_3$ is the subgroup of those permutations that normalize a set of three of the points. □

One easy consequence of Lemma 3.7 is that $\boldsymbol{N_0} = \widehat{\boldsymbol{M}}_{\mathbf{12}} \cap \mathrm{Perm}_\Delta(\mathbb{F}_3)$. In other words, the elements of $\boldsymbol{N_0}$ are the only ones in $\widehat{\boldsymbol{M}}_{\mathbf{12}}$ that permute the coordinates in Δ without sign changes. But this will not be needed later.

To simplify later calculations, we next describe $\mathscr{G}$ and the action of $\boldsymbol{N_0}$ on it in terms of (3×3) matrices over $\mathbb{F}_3$. In general, for a vector space V over a field K, we let $S_2(V)$ denote its symmetric power

$$S_2(V) = (V \otimes_K V)/\langle (v\otimes w) - (w \otimes v) \,|\, v,w \in V\rangle.$$

For $v,w \in V$, let $[v\otimes w] \in S_2(V)$ denote the class of $v \otimes w \in V \otimes_K V$, and write $v^{\otimes 2} = [v\otimes v]$ for short. When $\alpha \in \mathrm{Aut}_K(V)$, we let $S_2(\alpha) \in \mathrm{Aut}_K(S_2(V))$ be the automorphism $S_2(\alpha)([v\otimes w]) = [\alpha(v)\otimes \alpha(w)]$.

Definition 3.8. (a) Choose a map of sets $\lambda\colon I \longrightarrow \mathscr{T}$ such that for each $i \in I$, $\lambda(i) \neq 0$ and $(\lambda(i))(i) = 0$. Define a map of sets

$$\Phi_0\colon \mathscr{C} \cup \mathfrak{Gr}(\mathscr{T}) \xrightarrow{\qquad} S_2(\mathscr{T} \oplus \mathbb{F}_3)$$

by setting

$$\Phi_0(C_i) = (\lambda(i),0)^{\otimes 2} \qquad \text{and} \qquad \Phi_0(\mathfrak{Gr}(\xi)) = (\xi,1)^{\otimes 2}$$

for all $i \in I$ and all $\xi \in \mathscr{T}$.

(b) Define $\Theta_*\colon \boldsymbol{N_0} \longrightarrow \mathrm{Aut}(\mathscr{T} \oplus \mathbb{F}_3)$ by setting

$$\Theta_*(\mathfrak{tr}_\eta \boldsymbol{\tau}(\alpha))(\xi,a) = (\alpha(\xi)+a\eta, a)$$

for each $\eta,\xi \in \mathscr{T}$, $\alpha \in \mathrm{Aut}(\mathscr{T})$, and $a \in \mathbb{F}_3$.

We now check that Φ_0 and Θ_* extend to a natural isomorphism from the $\mathbb{F}_3\boldsymbol{N_0}$-module $\mathscr{G}$ to the group $S_2(\mathscr{T} \oplus \mathbb{F}_3)$ with action of a certain subgroup of $\mathrm{Aut}(\mathscr{T} \otimes \mathbb{F}_3)$.

Lemma 3.9. (a) *The map Φ_0 of Definition 3.8(a) is independent of the choice of λ, and extends to a surjective homomorphism $\overline{\Phi}\colon \overline{\mathscr{G}} \longrightarrow S_2(\mathscr{T} \oplus \mathbb{F}_3)$. This in turn restricts to an isomorphism Φ_* from $\mathscr{G}$ onto $S_2(\mathscr{T} \oplus \mathbb{F}_3)$.*

(b) *The map Θ_* of Definition 3.8(b) is an isomorphism from $\boldsymbol{N_0} \cong \mathscr{T} \rtimes \mathrm{Aut}(\mathscr{T})$ onto the group of all automorphisms of $\mathscr{T} \oplus \mathbb{F}_3$ that are the identity modulo $\mathscr{T} \oplus 0$.*

(c) *For each $\beta \in \boldsymbol{N_0}$ and each $\gamma \in \mathscr{G}$,*

$$\Phi_*(\beta(\gamma)) = S_2(\Theta_*(\beta))(\Phi_*(\gamma)). \tag{3.10}$$

Thus Θ_* and Φ_* define an isomorphism from $\mathscr{G}$ as an $\mathbb{F}_3\boldsymbol{N_0}$-module to $S_2(\mathscr{T} \oplus \mathbb{F}_3)$ with its natural structure as a module over $\Theta_*(\boldsymbol{N_0}) < \mathrm{Aut}(\mathscr{T} \oplus \mathbb{F}_3)$.

Proof. **(a)** For each $i \in I$, the choice of $\lambda(i)$ is unique up to sign. So $\overline{\Phi}(C_i) = (\lambda(i),0)^{\otimes 2}$ is independent of the choice of $\lambda(i)$.

We first check that $\sum_{i\in I} \Phi_0(C_i) = 0$. It suffices to show that $\sum_{i\in I} \lambda(i)^{\otimes 2} = 0$ in $S_2(\mathscr{T})$. Independently of our choices, $\{\lambda(i) \,|\, i \in I\}$ is a set of representatives of the four subspaces of dimension 1 in $\mathbb{F}_3^2$. So the $\lambda(i)$ are permuted up to sign by each $\alpha \in \mathrm{Aut}(\mathscr{T})$, and the sum of the $\lambda(i)^{\otimes 2}$ is fixed by each such α. Hence the sum must be zero. (Alternatively, this can be shown directly by choosing coordinates and then computing with matrices.)

We next check that the relations (3.5) hold for the images of the elements in $\mathscr{C} \cup \mathfrak{Gr}(\mathscr{T})$ under Φ_0 as defined above. So fix $\xi, \eta, \theta \in \mathscr{T}$ such that $\xi+\eta+\theta=0$. If $\xi=\eta=\theta$, then (3.5) clearly holds. Otherwise, $\xi-\eta \neq 0$, so there is a unique index $j \in I$ such that $(\xi-\eta)(j)=0$. Then $\xi-\eta=\pm\lambda(j)$, and so

$$\begin{aligned}(\xi,1)^{\otimes 2}+(\eta,1)^{\otimes 2}+(\theta,1)^{\otimes 2} &= (\xi,0)^{\otimes 2}+(\eta,0)^{\otimes 2}+(\theta,0)^{\otimes 2} \\ &= (\xi,0)^{\otimes 2}+(\eta,0)^{\otimes 2}+(-\xi-\eta,0)^{\otimes 2} = -(\xi-\eta,0)^{\otimes 2} \\ &= -(\lambda(j),0)^{\otimes 2} = \sum_{i\in I\smallsetminus\{j\}} (\lambda(i),0)^{\otimes 2},\end{aligned}$$

where the first equality holds since $\xi+\eta+\theta=0$, and the last one since $\sum_{i\in I}\Phi_0(C_i)=0$.

Thus Φ_0 extends to a homomorphism defined on a vector space over $\mathbb{F}_3$ with basis $\mathscr{C} \cup \mathfrak{Gr}(\mathscr{T})$, modulo the subspace generated by the relations (3.5). This quotient space is generated by the images of the C_i, as well as those of 0, ξ_1, and ξ_2 for any basis $\{\xi_1,\xi_2\}$ of $\mathscr{T}$, hence has dimension 7 and is isomorphic to $\overline{\mathscr{G}}$. So Φ_0 extends to a homomorphism $\overline{\Phi}$ from $\overline{\mathscr{G}}$ to $S_2(\mathscr{T}\oplus\mathbb{F}_3)$.

Now, $\overline{\Phi}(\langle\mathscr{C}\rangle) = \langle(\eta,0)^{\otimes 2}\rangle = S_2(\mathscr{T}\oplus 0)$ since $\mathscr{T}^\# = \{\lambda(i)^{\pm 1} \,|\, i\in I\}$. Hence

$$\overline{\Phi}(\boldsymbol{N_0}) = S_2(\mathscr{T}\oplus 0)\big\langle (\xi,1)^{\times 2} \,\big|\, \xi\in\mathscr{T}\big\rangle = S_2(\mathscr{T}\oplus\mathbb{F}_3).$$

Thus $\overline{\Phi}$ is onto, and a comparison of dimensions shows that $\mathrm{Ker}(\overline{\Phi}) = \langle e_\Delta\rangle$. Since $e_\Delta \notin \mathscr{G}$, $\overline{\Phi}$ restricts to an isomorphism Φ_* from $\mathscr{G}$ to $\mathrm{Sym}_3(\mathbb{F}_3)$.

(b) One easily checks that Θ_* as defined above restricts to homomorphisms on $\{\mathfrak{tr}_\eta \,|\, \eta\in\mathscr{T}\} \cong \mathscr{T}$ and on $\mathrm{Aut}(\mathscr{T})$. So it remains only to check conjugacy relations: for $\alpha\in\mathrm{Aut}(\mathscr{T})$ and $\eta\in\mathscr{T}$, we have

$$\begin{aligned}\Theta_*(\alpha)\big(\Theta_*(\mathfrak{tr}_\eta)(\Theta_*(\alpha)^{-1}(\xi,a))\big) &= \Theta_*(\alpha)(\alpha^{-1}(\xi)+a\eta,a) = (\xi, a\cdot\alpha(\eta),a) \\ &= \Theta_*(\mathfrak{tr}_{\alpha(\eta)})(\xi,a) = \Theta_*(\alpha\circ\mathfrak{tr}_\eta\circ\alpha^{-1})(\xi,a).\end{aligned}$$

Thus Θ_* is well defined on $\boldsymbol{N_0}$, and it clearly defines an isomorphism onto the group of all $\beta\in\mathrm{Aut}(\mathscr{T}\oplus\mathbb{F}_3)$ that are the identity modulo $\mathscr{T}\oplus 0$.

(c) For each $\xi,\eta\in\mathscr{T}$, $i\in I$, and $\alpha\in\mathrm{Aut}(\mathscr{T})$, we have

$$\begin{aligned}\overline{\Phi}(\mathfrak{tr}_\eta(\mathfrak{Gr}(\xi))) &= (\xi+\eta,1)^{\otimes 2} = (\Theta_*(\mathfrak{tr}_\eta)(\xi,1))^{\otimes 2} = S_2(\Theta_*(\mathfrak{tr}_\eta))(\overline{\Phi}(\mathfrak{Gr}(\xi))) \\ \overline{\Phi}(\boldsymbol{\tau}(\alpha)(\mathfrak{Gr}(\xi))) &= (\alpha(\xi),1)^{\otimes 2} = (\Theta_*(\boldsymbol{\tau}(\alpha))(\xi,1))^{\otimes 2} = S_2(\Theta_*(\boldsymbol{\tau}(\alpha)))(\overline{\Phi}(\mathfrak{Gr}(\xi))) \\ \overline{\Phi}(\mathfrak{tr}_\eta(C_i)) &= \overline{\Phi}(C_i) = (\lambda(i),0)^{\otimes 2} = S_2(\Theta_*(\mathfrak{tr}_\eta))(\overline{\Phi}(C_i)).\end{aligned}$$

Also, for all $\alpha\in\mathrm{Aut}(\mathscr{T})$ inducing the permutation $\sigma\in\Sigma_I$, and all $i\in I$,

$$\begin{aligned}\overline{\Phi}(\boldsymbol{\tau}(\alpha)(C_i)) = \overline{\Phi}(C_{\sigma(i)}) &= \big(\lambda(\sigma(i)),0\big)^{\otimes 2} \\ &= \big(\pm\Theta_*(\boldsymbol{\tau}(\alpha))(\lambda(i),0)\big)^{\otimes 2} = S_2(\Theta_*(\boldsymbol{\tau}(\alpha)))(\overline{\Phi}(C_i))\end{aligned}$$

where $\lambda(\sigma(i)) = \pm\boldsymbol{\tau}(\alpha)(\lambda(i))$ by definition (and uniqueness up to sign) of the $\lambda(i)$. Since $\overline{\mathscr{G}} = \langle\mathscr{C}\cup\mathfrak{Gr}(\mathscr{T})\rangle$ and $\boldsymbol{N_0} = \langle\mathfrak{tr}_\eta, \boldsymbol{\tau}(\alpha) \,|\, \eta\in\mathscr{T},\ \alpha\in\mathrm{Aut}(\mathscr{T})\rangle$, this proves (3.10). $\square$

To simplify computations still farther, we now describe elements in $\boldsymbol{N_0}$ and $\boldsymbol{A}$ as (3×3)-matrices over $\mathbb{F}_3$. Fix an isomorphism $\mathscr{T}\cong\mathbb{F}_3^2$ (e.g., by restriction to the first two coordinates), so that $\mathscr{T}\oplus\mathbb{F}_3$ is identified with $\mathbb{F}_3^3$ and $\mathrm{Aut}(\mathscr{T}\oplus\mathbb{F}_3)$ with $GL_3(\mathbb{F}_3)$. We then identify $S_2(\mathscr{T} \oplus \mathbb{F}_3)$ with the group $\mathrm{Sym}_3(\mathbb{F}_3)$ of symmetric (3×3) matrices over $\mathbb{F}_3$, by sending the class $[v \otimes w]$ (for $v, w \in \mathbb{F}_3^3$) to $\frac{1}{2}(v \cdot w^t + w \cdot v^t)$. More explicitly,

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes \begin{pmatrix} d \\ e \\ f \end{pmatrix}\right] \quad \text{is sent to} \quad \begin{pmatrix} ad & (ae+bd) & (af+cd)/2 \\ (ae+bd)/2 & be & (bf+ce)/2 \\ (af+cd)/2 & (bf+ce)/2 & cf \end{pmatrix}.$$

Let

$$\Phi\colon \mathscr{G} \xrightarrow{\;\cong\;} \mathrm{Sym}_3(\mathbb{F}_3) \tag{3.11}$$

$$\Theta\colon \boldsymbol{N_0} \xrightarrow{\;\cong\;} \left\{ \left(\begin{smallmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{smallmatrix}\right) \;\middle|\; \begin{matrix} a,b,c,d,e,f \in \mathbb{F}_3 \\ ae-bd \neq 0 \end{matrix} \right\} \le GL_3(\mathbb{F}_3) \tag{3.12}$$

be the composites of Φ_* and Θ_* with the isomorphisms induced by this identification $\mathscr{T} \cong \mathbb{F}_3^2$. Lemma 3.9(c) now takes the following form:

Lemma 3.13. *For each $\beta \in \boldsymbol{N_0}$ and each $\xi \in \mathscr{G}$,*

$$\Phi(\beta(\xi)) = \Theta(\beta)\Phi(\xi)\Theta(\beta)^t \in \mathrm{Sym}_3(\mathbb{F}_3).$$

As a first, very simple application, we describe the Jordan blocks for actions on $\boldsymbol{A}$.

Lemma 3.14. *There are exactly two conjugacy classes of elements of order 3 in $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$: those in one class act on $\mathscr{G}$ with three Jordan blocks of lengths $1, 2, 3$, and those in the other with two Jordan blocks of length 3. In particular, for each $x \in \widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ of order 3, $\mathrm{rk}(C_{\mathscr{G}}(x)) \le 3$.*

Proof. Each element of order 3 in M_{12} is the image of a unique element of order 3 in $2M_{12}$. So $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ has two conjugacy classes of elements of order 3 since M_{12} does (see, e.g., [Gr, Exercise 7.34(ii)]). With the help of Lemma 3.13, it is straightforward to check that $\Theta^{-1}\left(\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right)$ acts on $\mathscr{G}$ with three Jordan blocks of lengths $1, 2, 3$, and that $\Theta^{-1}\left(\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right)$ acts with two Jordan blocks of length 3. Thus these elements are in different classes, and each element of order 3 in $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ is conjugate to one of them and acts on $\mathscr{G}$ in one of these two ways. The last statement holds since the rank of $C_{\mathscr{G}}(x)$ is equal to the number of Jordan blocks. (See also [Gr, Exercise 7.37].) $\square$

The notation developed in this subsection is summarized in Table 3.15.

$$\boldsymbol{\Gamma} = \widehat{\boldsymbol{M}}_{\boldsymbol{12}} = \bigl\{\alpha \in \mathrm{Mon}_\Delta(\mathbb{F}_3) \bigm| \alpha(\mathscr{G}) = \mathscr{G}\bigr\} \cong 2M_{12}$$

$$\boldsymbol{N_0} = \mathfrak{tr}_{\mathscr{T}} \rtimes \boldsymbol{\tau}(\mathrm{Aut}(\mathscr{T})) \le \widehat{\boldsymbol{M}}_{\boldsymbol{12}}$$

$$\boldsymbol{N} = \boldsymbol{N_0} \times \{\pm\mathrm{Id}\} = \bigl\{\alpha \in \widehat{\boldsymbol{M}}_{\boldsymbol{12}} \bigm| \alpha \text{ permutes the } \mathscr{K}_i\bigr\}$$

$$\Theta\colon \boldsymbol{N_0} \xrightarrow{\;\cong\;} \bigl\{\left(\begin{smallmatrix} A & v \\ 0 & 1 \end{smallmatrix}\right) \bigm| A \in GL_2(3),\ v \in \mathbb{F}_3^2\bigr\} \le GL_3(3)$$

$$\boldsymbol{T} = \Theta^{-1}(UT_3(\mathbb{F}_3)) \in \mathrm{Syl}_3(\boldsymbol{N_0}) \subseteq \mathrm{Syl}_3(\boldsymbol{\Gamma})$$

$$\boldsymbol{A} = \Phi(\mathscr{G}) = \mathrm{Sym}_3(\mathbb{F}_3)$$

$$\beta(X) = \Theta(\beta)X\Theta(\beta)^t \quad \text{for } \beta \in \boldsymbol{N_0},\ X \in \boldsymbol{A}$$

TABLE 3.15. Notation used for certain subgroups of $\boldsymbol{\Gamma} = \widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ and their action on $\boldsymbol{A} = \Phi(\mathscr{G})$.

3.2. **Notation for the Todd modules of M_{11} and A_6.**

We next set up notation to work with the Todd modules of M_{11} and of $A_6 \cong O^2(M_{10})$. In particular, we get explicit descriptions of the actions of certain subgroups of A_6 and M_{11}.

Let $\mathscr{G} \le \mathbb{F}_3^{\Delta}$ be as in (3.4) and Notation 3.6. By [Gr, Lemma 7.12], $\mathscr{G}$ contains exactly 12 pairs $\{\pm\theta\}$ of elements of weight 12. Three of those pairs lie in $\langle\mathscr{C}\rangle$: the elements of the form $\sum_{i\in I}\varepsilon_i C_i$ for $\varepsilon_i \in \mathbb{F}_3^\times$ and $\sum_{i\in I}\varepsilon_i = 0$. (The other nine have the form $\pm(e_\Delta + \mathfrak{Gr}(\xi))$ for $\xi \in \mathscr{T}$.) By a direct check, for each basis $\{\xi,\eta\}$ of $\mathscr{T}$, the six elements

$$\big\{\pm((\xi,0)^{\otimes 2} + (\eta,0)^{\otimes 2}),\ \pm((\xi,0)^{\otimes 2} - (\eta,0)^{\otimes 2}) \pm [(\xi,0)\otimes(\eta,0)]\big\} \subseteq S_2(\mathscr{T}\oplus\mathbb{F}_3) \qquad (3.16)$$

are the images of the six elements of weight 12 in $\langle\mathscr{C}\rangle$ under the isomorphism

$$\Phi_*\colon \mathscr{G} \xrightarrow{\ \cong\ } S_2(\mathscr{T}\oplus\mathbb{F}_3)$$

of Lemma 3.9(a). We want to identify M_{11} as the subgroup of elements in $\widehat{\boldsymbol{M}}_{\mathbf{12}}$ that are the identity on one of these subspaces, and similarly for M_{10}.

To simplify these descriptions, we identify $\mathscr{T}$ with $\mathbb{F}_9$ via some arbitrarily chosen isomorphism. We adopt the following notation for elements of $\mathbb{F}_9$:

$$\begin{aligned} &\mathbb{F}_9 = \mathbb{F}_3[i] \text{ where } i^2 = -1 \\ &\zeta = 1 + i \text{ of order 8 in } \mathbb{F}_9^\times \\ &\phi \in \mathrm{Aut}(\mathbb{F}_9) \ \ :\quad \phi(a+bi) = a - bi \text{ for } a,b \in \mathbb{F}_3. \end{aligned} \qquad (3.17)$$

We also write $\overline{x} = \phi(x)$ for $x \in \mathbb{F}_9$.

Notation 3.18. *Assume Notation 3.6 and Table 3.15, and choose an $\mathbb{F}_3$-linear isomorphism $\kappa\colon \mathscr{T} \xrightarrow{\cong} \mathbb{F}_9$. Define elements $\theta_1,\theta_2,\theta_3 \in S_2(\mathscr{T}) \le S_2(\mathscr{T}\oplus\mathbb{F}_3)$ by setting*

$$\begin{aligned} \theta_1 &= S_2(\kappa)^{-1}([1\otimes 1 + i\otimes i]) \\ \theta_2 &= S_2(\kappa)^{-1}([1\otimes 1 - i\otimes i + 1\otimes i]) \\ \theta_3 &= S_2(\kappa)^{-1}([1\otimes 1 - i\otimes i - 1\otimes i]). \end{aligned}$$

Set $\theta_i^* = \Phi_*^{-1}(\theta_i) \in \mathscr{G}$. By (3.16), $\pm\theta_1^*$, $\pm\theta_2^*$, and $\pm\theta_3^*$ are elements of weight 12 in $\mathscr{G}$, and the only ones in $\langle\mathscr{C}\rangle \cap \mathscr{G}$.

Set $K_1 = \langle\theta_1^*\rangle$ and $K_2 = \langle\theta_2^*, \theta_3^*\rangle$, both subspaces of $\mathscr{G}$, and define

$$\widehat{\boldsymbol{M}}_{\mathbf{11}} = N_{\widehat{\boldsymbol{M}}_{\mathbf{12}}}(K_1) \qquad \text{and} \qquad \widehat{\boldsymbol{M}}_{\mathbf{10}} = N_{\widehat{\boldsymbol{M}}_{\mathbf{12}}}(K_2).$$

Also, set $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}^{\mathbf{0}} = O^{3'}(\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}})$ and $\boldsymbol{N}^{(\boldsymbol{\ell})} = \boldsymbol{N} \cap \widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$ for $\ell = 10, 11$, and set $\boldsymbol{T} = \mathfrak{tr}_{\mathscr{T}}$.

Finally, define $\lambda\colon \mathbb{F}_9^\times\langle\phi\rangle \longrightarrow \mathrm{Aut}(\mathscr{T})$ by setting $\lambda(u) = \kappa^{-1}(x \mapsto ux)\kappa$ for $u \in \mathbb{F}_9^\times$ and $\lambda(\phi) = \kappa^{-1}\phi\kappa$. (Recall that we compose from right to left.) For $x \in \mathbb{F}_9$ and $u \in \mathbb{F}_9^\times$, set

$$(\!(x)\!) = \mathfrak{tr}_{\kappa^{-1}(x)} \in \boldsymbol{T}, \quad [u] = \boldsymbol{\tau}(\lambda(u)) \in \boldsymbol{N}, \quad \text{and} \quad [\phi] = \boldsymbol{\tau}(\lambda(\phi)) \in \boldsymbol{N}.$$

Also, for $\xi \in \boldsymbol{N}_{\mathbf{0}}$, we write $-\xi = \xi\cdot(-\mathrm{Id}) \in \boldsymbol{N}$.

For easy reference, we summarize in Table 3.20 some of the basic properties of groups defined in Notation 3.18.

Lemma 3.19. *Assume Notation 3.18. Then for $\ell = 10, 11$, $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}^{\mathbf{0}} = C_{\widehat{\boldsymbol{M}}_{\mathbf{12}}}(K_{12-\ell}) = C_{\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}}(K_{12-\ell})$, and the groups $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$, $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}^{\mathbf{0}}$, $\boldsymbol{N}^{(\boldsymbol{\ell})}$, and $\boldsymbol{T}$ are as described in Table 3.20. In particular, $\boldsymbol{T} \in \mathrm{Syl}_3(\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}) = \mathrm{Syl}_3(\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}^{\mathbf{0}})$.*

	$\ell=10$	$\ell=11$
$\boldsymbol{T}$	$\mathfrak{tr}_{\mathscr{T}} = \langle (\!(x)\!) \mid x\in\mathbb{F}_9\rangle$	$\mathfrak{tr}_{\mathscr{T}} = \langle (\!(x)\!) \mid x\in\mathbb{F}_9\rangle$
$\boldsymbol{N^{(\ell)}}$	$\boldsymbol{T}\langle [\zeta], [\phi], -\mathrm{Id}\rangle$	$\boldsymbol{T}\langle [\zeta], [\phi], -\mathrm{Id}\rangle$
$\boldsymbol{N^{(\ell)}}\cap\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}}$	$\boldsymbol{T}\langle -[i]\rangle$	$\boldsymbol{T}\langle -[\zeta], [\phi]\rangle$
$\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}}\cong$	A_6	M_{11}
$\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}/\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}}\cong$	D_8	C_2

TABLE 3.20. In particular, $\boldsymbol{N^{(10)}}=\boldsymbol{N^{(11)}}\cong(E_9\rtimes SD_{16})\times C_2$.

Proof. By definition (see Notation 3.6(d)), each element of $\boldsymbol{N}$ normalizes the subspace $\langle\mathscr{C}\rangle\cap\mathscr{G}$, and hence permutes the six elements $\pm\theta_1,\pm\theta_2,\pm\theta_3$ (the only elements of weight 12 in $\langle\mathscr{C}\rangle\cap\mathscr{G}$). Some of these actions are described in Table 3.21.

$g\in\boldsymbol{N}$	$[\zeta]$	$[i]$	$[\phi]$	$-\mathrm{Id}$
${}^g\theta_1^*$	$-\theta_1^*$	θ_1^*	θ_1^*	$-\theta_1^*$
${}^g\theta_2^*$	$-\theta_3^*$	$-\theta_2^*$	θ_3^*	$-\theta_2^*$
${}^g\theta_3^*$	θ_2^*	$-\theta_3^*$	θ_2^*	$-\theta_3^*$

TABLE 3.21.

Consider, for example, the case ${}^{[\zeta]}\theta_2^*$. Set $\xi=\kappa^{-1}(1)$ and $\eta=\kappa^{-1}(i)$, where $\kappa\colon\mathscr{T}\xrightarrow{\cong}\mathbb{F}_9$ is as in Notation 3.18. Then

$$\Phi_*(\theta_2^*)=\theta_2=S_2(\kappa)^{-1}\bigl([1\otimes1-i\otimes i+1\otimes i]\bigr)=[\xi\otimes\xi-\eta\otimes\eta+\xi\otimes\eta].$$

Since $\zeta=1+i$ and $i\zeta=-1+i$, we get

$$\begin{aligned}\Phi_*({}^{[\zeta]}\theta_2^*)&=S_2(\kappa)^{-1}\bigl([(1+i)\otimes(1+i)-(-1+i)\otimes(-1+i)+(1+i)\otimes(-1+i)]\bigr)\\&=[(\xi+\eta)\otimes(\xi+\eta)-(-\xi+\eta)\otimes(-\xi+\eta)+(\xi+\eta)\otimes(-\xi+\eta)]\\&=[4(\xi\otimes\eta)-\xi\otimes\xi+\eta\times\eta]=-\Phi_*(\theta_3^*).\end{aligned}$$

Hence ${}^{[\zeta]}\theta_2^*=-\theta_3^*$. The other computations are similar, but simpler in most cases.

Recall that $\boldsymbol{N}=(\mathfrak{tr}_{\mathscr{T}}\rtimes\boldsymbol{\tau}(\mathrm{Aut}(\mathscr{T})))\times\{\pm\mathrm{Id}\}$ (Notation 3.6(d)), where $\mathrm{Aut}(\mathscr{T})\cong GL_2(3)\cong 2\Sigma_4$ by (3.3). Since the element $[-1]=[i]^2$ centralizes K_1K_2 by Table 3.21, each element of $\mathfrak{tr}_{\mathscr{T}}=[[-1],\mathfrak{tr}_{\mathscr{T}}]$ also centralizes K_1K_2. Also, each noncentral element of $O_2(\boldsymbol{\tau}(\mathrm{Aut}(\mathscr{T})))=\langle[i],[\zeta\phi]\rangle\cong Q_8$ fixes one of the θ_i^* and sends the other two to their negative, and hence each element of order 3 in $\boldsymbol{\tau}(\mathrm{Aut}(\mathscr{T}))$ acts by permuting the sets $\{\pm\theta_i^*\}$ $(i=1,2,3)$ cyclically. From this, we conclude that $\boldsymbol{N^{(10)}}=\boldsymbol{N^{(11)}}$ is as described in Table 3.20, and also that

$$C_{\boldsymbol{N^{(10)}}}(K_2)=\boldsymbol{T}\langle-[i]\rangle\quad\text{and}\quad C_{\boldsymbol{N^{(11)}}}(K_1)=\boldsymbol{T}\langle-[\zeta],[\phi]\rangle.$$

In particular, $\boldsymbol{N^{(10)}}/C_{\boldsymbol{N^{(10)}}}(K_2)\cong D_8$ and $\boldsymbol{N^{(11)}}/C_{\boldsymbol{N^{(11)}}}(K_1)\cong C_2$.

It remains only to show that $\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}}=C_{\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}}(K_{12-\ell})$. For $\ell=10$ or $\ell=11$, consider the action of $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}=N_{\widehat{\boldsymbol{M}}_{\boldsymbol{12}}}(K_{12-\ell})$ on $\mathscr{G}/K_{12-\ell}$. Since $\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{10}}\cong O^{3'}(M_{10})\cong A_6$ and $\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{11}}\cong M_{11}$ by definition of M_{10} and M_{11} as permutation groups, and since $\dim(\mathscr{G}/K_{12-\ell})=4$ or 5, respectively, this quotient is absolutely irreducible as an $\mathbb{F}_3\widehat{\boldsymbol{M}}{}^{\mathbf{0}}_{\boldsymbol{\ell}}$-module by Lemma 5.2. Hence $C_{\mathrm{Aut}(\mathscr{G}/K_{12-\ell})}(\widehat{\boldsymbol{M}}{}^{\mathbf{0}}_{\boldsymbol{\ell}}) = \{\pm\mathrm{Id}\}$, and so

$$|\boldsymbol{N}^{(\boldsymbol{\ell})}/C_{\boldsymbol{N}^{(\boldsymbol{\ell})}}(K_{12-\ell})| \le |\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}/C_{\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}}(K_{12-\ell})| \le |\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}/\widehat{\boldsymbol{M}}{}^{\mathbf{0}}_{\boldsymbol{\ell}}| \le 2\cdot|\mathrm{Out}(\widehat{\boldsymbol{M}}{}^{\mathbf{0}}_{\boldsymbol{\ell}})| \tag{3.22}$$

We just saw that $|\boldsymbol{N}^{(\mathbf{10})}/C_{\boldsymbol{N}^{(\mathbf{10})}}(K_2)| = 8 = 2\cdot|\mathrm{Aut}(A_6)|$ and $|\boldsymbol{N}^{(\mathbf{11})}/C_{\boldsymbol{N}^{(\mathbf{11})}}(K_1)| = 2 = 2\cdot|\mathrm{Aut}(M_{11})|$, and so the inequalities in (3.22) are all equalities. Hence $\widehat{\boldsymbol{M}}{}^{\mathbf{0}}_{\boldsymbol{\ell}} = C_{\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}}(K_{12-\ell}) = C_{\widehat{\boldsymbol{M}}_{\mathbf{12}}}(K_{12-\ell})$, and the descriptions of $\boldsymbol{N}^{(\boldsymbol{\ell})}\cap\widehat{\boldsymbol{M}}{}^{\mathbf{0}}_{\boldsymbol{\ell}}$ and $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}/\widehat{\boldsymbol{M}}{}^{\mathbf{0}}_{\boldsymbol{\ell}}$ in Table 3.20 all hold. $\square$

As seen in Lemma 5.2, there are three different representations that appear under Hypotheses 5.1: one of A_6 and two of M_{11}. We will refer to these throughout the rest of the section as the "A_6-case" (when $\boldsymbol{\Gamma}_{\mathbf{0}}\cong A_6$), the "$M_{11}$-case" (when $\boldsymbol{\Gamma}_{\mathbf{0}}\cong M_{11}$ and $\boldsymbol{A}$ is its Todd module), and the "M_{11}^*-case" (when $\boldsymbol{\Gamma}_{\mathbf{0}}\cong M_{11}$ and $\boldsymbol{A}$ is the dual Todd module).

Lemma 3.23. *Assume Notation 3.18. We summarize here the notation we use for the $\mathbb{F}_3\widehat{\boldsymbol{M}}_{\mathbf{10}}$- and $\mathbb{F}_3\widehat{\boldsymbol{M}}_{\mathbf{11}}$-modules we are working with, and describe explicitly the action of the subgroup $\boldsymbol{N}^{(\mathbf{10})}$ or $\boldsymbol{N}^{(\mathbf{11})}$.*

(a) (A_6-case) *We identify the Todd module for $\widehat{\boldsymbol{M}}_{\mathbf{10}}$ with $\boldsymbol{A}^{(\mathbf{10})}\overset{\mathrm{def}}{=}\mathbb{F}_3\times\mathbb{F}_9\times\mathbb{F}_3$ in such a way that $\boldsymbol{N}^{(\mathbf{10})}$ acts as follows:*

$$^{(\!(x)\!)}[\![a,b,c]\!] = [\![a,b-ax,c+\mathrm{Tr}(x\bar{b})-aN(x)]\!] \qquad \text{for } x\in\mathbb{F}_9$$
$$^{[u]}[\![a,b,c]\!] = [\![a,ub,N(u)c]\!] \qquad \text{for } u\in\mathbb{F}_9^\times$$
$$^{[\phi]}[\![a,b,c]\!] = [\![a,\bar{b},c]\!] \quad\text{and}\quad {}^{-\mathrm{Id}}[\![a,b,c]\!] = [\![-a,-b,-c]\!]\,.$$

(b) (M_{11}-case) *We identify the Todd module for $\widehat{\boldsymbol{M}}_{\mathbf{11}}$ with $\boldsymbol{A}^{(\mathbf{11})}\overset{\mathrm{def}}{=}\mathbb{F}_3\times\mathbb{F}_9\times\mathbb{F}_9$ in such a way that $\boldsymbol{N}^{(\mathbf{11})}$ acts as follows:*

$$^{(\!(x)\!)}[\![a,b,c]\!] = [\![a,b-ax,c+bx+ax^2]\!] \qquad \text{for } x\in\mathbb{F}_9$$
$$^{[u]}[\![a,b,c]\!] = [\![a,ub,u^2c]\!] \qquad \text{for } u\in\mathbb{F}_9^\times$$
$$^{[\phi]}[\![a,b,c]\!] = [\![a,\bar{b},\bar{c}]\!] \quad\text{and}\quad {}^{-\mathrm{Id}}[\![a,b,c]\!] = [\![-a,-b,-c]\!]\,.$$

(c) (M_{11}^*-case) *We identify the dual Todd module for $\widehat{\boldsymbol{M}}_{\mathbf{11}}$ with $\boldsymbol{A}^{(\mathbf{11})^*}\overset{\mathrm{def}}{=}\mathbb{F}_9\times\mathbb{F}_9\times\mathbb{F}_3$ in such a way that $\boldsymbol{N}^{(\mathbf{11})}$ acts as follows:*

$$^{(\!(x)\!)}[\![a,b,c]\!] = [\![a,b-ax,c+\mathrm{Tr}(bx+ax^2)]\!] \qquad \text{for } x\in\mathbb{F}_9$$
$$^{[u]}[\![a,b,c]\!] = [\![u^{-2}a,u^{-1}b,c]\!] \qquad \text{for } u\in\mathbb{F}_9^\times$$
$$^{[\phi]}[\![a,b,c]\!] = [\![\bar{a},\bar{b},c]\!] \quad\text{and}\quad {}^{-\mathrm{Id}}[\![a,b,c]\!] = [\![-a,-b,-c]\!]\,.$$

Proof. **(b)** Define

$$\widehat{\kappa}_{11}\colon S_2(\mathscr{T}\oplus\mathbb{F}_3) \longrightarrow \boldsymbol{A}^{(\mathbf{11})} = \mathbb{F}_3\times\mathbb{F}_9\times\mathbb{F}_9$$

by setting

$$\widehat{\kappa}_{11}\big([(\xi,r)\otimes(\eta,s)]\big) = [\![rs, r\kappa(\eta)+s\kappa(\xi), \kappa(\xi)\cdot\kappa(\eta)]\!]\,.$$

This is surjective since $\boldsymbol{A}^{(\mathbf{11})}$ is generated by the elements

$$\widehat{\kappa}_{11}([(0,1)\otimes(\eta,s)]) = [\![s,\kappa(\eta),0]\!] \qquad\text{and}\qquad \widehat{\kappa}_{11}([(1,0)\otimes(\eta,0)]) = [\![0,0,\kappa(\eta)]\!]\,.$$

Also, $\widehat{\kappa}_{11}(\theta_1)=0$, so $\mathrm{Ker}(\widehat{\kappa}_{11}\circ\Phi) = \langle\theta_1^*\rangle = K_1$ since they both are 1-dimensional. Thus the action of $\widehat{\boldsymbol{M}}_{\mathbf{12}}$ on $\mathscr{G}$ induces an action of $\widehat{\boldsymbol{M}}_{\mathbf{11}} = N_{\widehat{\boldsymbol{M}}_{\mathbf{12}}}(K_1)$ on $\mathscr{G}/K_1\cong\boldsymbol{A}^{(\mathbf{11})}$.

For $\theta \in \mathscr{T}$, $\mathbf{tr}_\theta(\xi, r) = (\xi + r\theta, r)$ and $\mathbf{tr}_\theta(\eta, s) = (\eta + s\theta, s)$. So if we set $x = \kappa(\theta)$ and $[\![a, b, c]\!] = \widehat{\kappa}_{11}([(\xi, r) \otimes (\eta, s)])$, then

$$\begin{aligned} {}^{\langle\!\langle x \rangle\!\rangle}[\![a, b, c]\!] &= \widehat{\kappa}_{11}\big([(\xi + r\theta, r) \otimes (\eta + s\theta, s)]\big) \\ &= [\![rs, (r\kappa(\eta) + s\kappa(\xi)) + 2rs\kappa(\theta), \kappa(\xi)\kappa(\eta) + \kappa(\theta)(r\kappa(\eta) + s\kappa(\xi)) + rs\kappa(\theta)^2]\!] \\ &= [\![a, b - ax, c + bx + ax^2]\!]\,. \end{aligned}$$

The other formulas follow by similar (but simpler) arguments.

(c) The description of the action of $\boldsymbol{N}^{\mathbf{(11)}}$ on $\boldsymbol{A}^{\mathbf{(11)}*}$ follows from that in (b), together with the relation $\langle {}^g\xi, \eta \rangle = \langle \xi, {}^{g^{-1}}\eta \rangle$ for $\xi \in \boldsymbol{A}^{\mathbf{(11)}*}$ and $\eta \in \boldsymbol{A}^{\mathbf{(11)}}$, where the nonsingular pairing

$$\boldsymbol{A}^{\mathbf{(11)}*} \times \boldsymbol{A}^{\mathbf{(11)}} = (\mathbb{F}_9 \times \mathbb{F}_9 \times \mathbb{F}_3) \times (\mathbb{F}_3 \times \mathbb{F}_9 \times \mathbb{F}_9) \xrightarrow{\ \langle -,- \rangle\ } \mathbb{F}_3$$

is defined by $\big\langle\, [\![a, b, z]\!]\,, [\![y, c, d]\!]\,\big\rangle = yz + \mathrm{Tr}(ad + bc)$.

(a) This proof is similar to that of (b), except that $\widehat{\kappa}_{11}$ is replaced by the map

$$\widehat{\kappa}_{10} \colon S_2(\mathscr{T} \oplus \mathbb{F}_3) \longrightarrow \boldsymbol{A}^{\mathbf{(10)}} = \mathbb{F}_3 \times \mathbb{F}_9 \times \mathbb{F}_3,$$

defined by setting

$$\widehat{\kappa}_{10}\big([(\xi, r) \otimes (\eta, s)]\big) = [\![rs, r\kappa(\eta) + s\kappa(\xi), \mathrm{Tr}(\kappa(\xi) \cdot \overline{\kappa(\eta)})]\!]\,.$$

This is easily seen to be surjective. For $i = 2, 3$, we have

$$\widehat{\kappa}_{10}(\theta_i^*) = [\![0, 0, \mathrm{Tr}(1 \cdot 1 - i \cdot \bar{\imath} \pm 1 \cdot \bar{\imath})]\!] = 0,$$

and so $\mathrm{Ker}(\widehat{\kappa}_{10}) = \langle \theta_2^*, \theta_3^* \rangle = K_2$ since they are both 2-dimensional. So the action of $\widehat{\boldsymbol{M}}_{\mathbf{12}}$ on $\mathscr{G}$ induces an action of $\widehat{\boldsymbol{M}}_{\mathbf{10}} = N_{\widehat{\boldsymbol{M}}_{\mathbf{12}}}(K_2)$ on $\mathscr{G}/K_2 \cong \boldsymbol{A}^{\mathbf{(11)}}$.

The formulas for ${}^{\langle\!\langle x \rangle\!\rangle}[\![a, b, c]\!]$, ${}^{[u]}[\![a, b, c]\!]$, and ${}^{[\phi]}[\![a, b, c]\!]$ follow from arguments similar to those used in case (b). $\square$

4. The Todd module for $2M_{12}$

We are now ready to look at fusion systems that involve the Todd module for $2M_{12}$. Throughout the section, we refer to the following assumptions:

Hypotheses 4.1. *Set $p = 3$. Let $\boldsymbol{\mathcal{F}}$ be a saturated fusion system over a finite 3-group $\boldsymbol{S}$, and let $\boldsymbol{A} \le \boldsymbol{S}$ be an elementary abelian subgroup such that $C_{\boldsymbol{S}}(\boldsymbol{A}) = \boldsymbol{A}$. Set $\boldsymbol{\Gamma} = \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ and $\boldsymbol{\Gamma_0} = O^{3'}(\boldsymbol{\Gamma})$, and assume that $\mathrm{rk}(\boldsymbol{A}) = 6$ and $\boldsymbol{\Gamma}_0 \cong 2M_{12}$.*

The main result in this section is Theorem 4.16, where we show that if $\boldsymbol{\mathcal{F}}$ satisfies these hypotheses, then either $\boldsymbol{A} \trianglelefteq \boldsymbol{\mathcal{F}}$, or $\boldsymbol{\mathcal{F}}$ is isomorphic to the 3-fusion system of the sporadic group Co_1.

Standard results in the representation theory of $2M_{12}$ show that in the above situation, $\boldsymbol{A}$ must be the Todd module for $\boldsymbol{\Gamma} = \boldsymbol{\Gamma_0}$ or its dual. In fact, we can assume in all cases that it is the Todd module.

Lemma 4.2. *Assume Hypotheses 4.1. Then $\boldsymbol{\Gamma} = \boldsymbol{\Gamma_0} \cong 2M_{12}$, $\boldsymbol{A}$ is the Todd module for $\boldsymbol{\Gamma}$, and $\boldsymbol{A}$ is absolutely irreducible as an $\mathbb{F}_3\boldsymbol{\Gamma}$-module.*

Proof. By § 4 and Table 5 in [Hu], the only 6-dimensional faithful $\mathbb{F}_3\boldsymbol{\Gamma_0}$-modules are the Todd module and its dual, and they are absolutely irreducible and not isomorphic. Also, $\mathrm{Out}(\boldsymbol{\Gamma_0}) \cong \mathrm{Out}(M_{12}) \cong C_2$, and composition with an outer automorphism of $\boldsymbol{\Gamma_0}$ sends the

Todd module to its dual. So the action of $\boldsymbol{\Gamma_0}$ on $\boldsymbol{A}$ does not extend to any extension of $\boldsymbol{\Gamma_0}$ by an outer automorphism, and $\boldsymbol{\Gamma} = \boldsymbol{\Gamma_0} \cdot C_{\boldsymbol{\Gamma}}(\boldsymbol{\Gamma_0})$. As subgroups of $\mathrm{Aut}(\boldsymbol{A})$, we have

$$C_{\boldsymbol{\Gamma}}(\boldsymbol{\Gamma_0}) \le \mathrm{Aut}_{\mathbb{F}_3\boldsymbol{\Gamma_0}}(\boldsymbol{A}) = \{\pm\mathrm{Id}\} = Z(\boldsymbol{\Gamma_0}),$$

where $\mathrm{Aut}_{\mathbb{F}_3\boldsymbol{\Gamma_0}}(\boldsymbol{A}) = \{\pm\mathrm{Id}\}$ since $\boldsymbol{A}$ is absolutely irreducible. Hence $\boldsymbol{\Gamma} = \boldsymbol{\Gamma_0} \cong 2M_{12}$.

Now, $\mathrm{Out}(\boldsymbol{\Gamma}) \cong \mathrm{Out}(M_{12}) \cong C_2$, and by § 4 in [Hu] again, an outer automorphism of $\boldsymbol{\Gamma}$ acts by exchanging the Todd module with its dual. So $(\boldsymbol{\Gamma}, \boldsymbol{A}^*) \cong (\boldsymbol{\Gamma}, \boldsymbol{A})$ as pairs, and we can assume that $\boldsymbol{A}$ is the Todd module for $\boldsymbol{\Gamma}$. □

We next check that under Hypotheses 4.1, $\boldsymbol{A}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{S}$ splits over $\boldsymbol{A}$. These are easy consequences of Lemma 3.14.

Lemma 4.3. *Assume that $\boldsymbol{A} \le \boldsymbol{S}$ and $\boldsymbol{\mathcal{F}}$ satisfy Hypotheses 4.1, and let M be a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ (see Proposition 1.12). Then*

(a) *$\boldsymbol{A}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ and hence normal in $\boldsymbol{S}$, and*

(b) *$\boldsymbol{S}$ and M both split over $\boldsymbol{A}$.*

Proof. By Lemma 4.2, we have $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) \cong \widehat{\boldsymbol{M}}_{\boldsymbol{12}}$, and $\boldsymbol{A} \cong \mathscr{G}$ as $\mathbb{F}_3\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$-modules.

(a) If $A^* < N_{\boldsymbol{S}}(\boldsymbol{A})$ is such that $A^* \cong E_{3^6}$ and $A^* \ne \boldsymbol{A}$, then for $x \in A^* \smallsetminus \boldsymbol{A}$, $\boldsymbol{A} \cap A^* \le C_{\boldsymbol{A}}(x)$, where $\mathrm{rk}(C_{\boldsymbol{A}}(x)) \le 3$ by Lemma 3.14 and since $c_x^{\boldsymbol{A}}$ has order 3 in $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$. Hence $\mathrm{rk}(\mathrm{Aut}_{A^*}(\boldsymbol{A})) \ge 3$, which is impossible since $\mathrm{rk}(\mathrm{Aut}_{\boldsymbol{S}^*}(\boldsymbol{A})) = \mathrm{rk}_3(2M_{12}) = 2$. So $\boldsymbol{A}$ is the only element of $\boldsymbol{A}^{\boldsymbol{\mathcal{F}}}$ contained in $N_{\boldsymbol{S}}(\boldsymbol{A})$. Hence $\boldsymbol{A}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ by Lemma 2.1.

(b) Choose $\theta \in M$ such that c_θ is the central involution in $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) \cong 2M_{12}$ (Lemma 4.2). Then $|\theta| = 2$ or 6, and after replacing θ by θ^3 if necessary, we can assume $|\theta| = 2$. Also, θ fixes at least one element in each coset $h\boldsymbol{A}$ of $\boldsymbol{A}$ in M since the cosets have odd order. Hence $M = \boldsymbol{A}C_M(\theta)$ and $\boldsymbol{S} = \boldsymbol{A}C_{\boldsymbol{S}}(\theta)$, while $\boldsymbol{A} \cap C_M(\theta) = 1$ since θ acts as $-\mathrm{Id}$ on $\boldsymbol{A}$. This proves that $C_M(\theta)$ and $C_{\boldsymbol{S}}(\theta)$ are splittings of M and $\boldsymbol{S}$ over $\boldsymbol{A}$. □

We use throughout this section the notation set up in Section 3.1 for working with the Todd module for $2M_{12}$, as summarized in Notation 4.4. In Subsection 4.1, we set up notation for some of the subgroups of $\boldsymbol{S}$ and $\boldsymbol{\Gamma}$ that we have to work with. All of this is then applied in Subsection 4.2 to prove Theorem 4.16 describing fusion systems satisfying Hypotheses 4.1.

Notation 4.4. *Assume Hypotheses 4.1 and Notation 3.6. Identify*

$$\boldsymbol{\Gamma} = \widehat{\boldsymbol{M}}_{\boldsymbol{12}} \cong 2M_{12} \qquad \text{and} \qquad \boldsymbol{A} = \Phi(\mathscr{G}) = \mathrm{Sym}_3(\mathbb{F}_3),$$

where $\widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ is as in Notation 3.6(a). Let $\boldsymbol{N_0} \le \widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ be as in Notation 3.6(d), set $\boldsymbol{N} = \boldsymbol{N_0} \times \{\pm\mathrm{Id}\}$, and let

$$\Theta\colon \boldsymbol{N_0} \xrightarrow{\ \cong\ } \left\{ \left(\begin{smallmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{smallmatrix}\right) \,\middle|\, a,b,c,d,e,f \in \mathbb{F}_3,\ ae \ne bd \right\} \le GL_3(\mathbb{F}_3)$$

be the isomorphism defined by (3.12). Thus

$$\beta(X) = \Theta(\beta)X\Theta(\beta)^t$$

for all $\beta \in \boldsymbol{N_0}$ and $X \in \boldsymbol{A}$ by Lemma 3.13. Finally, define

$$\boldsymbol{T} = \Theta^{-1}(UT_3(\mathbb{F}_3)) \in \mathrm{Syl}_3(\boldsymbol{N_0}) \subseteq \mathrm{Syl}_3(\boldsymbol{\Gamma}),$$

and set

$$M = \boldsymbol{A} \rtimes \boldsymbol{\Gamma} \qquad \text{and} \qquad \boldsymbol{S} = \boldsymbol{A} \rtimes \boldsymbol{T} \in \mathrm{Syl}_3(M).$$

4.1. **Some subgroups of $\boldsymbol{\Gamma}$ and $\boldsymbol{S}$.**

We begin by listing the additional notation that will be needed; in particular, notation to describe the subgroups of index 3 in $\boldsymbol{T}$.

Notation 4.5. *Define*

$$Z = Z(\boldsymbol{S}) = C_{\boldsymbol{A}}(\boldsymbol{T}) \qquad \text{and} \qquad A_* = [\boldsymbol{T}, \boldsymbol{A}].$$

Define elements $\eta_0, \eta_{\pm 1}, \eta_\infty, \widehat{\eta} \in \boldsymbol{T}$ as follows:

$$\eta_k = \Theta^{-1}\left(\left(\begin{smallmatrix} 1&1&0\\ 0&1&k\\ 0&0&1 \end{smallmatrix}\right)\right) \text{ (for } k \in \mathbb{F}_3), \qquad \eta_\infty = \Theta^{-1}\left(\left(\begin{smallmatrix} 1&0&0\\ 0&1&1\\ 0&0&1 \end{smallmatrix}\right)\right), \qquad \widehat{\eta} = \Theta^{-1}\left(\left(\begin{smallmatrix} 1&0&1\\ 0&1&0\\ 0&0&1 \end{smallmatrix}\right)\right).$$

Thus $\boldsymbol{T} = \langle \eta_0, \eta_\infty \rangle$ and $Z(\boldsymbol{T}) = \langle \widehat{\eta} \rangle$. For each $k \in \mathbb{F}_3 \cup \{\infty\}$, set

$$\begin{aligned} U_k &= \langle \widehat{\eta}, \eta_k \rangle \le \boldsymbol{T} \\ W_k &= \bigl\{ a \in \boldsymbol{A} \bigm| [a, U_k] \le Z = Z(\boldsymbol{S}) \bigr\} \le \boldsymbol{A} \qquad (\text{so } W_k/Z = C_{\boldsymbol{A}/Z}(U_k)) \\ Q_k &= W_k U_k \le \boldsymbol{S} \end{aligned}$$

For $k \in \mathbb{F}_3$, set

$$\mathcal{Q}_k = \bigl\{ Q \le \boldsymbol{S} \bigm| Q \cap \boldsymbol{A} = W_k,\ Q\boldsymbol{A} = U_k \boldsymbol{A} \bigr\}.$$

In addition, we set

$$\widehat{Q} = A_* U_\infty \cong 3^{3+4}.$$

For $1 \le i, j \le 3$ and $x \in \mathbb{F}_3$, let $a_{ij}^x \in \boldsymbol{A} = \mathrm{Sym}_3(\mathbb{F}_3)$ be the symmetric (3×3)-matrix with x in positions (i,j) and (j,i) (or $2x$ in position (i,i) if $i = j$) and 0 elsewhere, and set $a_{ij} = a_{ij}^1$.

The actions of the η_k on $\boldsymbol{A}$ are described explicitly in Table 4.6.

η	$\eta\left(\left(\begin{smallmatrix} t&u&r\\ u&v&s\\ r&s&a \end{smallmatrix}\right)\right)$	$\left[\eta, \left(\begin{smallmatrix} t&u&r\\ u&v&s\\ r&s&a \end{smallmatrix}\right)\right]$
$\eta_k = \left(\begin{smallmatrix} 1&1&0\\ 0&1&k\\ 0&0&1 \end{smallmatrix}\right)$ $(k \in \mathbb{F}_3)$	$\begin{pmatrix} t{-}u{+}v & u{+}v{+}k(r{+}s) & r{+}s\\ u{+}v{+}k(r{+}s) & v{-}ks{+}ak^2 & s{+}ak\\ r{+}s & s{+}ak & a \end{pmatrix}$	$\begin{pmatrix} -u{+}v & v{+}k(r{+}s) & s\\ v{+}k(r{+}s) & -ks{+}ak^2 & ak\\ s & ak & 0 \end{pmatrix}$
$\eta_\infty = \left(\begin{smallmatrix} 1&0&0\\ 0&1&1\\ 0&0&1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} t & u+r & r\\ u+r & v-s+a & s+a\\ r & s+a & a \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} 0 & r & 0\\ r & -s+a & a\\ 0 & a & 0 \end{smallmatrix}\right)$
$\widehat{\eta} = \left(\begin{smallmatrix} 1&0&1\\ 0&1&0\\ 0&0&1 \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} t-r+a & u+s & r+a\\ u+s & v & s\\ r+a & s & a \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -r+a & s & a\\ s & 0 & 0\\ a & 0 & 0 \end{smallmatrix}\right)$

TABLE 4.6.

Lemma 4.7. *Assume Notation 4.4 and 4.5.*

(a) *We have*

$$Z = \left\{ \left(\begin{smallmatrix} t&0&0\\ 0&0&0\\ 0&0&0 \end{smallmatrix}\right) \,\middle|\, t \in \mathbb{F}_3 \right\} \qquad \text{and} \qquad A_* = \left\{ \left(\begin{smallmatrix} t&u&r\\ u&v&s\\ r&s&0 \end{smallmatrix}\right) \,\middle|\, t, u, v, r, s \in \mathbb{F}_3 \right\},$$

and

$$\mathrm{Aut}_{N_{\boldsymbol{\mathcal{F}}}(A_*)}(\boldsymbol{A}) = \mathrm{Aut}_{N_{\boldsymbol{\Gamma}}(A_*)}(\boldsymbol{A}) \qquad \text{where} \qquad N_{\boldsymbol{\Gamma}}(A_*) = \boldsymbol{N}.$$

(b) *For each $k \in \mathbb{F}_3 \cup \{\infty\}$,*

$$W_k = \begin{cases} \left\{ \left(\begin{smallmatrix} t & u & r \\ u & -kr & 0 \\ r & 0 & 0 \end{smallmatrix}\right) \,\middle|\, r,t,u \in \mathbb{F}_3 \right\} & \text{if } k \in \mathbb{F}_3 \\ \left\{ \left(\begin{smallmatrix} t & u & 0 \\ u & v & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \,\middle|\, t,u,v \in \mathbb{F}_3 \right\} & \text{if } k = \infty \end{cases} \qquad C_{\boldsymbol{A}}(U_k) = \begin{cases} Z & \text{if } k \in \mathbb{F}_3 \\ W_\infty & \text{if } k = \infty \end{cases}$$

$$Q_k \cong \begin{cases} 3^{1+4}_+ & \text{if } k \in \mathbb{F}_3 \\ E_{3^5} & \text{if } k = \infty. \end{cases} \qquad N_{\boldsymbol{S}}(Q_k) = \begin{cases} \boldsymbol{S} & \text{if } k = 0 \\ A_*\boldsymbol{T} < \boldsymbol{S} & \text{if } k \neq 0. \end{cases}$$

(c) *More generally, if $k \in \mathbb{F}_3$ and $Q \in \mathcal{Q}_k$, then $N_{\boldsymbol{S}}(Q) \geq \boldsymbol{A}$ if $k = 0$, and $\boldsymbol{A} \cap N_{\boldsymbol{S}}(Q) = A_*$ if $k \neq 0$.*

Proof. The descriptions of Z and A_* follow immediately from the formulas in Notation 4.4. From this, we see that $A_* = [\boldsymbol{N_0}, \boldsymbol{A}]$ and hence is normalized by $\boldsymbol{N}$. Since $\boldsymbol{N}$ is a maximal subgroup of $\boldsymbol{\Gamma}$ by Lemma 3.7, it must be the full normalizer of A_*.

The formulas in point (b) follow easily from those in Table 4.6. (Note, for each $k \in \mathbb{F}_3 \cup \{\infty\}$, that $\boldsymbol{T}$ normalizes Q_k since it normalizes U_k and W_k.)

If $Q \in \mathcal{Q}_k$ for some $k \in \mathbb{F}_3$, then an element $a \in \boldsymbol{A}$ normalizes Q if and only if $[a, U_k] \leq W_k$, which holds for all $a \in \boldsymbol{A}$ if $k = 0$, but only for $a \in A_*$ if $k = \pm 1$. □

Note that for each $k \in \mathbb{F}_3$, the subgroup $W_k\langle \widehat{\eta}, a_{23}\eta_k \rangle$ lies in $\mathcal{Q}_k$ (since $(a_{23}\eta_k)^3 \in C_{\boldsymbol{A}}(\eta_k) \leq W_k$), but is not extraspecial since $[\widehat{\eta}, a_{23}\eta_k] = [\widehat{\eta}, a_{23}] \in W_k \smallsetminus Z$. Thus members of the $\mathcal{Q}_k$ need not be extraspecial. However, as shown in the next lemma, all subgroups of $\boldsymbol{S}$ not in $\boldsymbol{A}$ and isomorphic to E_{3^5} or 3^{1+4}_+ are members of $\mathcal{Q}_k$ for some k.

Lemma 4.8. *Assume Notation 4.4 and 4.5.*

(a) *There are exactly three abelian subgroups of $\boldsymbol{S}$ of order 3^5 not contained in $\boldsymbol{A}$, and all of them are conjugate to $Q_\infty \cong E_{3^5}$ by elements of $\boldsymbol{A} \smallsetminus A_*$.*

(b) *If $P \leq \boldsymbol{S}$ is extraspecial of order 3^5, then $Z(P) = Z$, and $P \in \mathcal{Q}_k$ for some $k \in \mathbb{F}_3$. If in addition, P is weakly closed in $N_{\boldsymbol{\mathcal{F}}}(Z)$, then $P = Q_0$.*

(c) *For each saturated fusion system $\mathcal{E}$ over $\boldsymbol{S}$ and each $k \in \mathbb{F}_3$, Q_k is $\mathcal{E}$-centric.*

Proof. **(a)** Assume $B \leq \boldsymbol{S}$ is abelian and such that $B \not\leq \boldsymbol{A}$ and $|B| = 3^5$. For each $\eta \in \boldsymbol{S} \smallsetminus \boldsymbol{A}$, $\mathrm{rk}(C_{\boldsymbol{A}}(\eta)) \leq 3$ by Lemma 3.14, so $\mathrm{rk}(B\boldsymbol{A}/\boldsymbol{A}) = 2$ and $\mathrm{rk}(B \cap \boldsymbol{A}) = 3$. Thus $B\boldsymbol{A} = U_k\boldsymbol{A}$ for some $k \in \mathbb{F}_3 \cup \{\infty\}$ such that $\mathrm{rk}(W_k) \geq \mathrm{rk}(C_{\boldsymbol{A}}(U_k)) \geq 3$, and $k = \infty$ by Lemma 4.7(a). By the same lemma, $B \cap \boldsymbol{A} = W_\infty$.

Thus $B = W_\infty\langle b_1\widehat{\eta}, b_2\eta_\infty \rangle$ for some $b_1, b_2 \in \boldsymbol{A}$ uniquely determined modulo W_∞. Since $[\widehat{\eta}, \eta_\infty] = 1$ and $\boldsymbol{A} \trianglelefteq \boldsymbol{S}$, we have

$$1 = [b_1\widehat{\eta}, b_2\eta_\infty] = b_1(\widehat{\eta}b_2\widehat{\eta}^{-1})(\eta_\infty b_1^{-1}\eta_\infty^{-1})b_2^{-1} = [\widehat{\eta}, b_2][b_1, \eta_\infty],$$

and hence

$$[\widehat{\eta}, b_2] = [\eta_\infty, b_1] \in [\widehat{\eta}, \boldsymbol{A}] \cap [\eta_\infty, \boldsymbol{A}] = \langle a_{12} \rangle.$$

So by Table 4.6 again, $b_1 \equiv a_{13}^x$ and $b_2 \equiv a_{23}^x \pmod{W_\infty}$ for some $x \in \mathbb{F}_3$.

In particular, there are at most three subgroups of $\boldsymbol{S}$ isomorphic to E_{3^5} and not in $\boldsymbol{A}$. Since $N_{\boldsymbol{S}}(Q_\infty) = A_*\boldsymbol{T}$ has index 3 in $\boldsymbol{S}$, there are exactly three such subgroups, and they are all conjugate to Q_∞ by elements of $\boldsymbol{A} \smallsetminus A_*$. More precisely, the three subgroups $W_\infty\langle a_{13}^x\widehat{\eta}, a_{23}^x\eta_\infty \rangle$ for $x \in \mathbb{F}_3$ all have the form ${}^{\beta}Q_\infty$ for some $\beta \in \langle a_{33} \rangle$.

(b) Assume that $P \leq \boldsymbol{S}$ is extraspecial of order 3^5, and set $P_0 = P \cap \boldsymbol{A}$. Then P_0 and P/P_0 are both elementary abelian (since $[P, P] = Z(P) \leq P_0$), and hence $P_0 \cong E_{27}$ and

$P/P_0 \cong E_9$. So $P\boldsymbol{A} = U_k\boldsymbol{A}$ for some $k \in \mathbb{F}_3 \cup \{\infty\}$, and $Z(P) \le C_{\boldsymbol{A}}(U_k)$. Since $U_k = \langle \widehat{\eta}, \eta_k \rangle$ and $C_{\boldsymbol{A}}(\widehat{\eta}) = W_\infty$, this means that $Z(P) \le C_{W_\infty}(\eta_k)$, and hence $Z(P) = Z$ if $k \in \mathbb{F}_3$ (while $C_{\boldsymbol{A}}(U_\infty) = W_\infty$). So if $k \ne \infty$, then $[P_0, U_k] = Z$, and hence $P_0 \le W_k$ in this case, with equality since $\mathrm{rk}(W_k) = 3$ for each k (Lemma 4.7). Thus $P \in \mathcal{Q}_k$ if $k \in \mathbb{F}_3$.

Conjugation by the element $\left(\begin{smallmatrix} -I & 0 \\ 0 & 1 \end{smallmatrix}\right) \in N$ lies in $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S}) = \mathrm{Aut}_{N_{\boldsymbol{\mathcal{F}}}(Z)}(\boldsymbol{S})$, and its action on $\boldsymbol{S}$ exchanges the sets $\mathcal{Q}_1$ and $\mathcal{Q}_{-1}$. So no member of either of these is weakly closed in $N_{\boldsymbol{\mathcal{F}}}(Z)$. Each member of $\mathcal{Q}_0$ has the form $Q = W_0\langle g_1\eta_0, g_2\widehat{\eta}\rangle$ for some $g_1, g_2 \in \boldsymbol{A}$, and $-\mathrm{Id} \in N$ sends Q to $W_0\langle g_1^{-1}\eta_0, g_2^{-1}\widehat{\eta}\rangle$. Since $c_{-\mathrm{Id}} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S}) = \mathrm{Aut}_{N_{\boldsymbol{\mathcal{F}}}(Z)}(\boldsymbol{S})$, Q is weakly closed only if $g_i \equiv g_i^{-1} \pmod{W_0}$ for $i = 1, 2$, which occurs only if $g_1, g_2 \in W_0$ and hence $Q = Q_0$. Thus Q_0 is the only member of $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_{-1}$ that could be weakly closed in $N_{\boldsymbol{\mathcal{F}}}(Z)$.

If $k = \infty$, then

$$Z(P) \le C_{\boldsymbol{A}}(U_\infty) \cap [\widehat{\eta}, \boldsymbol{A}] \cap [\eta_\infty, \boldsymbol{A}] = \langle a_{12} \rangle$$

by Table 4.6, and so

$$P_0 \le \bigl\{ a \in \boldsymbol{A} \bigm| [U_\infty, a] \le Z(P) \bigr\} = W_\infty$$

with equality since $\mathrm{rk}(W_\infty) = 3 = \mathrm{rk}(P_0)$. But $[U_\infty, W_\infty] = 1$, so $W_\infty \le Z(P)$, a contradiction.

(c) For each $k \in \mathbb{F}_3$ and each $Q \in \mathcal{Q}_k$, $C_{\boldsymbol{S}}(Q) \le C_{\boldsymbol{A}U_k}(W_k) = \boldsymbol{A}$ since $Q_k = U_kW_k$ is extraspecial (Lemma 4.7(b)), and hence $C_{\boldsymbol{S}}(Q) = C_{\boldsymbol{A}}(U_k) = Z$ by the same lemma. Since $(Q_k)^{\boldsymbol{\mathcal{F}}} \subseteq \mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ by (b), this proves that Q_k is $\mathcal{E}$-centric for each saturated fusion system $\mathcal{E}$ over $\boldsymbol{S}$. □

Point (c) in Lemma 4.8 is not true if one replaces Q_k (for $k \in \mathbb{F}_3$) by Q_∞. If $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{S}$ satisfy Hypotheses 4.1, then one can show that $\widehat{Q} \trianglelefteq C_{\boldsymbol{\mathcal{F}}}(W_\infty)$, and that $\mathrm{Out}_{C_{\boldsymbol{\mathcal{F}}}(W_\infty)}(\widehat{Q}) \cong 2A_4$. (Since $\boldsymbol{\mathcal{F}}$ is isomorphic to the fusion system of Co_1 by Theorem 4.16, this follows from the structure of $C_{Co_1}(W_\infty) \cong \widehat{Q}.2A_4$.) The subgroup $\widehat{Q}$ contains exactly four elementary abelian subgroups of rank 5 (the three described in Lemma 4.8 and A_*), and they are permuted transitively by $\mathrm{Out}_{C_{\boldsymbol{\mathcal{F}}}(W_\infty)}(\widehat{Q})$. So $Q_\infty \in (A_*)^{\boldsymbol{\mathcal{F}}}$, and hence is not $\boldsymbol{\mathcal{F}}$-centric.

4.2. Fusion systems involving the Todd module for $2M_{12}$.

We now begin to apply results from Section 2. Recall that our goal is to describe all fusion systems that satisfy Hypotheses 4.1 with $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$.

Proposition 4.9. *Assume Hypotheses 4.1 with $\boldsymbol{\Gamma} = \widehat{\boldsymbol{M}}_{\boldsymbol{12}}$ and $\boldsymbol{A}$ as in Notation 4.4, and set $Z = Z(\boldsymbol{S})$. Then $\boldsymbol{\mathcal{F}} = \langle C_{\boldsymbol{\mathcal{F}}}(Z), N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})\rangle$.*

Proof. Assume otherwise. By Proposition 2.3, there are subgroups $X \in Z^{\boldsymbol{\mathcal{F}}}$ and $R \in \mathbf{E}_{\boldsymbol{\mathcal{F}}}$ such that $X \nleq \boldsymbol{A}$, $R = C_{\boldsymbol{S}}(X) = N_{\boldsymbol{S}}(X)$, and $Z = \alpha(X)$ for some $\alpha \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$. Fix $x \in X \smallsetminus \boldsymbol{A}$. In all cases, $R \cap \boldsymbol{A} = C_{\boldsymbol{A}}(X) = C_{\boldsymbol{A}}(x)$, since $|X| = |Z| = 3$ and hence $X = \langle x \rangle$. Also, $|x| = 3$ since $x \in X \in Z^{\boldsymbol{\mathcal{F}}}$ where Z has order 3. Set $R_0 = R \cap \boldsymbol{A}$.

Case 1: Assume first that $|R\boldsymbol{A}/\boldsymbol{A}| = 3$, so that $R\boldsymbol{A} = \boldsymbol{A}\langle x \rangle$ and $R = C_{\boldsymbol{S}}(X) = C_{\boldsymbol{A}}(x)\langle x \rangle$. Then $\mathrm{Aut}_{\boldsymbol{A}}(R) \cong C_{\boldsymbol{A}/R_0}(x) \cong E_{3^m}$, where m is the number of Jordan blocks of length at least 2 for the action of x on $\boldsymbol{A}$, and $m = 2$ by Lemma 3.14.

Thus $|\mathrm{Out}_{\boldsymbol{A}}(R)| = 9$. Since $\mathrm{Out}_{\boldsymbol{A}}(R)$ acts trivially on R_0 and $|R : R_0| = 3$, this contradicts Lemma B.7.

Case 2: Assume that $|R\boldsymbol{A}/\boldsymbol{A}| = 9$, and hence that $\mathrm{Aut}_R(\boldsymbol{A}) = U_k$ for some $k \in \mathbb{F}_3 \cup \{\infty\}$. If $k \in \mathbb{F}_3$, then $Z = C_{\boldsymbol{A}}(R) < C_{\boldsymbol{A}}(x)$ by Lemma 4.7(b), and hence $Z \le [R, C_{\boldsymbol{A}}(x)] \le [R, R]$. Since $X \nleq [R, R]$, no automorphism of R sends X to Z.

Now assume $k = \infty$, so $R_0 = C_{\boldsymbol{A}}(x) = C_{\boldsymbol{A}}(R) \cong E_{27}$ by Lemma 4.7(b) again. Also, $\mathrm{Out}_{\boldsymbol{A}}(R) \cong C_{\boldsymbol{A}/R_0}(U_\infty) \cong E_9$ (see Table 4.6). So by Lemma B.6(b), for each characteristic subgroup $P \le R$, either $|P| \ge 3^4$ or $|R/P| \ge 3^4$. Since $|R| = 3^5$, and since R is not extraspecial by Lemma 4.8(b), this implies that $R \cong E_{3^5}$.

Set $B = \mathrm{Out}_{\boldsymbol{A}}(R) \cong E_9$, so that $B \le \mathrm{Out}_{\boldsymbol{S}}(R)$. Let $H < \mathrm{Out}_{\mathcal{F}}(R)$ be a strongly 3-embedded subgroup that contains $\mathrm{Out}_S(R)$ (recall $R \in \mathbf{E}_{\mathcal{F}}$), fix $g \in \mathrm{Out}_{\mathcal{F}}(R) \smallsetminus H$, and set $L = \langle B, {}^gB\rangle$. Then $L \not\le H$ and $3 \mid |H \cap L|$, so by Lemma B.2(b), the subgroup $H \cap L$ is strongly p-embedded in L.

Since $\mathrm{rk}(C_R(B)) = 3$ and $\mathrm{rk}(R) = 5$, we have $\mathrm{rk}(C_R(L)) = \mathrm{rk}\big(C_R(B) \cap C_R({}^gB)\big) \ge 1$. Also, $\mathrm{rk}(R/C_R(L)) \ge 4$ by Lemma B.6(b) again, so $\mathrm{rk}(C_R(L)) = 1$, and $R/C_R(L)$ is a faithful 4-dimensional representation of L. For each $x \in B^\#$, $\mathrm{rk}([x, R]) = \mathrm{rk}([x, U_\infty]) = 2$, and so $[x, R/C_R(L)]$ has rank 1 or 2, and x acts on $R/C_R(L)$ with Jordan blocks of lengths $2 + 2$ or $2 + 1 + 1$. By Proposition B.10, $L \cong SL_2(9)$ with the natural action on $R/C_R(L)$, and hence $\mathrm{rk}([x, R/C_R(L)]) = 2$ for each $x \in B^\#$. Thus $C_R(L) \cap [x, R] = 1$ for each $x \in B^\#$. But this is impossible: from Table 4.6, we see that the subgroups $[x, R]$ are precisely the four subgroups of rank 2 in $W_\infty \cong E_{27}$ that contain $\langle a_{12}\rangle$, and hence each element of W_∞ lies in at least one of them.

Case 3: Finally, assume that $|R\boldsymbol{A}/\boldsymbol{A}| > 9$. Then $R\boldsymbol{A}/\boldsymbol{A} = \boldsymbol{S}/\boldsymbol{A} \cong 3^{1+2}_+$, and $\boldsymbol{A}X = \boldsymbol{A}\langle\widehat{\eta}\rangle$. From Table 4.6, we see that $R_0 = C_{\boldsymbol{A}}(\widehat{\eta}) = Z\langle a_{12}, a_{22}\rangle \cong E_{27}$.

From the formulas in Table 4.6 again, we see that $Z\langle a_{12}\rangle \le [\boldsymbol{T}, R_0] \le [R, R]$, and hence that $Z \le [R, [R, R]]$. Since $[R, [R, R]] \le \boldsymbol{A}$, it does not contain X, so no automorphism of R sends X to Z, contradicting our assumptions. □

We next show that Q_0 is normal in $C_{\boldsymbol{\mathcal{F}}}(Z)$. The following lemma is a first step towards doing this. From now on, we set $\boldsymbol{Q} = Q_0$, since this subgroup plays a central role in studying these fusion systems satisfying Hypotheses 4.1.

Lemma 4.10. *Assume Hypotheses 4.1, and Notation 4.4 and 4.5, and set $\boldsymbol{Q} = Q_0$. Then*

(a) *$\boldsymbol{Q}$ is weakly closed in $\boldsymbol{\mathcal{F}}$;*

(b) *$\boldsymbol{Q}$ is normal in $N_{N_{\boldsymbol{\mathcal{F}}(\boldsymbol{A})}}(Z)$;*

(c) *$C_{\boldsymbol{\Gamma}}(Z) \cong E_9 \rtimes GL_2(3)$ and $N_{\boldsymbol{\Gamma}}(U_0) = N_{\boldsymbol{\Gamma}}(Z) \cong (E_9 \rtimes GL_2(3)) \times C_2$; and*

(d) *Z and W_0 are the only proper nontrivial subspaces of $\boldsymbol{A}$ invariant under the action of $C_{\boldsymbol{\Gamma}}(Z)$.*

Proof. **(c)** Since $Z = C_{\boldsymbol{A}}(U_0)$ (see Table 4.6), we have $N_{\boldsymbol{\Gamma}}(U_0) \le N_{\boldsymbol{\Gamma}}(Z)$. Also, $N_{\boldsymbol{\Gamma}}(U_0) \ge N_{\boldsymbol{N}}(U_0) \cong \boldsymbol{T} \rtimes E_8$, so the index of $N_{\boldsymbol{\Gamma}}(U_0)$ in $\boldsymbol{\Gamma}$ divides 880. By [Gr, Lemma 7.12 & Exercise 7.36], the orbits of $\boldsymbol{\Gamma}$ acting on the projective space $P(\boldsymbol{A})$ have lengths 132, 220, and 12, so Z must be in an orbit of length 220, and hence $|N_{\boldsymbol{\Gamma}}(Z)| = 3^2 \cdot 96 = |\boldsymbol{N}|$.

Recall (Lemma 3.14) that there are two conjugacy classes of elements of order 3 in $\boldsymbol{\Gamma}$, differing by the number of Jordan blocks for their actions on $\boldsymbol{A}$. Thus all elements in $U_0^\#$ and $U_\infty^\#$ are in one of the classes, while elements in $U_k \smallsetminus \langle\widehat{\eta}\rangle$ for $k \in \{\pm 1\}$ are in the other. Since $C_{\boldsymbol{A}}(U_0) = Z$ while $C_{\boldsymbol{A}}(U_\infty) = W_\infty$ by Lemma 4.7(b), U_0 and U_∞ are not $\boldsymbol{\Gamma}$-conjugate.

As noted earlier (see [Hu, §4]), while $\boldsymbol{A}$ is not isomorphic to its dual $\boldsymbol{A}^*$ as $\mathbb{F}_3\boldsymbol{\Gamma}$-modules, the pairs $(\boldsymbol{\Gamma}, \boldsymbol{A})$ and $(\boldsymbol{\Gamma}, \boldsymbol{A}^*)$ are isomorphic via an outer automorphism $\alpha \in \mathrm{Aut}(\boldsymbol{\Gamma}) \smallsetminus \mathrm{Inn}(\boldsymbol{\Gamma})$. Hence by Table 4.6,

$$\mathrm{rk}(C_{\boldsymbol{A}}(U_0)) = 1 \quad \text{and} \quad \mathrm{rk}(C_{\boldsymbol{A}}(\alpha(U_0))) = \mathrm{rk}(C_{\boldsymbol{A}^*}(U_0)) = \mathrm{rk}(\boldsymbol{A}/[U_0, \boldsymbol{A}]) = 3,$$

so $\alpha(U_0)$ is not $\boldsymbol{\Gamma}$-conjugate to U_0. Since all elements of order 3 in $\alpha(U_0)$ are conjugate to each other, $\alpha(U_0)$ must be $\boldsymbol{\Gamma}$-conjugate to U_∞. Thus α exchanges the classes of U_0 and U_∞.

By the description of the action of $\boldsymbol{N}$ on $\boldsymbol{A}$ in Notation 4.4, $\boldsymbol{N}$ normalizes the subgroup A_* of index 3 in $\boldsymbol{A}$. So it also normalizes a subgroup of order 3 in the dual space $\boldsymbol{A}^*$, and hence $\alpha(\boldsymbol{N}) \le N_{\boldsymbol{\Gamma}}(X)$ for some $X \le \boldsymbol{A}$ of order 3. The length of the orbit of X under the action of $\boldsymbol{\Gamma}$ divides $|\boldsymbol{\Gamma} : \boldsymbol{N}| = 220$, so X is in the orbit of Z by earlier remarks, and $\alpha(\boldsymbol{N}) = N_{\boldsymbol{\Gamma}}(X)$ is $\boldsymbol{\Gamma}$-conjugate to $N_{\boldsymbol{\Gamma}}(Z)$. Thus $N_{\boldsymbol{\Gamma}}(Z) \cong \boldsymbol{N} \cong (E_9 \rtimes GL_2(3)) \times C_2$. Since $\boldsymbol{N_0}$ acts via the identity on $\boldsymbol{A}/A_*$, a similar argument shows that $C_{\boldsymbol{\Gamma}}(Z) \cong \boldsymbol{N_0}$. Finally, since $U_\infty = O_3(\boldsymbol{N})$ and $\alpha(U_\infty)$ is $\boldsymbol{\Gamma}$-conjugate to U_0, we get that $O_3(N_{\boldsymbol{\Gamma}}(Z))$ is $\boldsymbol{\Gamma}$-conjugate to U_0, so $|N_{\boldsymbol{\Gamma}}(U_0)| = |N_{\boldsymbol{\Gamma}}(\alpha(U_\infty))| \ge |N_{\boldsymbol{\Gamma}}(Z)|$. Since $N_{\boldsymbol{\Gamma}}(U_0) \le N_{\boldsymbol{\Gamma}}(Z)$, they must be equal.

(d) Since $C_{\boldsymbol{\Gamma}}(Z)$ has index 2 in $N_{\boldsymbol{\Gamma}}(Z) = N_{\boldsymbol{\Gamma}}(U_0)$ by (c), Z and W_0 are both invariant under its action on $\boldsymbol{A}$ (recall $W_0/Z = C_{\boldsymbol{A}/Z}(U_0)$ by definition). We must show that there are no other invariant subgroups.

As noted in the proof of (c), the action of $C_{\boldsymbol{\Gamma}}(Z)$ on $\boldsymbol{A}$ is (up to isomorphism) dual to the action of $\boldsymbol{N_0} \cong E_9 \rtimes GL_2(3)$ on $\boldsymbol{A}$. Set

$$B = \Theta^{-1}\bigl(\bigl\{\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right) \bigm| A \in GL_2(3)\bigr\}\bigr) < \boldsymbol{N_0}.$$

Then $\boldsymbol{A}$ splits as a direct sum of the three irreducible $\mathbb{F}_3B$-submodules

$$W_\infty = \left\{ \left(\begin{smallmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \middle| \, a,b,c \in \mathbb{F}_3 \right\}, \quad \left\{ \left(\begin{smallmatrix} 0 & 0 & x \\ 0 & 0 & y \\ x & y & 0 \end{smallmatrix}\right) \middle| \, x,y \in \mathbb{F}_3 \right\}, \quad \left\{ \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{smallmatrix}\right) \middle| \, z \in \mathbb{F}_3 \right\},$$

of which only W_∞ is $\boldsymbol{N_0}$-invariant. Since $\boldsymbol{N_0} = U_\infty B$, it now follows that the only proper nontrivial $\mathbb{F}_3\boldsymbol{N_0}$-submodules are W_∞ and A_*, and hence (after dualizing) that $\boldsymbol{A}$ also has only two proper nontrivial $\mathbb{F}_3C_{\boldsymbol{\Gamma}}(Z)$-submodules.

(b) Since $M = \boldsymbol{A} \rtimes \boldsymbol{\Gamma}$ is a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ (Lemma 4.3(b)), it suffices to show that $\boldsymbol{Q} \trianglelefteq N_M(Z) = \boldsymbol{A}N_{\boldsymbol{\Gamma}}(Z)$. Since $[\boldsymbol{Q}, \boldsymbol{A}] = [U_0, \boldsymbol{A}] = W_0 \le \boldsymbol{Q}$, where the second equality holds by Table 4.6, we have $\boldsymbol{A} \le N_M(\boldsymbol{Q})$. Also, $N_{\boldsymbol{\Gamma}}(Z) = N_{\boldsymbol{\Gamma}}(U_0)$ by (c), this group normalizes W_0 since U_0 normalizes $W_0 = [U_0, \boldsymbol{A}]$, and hence $N_{\boldsymbol{\Gamma}}(Z)$ also normalizes $\boldsymbol{Q} = U_0W_0$. So $\boldsymbol{Q} \trianglelefteq \boldsymbol{A}N_{\boldsymbol{\Gamma}}(Z)$.

(a) We first check that

$$\boldsymbol{Q}^{\boldsymbol{\mathcal{F}}} \cap \mathcal{Q}_0 = \{\boldsymbol{Q}\}. \tag{4.11}$$

Assume otherwise: assume $P \in \boldsymbol{Q}^{\boldsymbol{\mathcal{F}}} \cap \mathcal{Q}_0$ and $P \ne \boldsymbol{Q}$. By Lemma 1.2, there is $\varphi \in \mathrm{Hom}_{\boldsymbol{\mathcal{F}}}(N_{\boldsymbol{S}}(P), \boldsymbol{S})$ such that $\varphi(P) = \boldsymbol{Q}$, and $\boldsymbol{A} \le N_{\boldsymbol{S}}(P)$ by Lemma 4.7(c). Then $\varphi(\boldsymbol{A}) = \boldsymbol{A}$ since $\boldsymbol{A}$ is weakly closed (Lemma 4.3(a)), and $\varphi(Z) = Z$ since $Z = Z(N_{\boldsymbol{S}}(P)) = Z(\boldsymbol{S})$. (Note that $N_{\boldsymbol{S}}(P) = U_0\boldsymbol{A}$ or $\boldsymbol{S}$.) Thus $\varphi \in \mathrm{Mor}(N_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(Z))$, so $\varphi(\boldsymbol{Q}) = \boldsymbol{Q}$ by (b), contradicting our assumption that $P \ne \boldsymbol{Q}$.

If $\boldsymbol{Q}$ is not weakly closed, then by Lemma 2.8, there are $R \in \mathbf{E}_{\boldsymbol{\mathcal{F}}} \cup \{\boldsymbol{S}\}$, $\alpha \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$, and $P \le R$ such that $R \ge \boldsymbol{Q}$, $P = \alpha(\boldsymbol{Q}) \ne \boldsymbol{Q}$, and $R = N_{\boldsymbol{S}}(P)$. Then $P \notin \mathcal{Q}_0$ by (4.11), so by Lemma 4.8(b), there is $k \in \{\pm 1\}$ such that $P \in \mathcal{Q}_k$. By Lemma 4.7(c) again, $R \cap \boldsymbol{A} = N_{\boldsymbol{S}}(P) \cap \boldsymbol{A} = A_*$. Also, $R\boldsymbol{A}$ contains both $\boldsymbol{Q}\boldsymbol{A} = U_0\boldsymbol{A}$ and $P\boldsymbol{A} = U_k\boldsymbol{A}$, so $R\boldsymbol{A} = \boldsymbol{S}$ and $|\boldsymbol{S}/R| = 3$. In particular, $R \trianglelefteq \boldsymbol{S}$.

We next claim that

$$\beta \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R),\ \beta(A_*) = A_* \implies \beta(\boldsymbol{Q}) = \boldsymbol{Q}. \tag{4.12}$$

Fix such a β. Since $\beta(A_*) = A_*$ and $\boldsymbol{A}$ is weakly closed, $\beta|_{A_*}$ extends to some $\widehat{\beta} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) = \boldsymbol{\Gamma}$ by Lemma 2.2(b). Also, $\beta(Z) = Z$ since $Z = Z(R)$, so $\widehat{\beta} \in N_{\boldsymbol{\Gamma}}(Z) = N_{\boldsymbol{\Gamma}}(U_0)$ by (c), and $\widehat{\beta}$ normalizes $C_{\boldsymbol{A}/Z}(U_0) = W_0/Z$. So $\beta(W_0) = W_0$, hence $\beta(\boldsymbol{Q}) \cap A_* = \beta(W_0) = W_0$, and $\beta(\boldsymbol{Q}) \in \mathcal{Q}_0$ by Lemma 4.8(b) again. So $\beta(\boldsymbol{Q}) = \boldsymbol{Q}$ by (4.11), proving (4.12).

In particular, $\alpha(A_*) \neq A_* = R \cap \boldsymbol{A}$ by (4.12) and since $\alpha(\boldsymbol{Q}) \neq \boldsymbol{Q}$, so $\alpha(A_*) \not\leq \boldsymbol{A}$, and by Lemma 4.8(a), $\alpha(A_*)$ is one of the three subgroups $\boldsymbol{A}$-conjugate to Q_∞. Since $R \trianglelefteq \boldsymbol{S}$, all three of these subgroups are in the $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$-orbit of $\boldsymbol{Q}$. In particular, $Q_\infty = U_\infty W_\infty \leq R$, so $R \geq U_\infty \boldsymbol{Q} A_* = \boldsymbol{T} A_*$, with equality since both have index 3 in $\boldsymbol{S}$.

Let $\mathrm{Aut}^0_{\boldsymbol{\mathcal{F}}}(R) \leq \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$ be the stabilizer of A_*. We just saw that the $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$-orbit of A_* consists of A_* together with the three subgroups conjugate to Q_∞ by elements of $\boldsymbol{A}$. So $\mathrm{Aut}^0_{\boldsymbol{\mathcal{F}}}(R)$ has index 4 in $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$. By (4.12), $\beta(\boldsymbol{Q}) = \boldsymbol{Q}$ for each $\beta \in \mathrm{Aut}^0_{\boldsymbol{\mathcal{F}}}(R)$, and hence the $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$-orbit of $\boldsymbol{Q}$ has order at most 4. Since $R \trianglelefteq \boldsymbol{S}$, all three members of the $\boldsymbol{A}$-conjugacy class of $P \in \mathcal{Q}_k$ lie in this orbit. Also, the element $\Theta^{-1}\left(\left(\begin{smallmatrix} -I & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \in N_{\boldsymbol{N_0}}(\boldsymbol{T}) \leq M$ exchanges the two classes $\mathcal{Q}_1$ and $\mathcal{Q}_{-1}$ and normalizes $R = \boldsymbol{T} A_*$, so the $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$-orbit of $\boldsymbol{Q}$ has at least three members from each of these classes. Since this contradicts the earlier observation that the orbit has at most four members, we conclude that $\boldsymbol{Q}$ is weakly closed in $\boldsymbol{\mathcal{F}}$. □

We are now ready to prove that $\boldsymbol{Q} \trianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)$.

Lemma 4.13. *Assume Hypotheses 4.1 and Notation 4.5, and again set $\boldsymbol{Q} = Q_0$. Then $\boldsymbol{Q} \trianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)$.*

Proof. For $1 \leq i \leq j \leq 3$, let $A_{ij} \leq \boldsymbol{A}$ be the subgroup of those elements represented by symmetric (3×3)-matrices with entries 0 except possibly in positions (i,j) and (j,i). We also set $\Delta = W_0 A_{22} = W_\infty A_{13}$, since this "triangular shaped" subgroup appears frequently in the arguments below.

Define inductively $Z = B_0 < B_1 < B_2 < B_3 < B_4 = B = \boldsymbol{Q}$ by setting $B_i/B_{i-1} = C_{\boldsymbol{Q}/B_{i-1}}(\boldsymbol{S})$. Thus

$$B_0 = A_{11}, \quad B_1 = B_0 A_{12}, \quad B_2 = W_0 = B_1 A_{13}, \quad B_3 = B_2\langle \widehat{\eta}\rangle, \quad B_4 = \boldsymbol{Q} = B_3\langle \eta_0\rangle,$$

and $B_i \trianglelefteq \boldsymbol{S}$ for each i since Z and $\boldsymbol{Q}$ are normal.

Assume $\boldsymbol{Q} \ntrianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)$. Then $\boldsymbol{Q}/Z \ntrianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)/Z$ by Lemma 1.20 and since $Z \leq Z(C_{\boldsymbol{\mathcal{F}}}(Z))$. By Proposition 2.9, applied with $C_{\boldsymbol{\mathcal{F}}}(Z)/Z$ and $\boldsymbol{Q}/Z$ in the role of $\mathcal{F}$ and A, there are $\ell \leq 2$, $R \leq \boldsymbol{S}$, and $\alpha \in \mathrm{Aut}_{C_{\mathcal{F}}(Z)}(R)$ such that

((1)) $R \geq B_{\ell+1}$, $\alpha(B_i) = B_i$ for all $i \leq \ell$, and $X \stackrel{\text{def}}{=} \alpha(B_{\ell+1}) \not\leq \boldsymbol{Q}$;

((2)) $R = N_S(X)$ and $R/B_\ell = C_{S/B_\ell}(X/B_\ell)$; and

((3)) if $\ell = 0$, then $R \in \mathbf{E}_{C_{\mathcal{F}}(Z)}$ and $R/Z \in \mathbf{E}_{C_{\mathcal{F}}(Z)/Z}$.

Note, in **((3))**, that $R \in \mathbf{E}_{C_{\mathcal{F}}(Z)}$ by Lemma 1.19 together with Proposition 2.9.

We will show that this is impossible. Fix an element $t \in X \smallsetminus \boldsymbol{Q} = \alpha(B_{\ell+1}) \smallsetminus \boldsymbol{Q}$. Thus $X = B_\ell\langle t\rangle$ (recall $B_\ell \trianglelefteq \mathcal{F}_\ell$). Set $R_0 = R \cap \boldsymbol{A}$, so that $R_0 = N_{\boldsymbol{A}}(X)$ and $R_0/B_\ell = C_{\boldsymbol{A}/B_\ell}(X/B_\ell)$ by **((2))**. We claim that

((4)) $R \ngeq \boldsymbol{A}$ and hence $R_0 \neq \boldsymbol{A}$ and $t \notin \boldsymbol{A}$;

((5)) $|t| = 3$; and

((6)) $t \notin \widehat{\eta}\boldsymbol{A}$ implies $R \leq \boldsymbol{A}\langle t, \widehat{\eta}\rangle$.

To see these, note first that if $R \geq \boldsymbol{A}$, then $\alpha \in \mathrm{Aut}_{\mathcal{F}_\ell}(R) \subseteq \mathrm{Mor}(C_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(Z))$ since $\mathcal{F}_\ell \leq C_{\boldsymbol{\mathcal{F}}}(Z)$ and $\boldsymbol{A}$ is weakly closed (Lemma 4.3(a)). So $\alpha(R \cap \boldsymbol{Q}) = R \cap \boldsymbol{Q}$ since $\boldsymbol{Q} \trianglelefteq N_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(Z)$ by Lemma 4.10(b), contradicting the assumption that $t \in \alpha(B_{\ell+1}) \smallsetminus \boldsymbol{Q}$. Hence $R \ngeq \boldsymbol{A}$. Also, $B_\ell \leq \boldsymbol{A}$, while $X = B_\ell\langle t\rangle \not\leq \boldsymbol{A}$ since $\boldsymbol{A} \neq R_0 = N_{\boldsymbol{A}}(X)$, so $t \notin \boldsymbol{A}$, finishing the proof of **((4))**.

Since $B_{\ell+1} \le \boldsymbol{Q}$ has exponent 3, so does $X = \alpha(B_{\ell+1})$. Hence $|t| = 3$, proving **((5))**. If $t \notin \widehat{\eta}\boldsymbol{A}$, then $R\boldsymbol{A}/\boldsymbol{A} \le C_{\boldsymbol{S}/\boldsymbol{A}}(t) = \langle t\boldsymbol{A}, \widehat{\eta}\boldsymbol{A}\rangle$, so $R \le \boldsymbol{A}\langle t, \widehat{\eta}\rangle$, proving **((6))**.

Since $t \in \boldsymbol{S} \smallsetminus \boldsymbol{A}$ by **((4))**, and each element in $\boldsymbol{S} \smallsetminus \boldsymbol{A}$ is $\boldsymbol{S}$-conjugate to an element of $\eta\boldsymbol{A}$ for $\eta = \widehat{\eta}^{\pm1}$ or $\eta_k^{\pm1}$ for $k \in \mathbb{F}_3 \cup \{\infty\}$, we can arrange that $t \in \eta\boldsymbol{A}$ for $\eta \in \{\widehat{\eta}, \eta_\infty, \eta_0, \eta_{\pm1}\}$. The proof now splits up naturally into different cases, depending on the class $t\boldsymbol{A}$ and on ℓ. The following arguments, covering all possible pairs $(t\boldsymbol{A}, \ell)$, are summarized in Table 4.14.

ℓ	$\ell = 0$	$\ell = 1$	$\ell = 2$
B_ℓ	$\left\{\left(\begin{smallmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\right\}$	$\left\{\left(\begin{smallmatrix} * & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\right\}$	$\left\{\left(\begin{smallmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{smallmatrix}\right)\right\}$
$t \in \widehat{\eta}\boldsymbol{A}$	$R_0 = \Delta = \left\{\left(\begin{smallmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{smallmatrix}\right)\right\}$ $\alpha^{-1}(t) \in [R,R] \implies t \in [R,R] \implies$ $R = \Delta\langle t,u,v\rangle$ with $u \in \eta_0\boldsymbol{A},\ v \in \eta_\infty\boldsymbol{A}$ $\left.\begin{matrix} Z(R/B_1) = R_0\langle t\rangle/B_1 \cong E_{27} \\ Z(R/Z\langle t\rangle) = B_1\langle t\rangle/Z\langle t\rangle \cong C_3 \end{matrix}\right\}$ imposs.	$R_0 = A_* = \left\{\left(\begin{smallmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{smallmatrix}\right)\right\}$ $A_* \cong E_{3^5},\ \alpha(A_*) \nleq \boldsymbol{A}$ $\implies R \ge A_*Q_\infty$ $\alpha(A_*) \trianglelefteq R \implies R = A_*\boldsymbol{T}$ $R \ge \boldsymbol{Q}$ w.cl. $\Rightarrow \alpha(\boldsymbol{Q}) = \boldsymbol{Q}$	$R_0 = \boldsymbol{A}$ impossible by **((4))**
$t \in \eta_0\boldsymbol{A}$	$R_0 = \left\{\left(\begin{smallmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & * \end{smallmatrix}\right)\right\},\ R = R_0\langle t,u\rangle$ where $u \in \widehat{\eta}A_*,\ [t,u] = t^3 = u^3 = 1$ $[A_{23}, R] \le B_2\langle t\rangle = Z_2(R)$ $[A_{23}, B_2\langle t\rangle] \le ZA_{13} = Z(Z_2(R)$ $\implies R \notin \mathbf{E}_{C_{\mathcal{F}}(Z)}$ (Lem. B.9) imp. by **((3))**	$R_0 = \left\{\left(\begin{smallmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{smallmatrix}\right)\right\}$ $R = R_0\langle t,u\rangle$ with $u \in \widehat{\eta}A_*$ $\implies \alpha(t) \in B_2 \le [R,R]$ while $t \notin [R,R]$	$R_0 = \boldsymbol{A}$ impossible by **((4))**
$t \in \eta_k\boldsymbol{A},\ k \in \{\pm1, \infty\}$	$R_0 = W_k = \left\{\left(\begin{smallmatrix} * & * & r \\ * & -kr & 0 \\ r & 0 & 0 \end{smallmatrix}\right)\right\}$ or $\left\{\left(\begin{smallmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\right\}$ $R = R_0\langle t,u\rangle$ where $[t,u] \in Z$ $R \in \mathcal{Q}_k,\ R/Z \cong E_{3^4}$ Prop.B.10 $\Rightarrow \mathrm{Aut}_{C_{\mathcal{F}}(Z)/Z}(R/Z) \cong (P)SL_2(q)$ $\lvert\mathrm{Aut}_{\boldsymbol{S}/Z}(R/Z)\rvert = \lvert N_{\boldsymbol{S}}(R)/R\rvert = 3^3$ $\implies \mathrm{Aut}_{\boldsymbol{S}/Z}(R/Z) \nleq \mathrm{Aut}_{C_{\mathcal{F}}(Z)/Z}(R/Z)$	$R_0 = \Delta = \left\{\left(\begin{smallmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{smallmatrix}\right)\right\}$ $R = \Delta\langle t,u\rangle$ for some $t \in \eta_k\boldsymbol{A},\ u \in \widehat{\eta}\boldsymbol{A},\ [t,u] \in Z$ Set $x = \alpha^{-1}(t) \in B_{\ell+1} \smallsetminus B_\ell$: then $C_R(x) \cong C_R(t) \le C_\Delta(\eta_k)\langle t, u'\rangle$ for $u' \in u\Delta$: nonabelian of order 3^4 $C_R(x) \ge \Delta \cong E_{3^4}$ if $\ell = 1$ $C_R(x) \ge W_\infty\langle x\rangle \cong E_{3^4}$ if $\ell = 2$	

TABLE 4.14. In all cases, $R_0 = R \cap \boldsymbol{A}$, where $R/B_\ell = C_{\boldsymbol{S}/B_\ell}(t)$. In the matrices used to describe R_0, a "$*$" means an arbitrary element of $\mathbb{F}_3$, independent of the other entries.

$\boldsymbol{t \in \widehat{\eta}A}$**:** Since $[\widehat{\eta}, \boldsymbol{A}] = B_2 = [\eta_0, \boldsymbol{A}]$, $[t,u] = 1$ for some element $u \in \eta_0\boldsymbol{A}$, and hence $R \ge R_0\langle t,u\rangle$.

- **If $\boldsymbol{\ell = 0}$,** then $R_0 = \Delta$. So $\alpha^{-1}(t) \in B_1 = [\Delta, \eta_0] = [R_0, u] \le [R,R]$, and hence $t \in [R,R]$. This implies that $R = \Delta\langle t,u,v\rangle$ for some $v \in \eta_\infty\boldsymbol{A}$, and hence that $Z(R) = Z$ and $Z_2(R) = B_1\langle t\rangle \cong E_{27}$.

  By the above relations, we have $Z(R/B_1) = \Delta\langle t\rangle/B_1 \cong E_{27}$, while $Z(R/(Z\langle t\rangle)) = B_1\langle t\rangle/Z\langle t\rangle \cong C_3$. So no $\alpha \in \mathrm{Aut}(R)$ sends B_1 into $Z\langle t\rangle$.

- **If $\boldsymbol{\ell = 1}$,** then $R_0 = A_* = \Delta A_{23} \cong E_{3^5}$. Set $E = \alpha(A_*)$. Then $t \in \alpha(B_2) \le E$, so $E \cong E_{3^5}$ is not contained in $\boldsymbol{A}$, and E is $\boldsymbol{A}$-conjugate to $Q_\infty = W_\infty\langle\widehat{\eta}, \eta_\infty\rangle$ by Lemma 4.8(a). Since $\widehat{Q} = A_*Q_\infty \trianglelefteq \boldsymbol{S}$, this implies that $R \ge \widehat{Q}$. Thus $R = \widehat{Q}\langle u\rangle = A_*\langle\widehat{\eta}, \eta_\infty, u\rangle$, and has index 3 in $\boldsymbol{S}$.

Let $a \in A_{33}$ be such that $u \in \eta_0 a A_*$. The element η_0 normalizes both A_* and $Q_\infty = W_\infty\langle\widehat{\eta},\eta_\infty\rangle$. Hence η_0 normalizes each of the four subgroups of $\widehat{Q}$ isomorphic to E_{3^5}, while A_{33} normalizes A_* and permutes the other three transitively. Since $A_* \trianglelefteq R$, we must have $E = \alpha(A_*) \trianglelefteq R$, and this is possible only if $a = 1$. Thus $R = \widehat{Q}\langle\eta_0\rangle = A_*\boldsymbol{T}$.

In particular, $\boldsymbol{Q} = B_2\langle\widehat{\eta},\eta_0\rangle \le R$, and $\alpha(\boldsymbol{Q}) = \boldsymbol{Q}$ since $\boldsymbol{Q}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ by Lemma 4.10(a). This contradicts the assumption that $\alpha(B_2) = B_1\langle t\rangle \nleq \boldsymbol{Q}$.

If $\boldsymbol{\ell = 2}$, then $R_0 = \boldsymbol{A}$, contradicting (**(4)**).

$\boldsymbol{t \in \eta_0 A}$: Since $[\widehat{\eta},\boldsymbol{A}] = B_2 = [\eta_0, A_*]$, t commutes with some element $u \in \widehat{\eta}A_*$. Thus $R = R_0\langle t,u\rangle$ by (**(6)**), where $u \in \widehat{\eta}A_*$, and $[t,u] = u^3 = 1$.

If $\boldsymbol{\ell = 0}$, then $R_0 = B_2A_{33}$ (recall $W_0 = B_2$). So $Z(R) = Z$, and $R/Z \cong 3^{1+2}_+ \times E_9$. Then $Z_2(R) = B_2\langle t\rangle \cong 3^{1+2}_+ \times C_3$ and $Z(Z_2(R)) = Z(B_2\langle t\rangle) = ZA_{13}$, and so both of these are characteristic in R.

Since $[A_{23}, R] \le B_2 \le Z_2(R)$ and $[A_{23}, Z_2(R)] = [A_{23}, t] = A_{13} \le Z(Z_2(R))$ (and since $[A_{23}, Z(Z_2(R))] = 1$), we have $R \notin \mathbf{E}_{C_{\mathcal{F}}(Z)}$ by Lemma B.9, contradicting (**(3)**).

If $\boldsymbol{\ell = 1}$, then $R_0 = B_2A_{22}A_{33} \cong E_{3^5}$. So $\alpha(t) \in B_2 \le [R_0,\langle t,u\rangle] \le [R,R]$, while $t \notin [R,R]$, a contradiction.

If $\boldsymbol{\ell = 2}$, then $R_0 = \boldsymbol{A}$, contradicting (**(4)**).

$\boldsymbol{t \in \eta_k A}$ for $\boldsymbol{k = \infty, \pm 1}$: We have $W_k \le R_0 \le \Delta$ in all cases. Since $|t| = 3$ by (**(5)**), we have $t \in \eta_k A_*$, and $t \in \eta_k\Delta$ if $k = \pm 1$. This follows from Lemma A.5, together with the formulas in Table 4.6. So if $k = \pm 1$, then $[\widehat{\eta}, t] \in [\widehat{\eta}, \Delta] = Z$, and we set $u = \widehat{\eta} \in R$. If $k = \infty$, then $[\widehat{\eta}, t] \in [\widehat{\eta}, A_*] = B_1$, and $[u, R_0\langle t\rangle] \le Z$ (hence $u \in R$) for some $u \in \widehat{\eta}A_{13}$. In all cases, $[t,u] \in Z$, and $R = R_0\langle t,u\rangle$ by (**(6)**).

If $\boldsymbol{\ell = 0}$, then $R_0 = W_k$, and so $R \in \mathcal{Q}_k$, and $R/Z \cong E_{3^4}$ in all cases. Since $R/Z \in \mathbf{E}_{C_{\mathcal{F}}(Z)/Z}$ by (**(3)**), the group $\mathrm{Aut}_{C_{\mathcal{F}}(Z)/Z}(R/Z) \le GL_4(3)$ has a strongly embedded subgroup, and hence $O^{3'}(\mathrm{Aut}_{C_{\mathcal{F}}(Z)/Z}(R/Z)) \cong SL_2(9)$ or $PSL_2(9)$ by Proposition B.10. So $\mathrm{Aut}_{\boldsymbol{S}/Z}(R/Z) \cong N_{\boldsymbol{S}}(R)/R \cong E_9$: a Sylow 3-subgroup of *(P)SL*$_2(9)$.

In all cases, $N_{\boldsymbol{S}}(R) \cap \boldsymbol{A} = A_*$. If $k = \pm 1$, then $N_{\boldsymbol{S}}(R) = A_*\langle t,\widehat{\eta},\eta_0\rangle$, so $|N_{\boldsymbol{S}}(R)/R| = 3^3$. If $k = \infty$, then $t \in Z(R)$ since $\alpha^{-1}(t) \in B_1 \le Z(R)$, so $R \cong E_{3^5}$ and is $\boldsymbol{S}$-conjugate to Q_∞ by Lemma 4.8(a). So $|N_{\boldsymbol{S}}(R)/R| = |N_{\boldsymbol{S}}(Q_\infty)/Q_\infty| = 3^3$, and we also get a contradiction in this case.

If $\boldsymbol{\ell = 1}$ or $\boldsymbol{2}$, then $R_0 = \Delta$ and $R = \Delta\langle t,u\rangle$ where $u \in \widehat{\eta}A_{13}$ and $[t,u] \in Z$. Set $x = \alpha^{-1}(t) \in B_{\ell+1} \smallsetminus B_\ell$. Then $C_R(x) \cong C_R(t)$, where either $C_R(t) = C_\Delta(t)\langle t\rangle \cong E_{27}$, or $C_\Delta(t)\langle t,u\rangle$ is nonabelian of order 3^4. If $\ell = 1$, then $x \in \boldsymbol{A}$, so $C_R(x) \ge R_0 \cong E_{3^4}$. If $\ell = 2$, then $x \in \widehat{\eta}B_2 \subseteq \widehat{\eta}\Delta$ (and $x \in R$), so $C_R(x) \ge W_\infty\langle x\rangle \cong E_{3^4}$. So this is impossible in either case. $\square$

We can now determine $\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$. Let $Sp_4^*(3) \le GL_4(3)$ denote the group of matrices that preserve a symplectic form up to sign. Thus $Sp_4^*(3)$ contains $Sp_4(3)$ with index 2.

Lemma 4.15. *Assume Hypotheses 4.1 and Notation 4.5. Then*

$$\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q}) = \mathrm{Out}(\boldsymbol{Q}) \cong Sp_4^*(3).$$

Also,

$$\begin{aligned}\mathrm{Out}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(\boldsymbol{Q}) &\cong N_M(\boldsymbol{Q})/\boldsymbol{Q} = \boldsymbol{A}N_M(U_0)/W_0U_0 \\ &\cong (\boldsymbol{A}/W_0) \rtimes (N_M(U_0)/U_0) \cong E_{27} \rtimes (GL_2(3) \times C_2).\end{aligned}$$

where the action of $C_M(U_0)/U_0 \cong GL_2(3)$ on $O_3(\mathrm{Out}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(\boldsymbol{Q})) \cong \boldsymbol{A}/W_0$ is irreducible.

Proof. The model M for $N_{\mathcal{F}}(\boldsymbol{A})$ is a semidirect product of $\boldsymbol{A}$ by $\boldsymbol{\Gamma} = \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) \cong 2M_{12}$ (Lemmas 4.2 and 4.3(b)). Since $\boldsymbol{Q}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ by Lemma 4.10(a), we have $N_M(\boldsymbol{Q}) = N_M(\boldsymbol{A}U_0) = \boldsymbol{A}N_{\boldsymbol{\Gamma}}(U_0)$, where $N_{\boldsymbol{\Gamma}}(U_0) \cong (E_9 \rtimes GL_2(3)) \times C_2$ by Lemma 4.10(c). The description of $\mathrm{Out}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(\boldsymbol{Q}) \cong N_M(\boldsymbol{Q})/\boldsymbol{Q}$ is now immediate, where the action of $C_M(U_0)/U_0$ on $\boldsymbol{A}/W_0$ is irreducible by Lemma 4.10(d).

Since $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) < \boldsymbol{\mathcal{F}}$ by assumption and $\boldsymbol{\mathcal{F}} = \langle C_{\boldsymbol{\mathcal{F}}}(Z), N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})\rangle$ by Proposition 4.9, we have $N_{\boldsymbol{\mathcal{F}}}(Z) > N_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(Z)$. Since $\boldsymbol{Q}$ is $\boldsymbol{\mathcal{F}}$-centric by Lemma 4.8(c) and normal in $N_{\boldsymbol{\mathcal{F}}}(Z)$ by Lemma 4.13, $N_{\boldsymbol{\mathcal{F}}}(Z)$ constrained and $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q}) > \mathrm{Aut}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(\boldsymbol{Q})$. Since $\mathrm{Out}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(\boldsymbol{Q})$ is maximal in $\mathrm{Out}(\boldsymbol{Q})$, we conclude that $\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q}) = \mathrm{Out}(\boldsymbol{Q}) \cong Sp_4^*(3)$. □

We are now ready to identify all fusion systems satisfying Hypotheses 4.1.

Theorem 4.16. *Let $\boldsymbol{\mathcal{F}}$ be a saturated fusion system over a finite 3-group $\boldsymbol{S}$ with a subgroup $\boldsymbol{A} \le \boldsymbol{S}$ such that $\boldsymbol{A} \cong E_{3^6}$, $C_{\boldsymbol{S}}(\boldsymbol{A}) = \boldsymbol{A}$, and $O^{3'}(\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})) \cong 2M_{12}$. Assume also that $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$. Then $\boldsymbol{A} \trianglelefteq \boldsymbol{S}$, $\boldsymbol{S}$ splits over $\boldsymbol{A}$, and $\boldsymbol{\mathcal{F}}$ is simple and isomorphic to the 3-fusion system of Co_1.*

Proof. By Lemma 4.2, $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) \cong 2M_{12}$ and acts on $\boldsymbol{A}$ as the Todd module. By Lemma 4.3, $\boldsymbol{A}$ is normal in $\boldsymbol{S}$ and weakly closed in $\boldsymbol{\mathcal{F}}$, and $\boldsymbol{S} \cong \boldsymbol{A} \rtimes \boldsymbol{T}$ where $\boldsymbol{T} \in \mathrm{Syl}_3(\boldsymbol{\Gamma})$ is defined in Notation 4.4. So we are in the situation of Notation 4.4 and 4.5, and can use the terminology listed there. Set $\boldsymbol{Q} = Q_0$; then $\boldsymbol{Q} \trianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)$ by Lemma 4.13, and is the only subgroup of $\boldsymbol{S}$ isomorphic to 3_+^{1+4} and weakly closed in $N_{\boldsymbol{\mathcal{F}}}(Z)$ by Lemma 4.8(b).

Set $G^* = Co_1$, fix $S^* \in \mathrm{Syl}_3(G)$, and let $A^* \trianglelefteq S^*$ be the unique subgroup isomorphic to E_{3^6}. Set $Z^* = C_{A^*}(S^*) = Z(S^*)$. By [Cu, Theorem 3.1] (see also the discussion about the subgroup !333 on p. 424), the fusion system $\mathcal{F}_{S^*}(G^*)$ satisfies Hypotheses 4.1.

Let M be a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ (see Proposition 1.12), and set $M^* = N_{G^*}(A^*)$. By Lemmas 4.2 and 4.3(b), M and M^* are both semidirect products of E_{3^6} by $2M_{12}$ acting as the Todd module, so there is an isomorphism $\varphi\colon M^* \xrightarrow{\cong} M$ such that $\varphi(S^*) = \boldsymbol{S}$. Set $\mathcal{F}^* = {}^{\varphi}(\mathcal{F}_{S^*}(G^*))$. Thus $\mathcal{F}^*$ is a fusion system over $\boldsymbol{S}$ isomorphic to $\mathcal{F}_{S^*}(G^*)$. We will show that $\mathcal{F}^* = \mathcal{F}$. By construction, $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) = N_{\mathcal{F}^*}(\boldsymbol{A})$.

Set $\mathcal{F}_1 = C_{\boldsymbol{\mathcal{F}}}(Z)$, $\mathcal{F}_2 = C_{\mathcal{F}^*}(Z)$, and $\mathcal{E} = C_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(Z)$. Since $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) = N_{\mathcal{F}^*}(\boldsymbol{A})$, $\mathcal{E}$ is contained in $\mathcal{F}_2$ as well as in $\mathcal{F}_1$. All three of these are fusion systems over $\boldsymbol{S}$, and $\boldsymbol{Q}$ is centric and normal in each of them by Lemmas 4.8(c) and 4.13. Also, $\mathrm{Out}_{\mathcal{F}_1}(\boldsymbol{Q}) = \mathrm{Out}_{\mathcal{F}_2}(\boldsymbol{Q}) \cong Sp_4(3)$ since they have index 2 in $\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$ and $\mathrm{Out}_{\mathcal{F}^*}(\boldsymbol{Q})$, respectively, where $\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q}) = \mathrm{Out}_{\mathcal{F}^*}(\boldsymbol{Q}) = \mathrm{Out}(\boldsymbol{Q})$ by Lemma 4.15.

By Lemma 4.15,

$$\mathrm{Out}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(\boldsymbol{Q}) = \mathrm{Aut}_{\boldsymbol{A}}(\boldsymbol{Q}) \rtimes (N_{\boldsymbol{\Gamma}}(Z)/U_0) \cong E_{27} \rtimes (GL_2(3) \times C_2),$$

where the action of $C_{\boldsymbol{\Gamma}}(Z)/U_0 \cong GL_2(3)$ on $\mathrm{Aut}_{\boldsymbol{A}}(\boldsymbol{Q}) \cong \boldsymbol{A}/W_0$ is irreducible. In particular, $\mathrm{Out}_{\mathcal{E}}(\boldsymbol{Q})$ has no normal subgroup of index 3, and hence

$$H^1(\mathrm{Out}_{\mathcal{E}}(\boldsymbol{Q}); Z(\boldsymbol{Q})) \cong \mathrm{Hom}(E_{27} \rtimes GL_2(3), \mathbb{Z}/3) = 0.$$

So $\mathcal{F}_1 = \mathcal{F}_2$ by Proposition 2.11.

Thus $C_{\boldsymbol{\mathcal{F}}}(Z) = C_{\mathcal{F}^*}(Z)$ and $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) = N_{\mathcal{F}^*}(\boldsymbol{A})$. Since $\boldsymbol{\mathcal{F}} = \langle C_{\boldsymbol{\mathcal{F}}}(Z), N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})\rangle$ by Proposition 4.9 again, and similarly for $\mathcal{F}^*$, we have $\boldsymbol{\mathcal{F}} = \mathcal{F}^*$.

The 3-fusion system of Co_1 was shown to be simple by Aschbacher [A4, 16.10] (see also [OR2, Theorem A]). □

5. Todd modules for M_{10} and M_{11}

We now look at Todd modules for the Mathieu groups M_{11} and M_{10}. More generally, rather than looking only at M_{10}-representations, we work with representations of extensions of $O^{3'}(M_{10}) \cong A_6$. We want to determine all saturated fusion systems over finite 3-groups which involve these modules. Throughout the section, we refer to the following hypotheses.

Hypotheses 5.1. *Set $p = 3$. Let $\boldsymbol{\mathcal{F}}$ be a saturated fusion system over a finite 3-group $\boldsymbol{S}$, and let $\boldsymbol{A} \le \boldsymbol{S}$ be an elementary abelian subgroup such that $C_{\boldsymbol{S}}(\boldsymbol{A}) = \boldsymbol{A}$. Set $\boldsymbol{\Gamma} = \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ and $\boldsymbol{\Gamma_0} = O^{3'}(\boldsymbol{\Gamma})$, and assume that one of the following holds:*

(i) $\mathrm{rk}(\boldsymbol{A}) = 4$ *and* $\boldsymbol{\Gamma_0} \cong A_6$, *or*

(ii) $\mathrm{rk}(\boldsymbol{A}) = 5$ *and* $\boldsymbol{\Gamma_0} \cong M_{11}$.

We will see in Lemma 5.5 that $\boldsymbol{A}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ under these assumptions.

The irreducible $\mathbb{F}_3A_6$- and $\mathbb{F}_3M_{11}$-modules are, of course, very well known. In particular, there are only three modules that we need to consider.

Lemma 5.2. *There are exactly one isomorphism class of faithful 4-dimensional $\mathbb{F}_3A_6$-modules, and exactly two isomorphism classes of faithful 5-dimensional $\mathbb{F}_3M_{11}$-modules. All of these modules are absolutely irreducible.*

Proof. We refer for simplicity to [JLPW, p. [4]] for the table of characters of A_6 in characteristic 3: there are none of degree 2, two of degree 3 which are not realized as $\mathbb{F}_3A_6$-modules (since $GL_3(3)$ has order prime to 5), and one of degree 4 which is realized (as the natural module for A_6). This proves the claim for $\mathbb{F}_3A_6$-modules.

By [Ja, § 7A], there are exactly two isomorphism classes of irreducible 5-dimensional $\overline{\mathbb{F}}_3M_{11}$-modules, one the dual of the other. In both cases, these are the smallest degrees of nontrivial Brauer characters. It is well known that they can be realized as $\mathbb{F}_3M_{11}$-modules; we give one explicit construction in Lemma 3.23(b,c). □

Note: Of the two distinct 5-dimensional $\mathbb{F}_3M_{11}$-modules, what we call the "Todd module" is the one that has a set of eleven 1-dimensional subspaces permuted by M_{11}. That one of the modules has this form is clear by the construction in Notation 3.18.

As noted in the proof of Lemma 5.2, the 4-dimensional $\mathbb{F}_3A_6$-module is the natural module for A_6: a subquotient of the 6-dimensional permutation module. However, for our constructions here (e.g., when we want to extend it to an $\mathbb{F}_3\mathrm{Aut}(A_6)$-module), it will be easier to work with it as a quotient module of the Todd module for $2M_{12}$ described in Section 4.

5.1. **Preliminary results.**

The main goal in this subsection is to show that $\boldsymbol{\mathcal{F}} = \langle C_{\boldsymbol{\mathcal{F}}}(Z), N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})\rangle$ whenever Hypotheses 5.1 hold (Proposition 5.8). But we first describe more explicitly how the notation of Section 3.2 is used in the situation of Hypotheses 5.1. Recall that $\boldsymbol{T} \in \mathrm{Syl}_3(\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}})$ by Lemma 3.19.

Notation 5.3. *Assume Hypotheses 5.1 and Notation 3.18 as well as the notation in Lemma 3.23. Identify $\boldsymbol{\Gamma_0}$ with $\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}} = O^{3'}(\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}})$ for $\ell = 10$ or 11 in such a way that $\boldsymbol{T} = \mathrm{Aut}_{\boldsymbol{S}}(\boldsymbol{A})$, and identify $\boldsymbol{A}$ with $\boldsymbol{A}^{\boldsymbol{(\ell)}}$ or (in the M_{11}^*-case) with $\boldsymbol{A}^{\boldsymbol{(11)}^*}$. Thus $Z = Z(\boldsymbol{S}) = C_{\boldsymbol{A}}(\boldsymbol{T})$. Finally, set $A_* = [\boldsymbol{S}, \boldsymbol{A}] = [\boldsymbol{T}, \boldsymbol{A}]$.*

	A_6-case	M_{11}-case	M_{11}^*-case
$\boldsymbol{A}$	$\mathbb{F}_3 \times \mathbb{F}_9 \times \mathbb{F}_3$	$\mathbb{F}_3 \times \mathbb{F}_9 \times \mathbb{F}_9$	$\mathbb{F}_9 \times \mathbb{F}_9 \times \mathbb{F}_3$
$\big[s, \llbracket a,b,c \rrbracket\big]$	$\llbracket 0, -ax, \mathrm{Tr}(\bar{b}x) - aN(x) \rrbracket$	$\llbracket 0, -ax, bx+ax^2 \rrbracket$	$\llbracket 0, -ax, \mathrm{Tr}(bx+ax^2) \rrbracket$
$A_* = [\boldsymbol{T}, \boldsymbol{A}]$	$0 \times \mathbb{F}_9 \times \mathbb{F}_3$	$0 \times \mathbb{F}_9 \times \mathbb{F}_9$	$0 \times \mathbb{F}_9 \times \mathbb{F}_3$
$[s, \boldsymbol{A}]$	$\{\llbracket 0, ax, c \rrbracket \mid a, c \in \mathbb{F}_3\}$	$\{\llbracket 0, ax, c \rrbracket \mid a \in \mathbb{F}_3,\ c \in \mathbb{F}_9\}$	$0 \times \mathbb{F}_9 \times \mathbb{F}_3$
$C_{\boldsymbol{A}}(\boldsymbol{T}) = Z(\boldsymbol{S})$	$0 \times 0 \times \mathbb{F}_3$	$0 \times 0 \times \mathbb{F}_9$	$0 \times 0 \times \mathbb{F}_3$
$C_{\boldsymbol{A}}(s) = Z(\boldsymbol{A}\langle s \rangle)$	$\{\llbracket 0, b, c \rrbracket \mid \mathrm{Tr}(b\overline{x}) = 0\}$	$0 \times 0 \times \mathbb{F}_9$	$\{\llbracket 0, b, c \rrbracket \mid \mathrm{Tr}(bx) = 0\}$
Jd. bl. lth. of c_s	$3+1$	$3+2$	$3+2$

TABLE 5.4. In all cases, $s \in \boldsymbol{S} \smallsetminus \boldsymbol{A}$, and $x \in \mathbb{F}_9$ is such that $c_s = (\!(x)\!) \in \boldsymbol{T}$. The last line gives the Jordan block lengths for the action of s on $\boldsymbol{A}$.

For later reference, we collect in Table 5.4 some easy computations involving some of the subgroups of $\boldsymbol{A}$ and $\boldsymbol{\Gamma}$ defined above.

The next lemma gives a first easy consequence of the computations in Table 5.4.

Lemma 5.5. *Assume that $\boldsymbol{A} \le \boldsymbol{S}$ and $\boldsymbol{\mathcal{F}}$ satisfy Hypotheses 5.1. Then $\boldsymbol{A}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ and in particular is normal in $\boldsymbol{S}$.*

Proof. By Lemma 5.2, $\boldsymbol{A}$ is one of the $\mathbb{F}_3\boldsymbol{\Gamma_0}$-modules described in Lemma 3.23. From that lemma and Table 5.4, we see that in all of these cases, $N_{\boldsymbol{S}}(\boldsymbol{A})/\boldsymbol{A} \cong E_9$, $|C_{\boldsymbol{A}}(x)| = 9$ for each $x \in N_{\boldsymbol{S}}(\boldsymbol{A}) \smallsetminus \boldsymbol{A}$, and $|\boldsymbol{A} : C_{\boldsymbol{A}}(N_{\boldsymbol{S}}(\boldsymbol{A}))| \ge 3^3$. So $\boldsymbol{A}$ is the unique abelian subgroup of index 9 in $N_{\boldsymbol{S}}(\boldsymbol{A})$, and hence by Lemma 2.1 is weakly closed in $\boldsymbol{\mathcal{F}}$. □

The following properties will also be needed.

Lemma 5.6. *Assume Hypotheses 5.1 and Notation 5.3.*

(a) *In the A_6- and M_{11}-cases, for $x \in \boldsymbol{S} \smallsetminus \boldsymbol{A}$ and $a \in \boldsymbol{A}$, we have $(ax)^3 = x^3$ if and only if $a \in A_*$. In all cases, $x \in \boldsymbol{S} \smallsetminus \boldsymbol{A}$ and $a \in A_*$ implies $(ax)^3 = x^3$.*

(b) *In all cases, if $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$, then $[\boldsymbol{S}, \boldsymbol{S}] = A_*$.*

Proof. **(a)** By Lemma A.5, for $a \in \boldsymbol{A}$ and $x \in \boldsymbol{S} \smallsetminus \boldsymbol{A}$, $x^3 = (ax)^3$ if and only if $[x, [x, a]] = 1$; i.e., if $[x, a] \in C_{\boldsymbol{A}}(x)$. By Table 5.4, this holds if and only if $a \in A_*$ in the A_6- and M_{11}-cases, while $[x, A_*] = Z \le C_{\boldsymbol{A}}(x)$ in the M_{11}^*-case.

(b) Assume otherwise: assume $[\boldsymbol{S}, \boldsymbol{S}] > A_* = [\boldsymbol{S}, \boldsymbol{A}]$. Then since $\boldsymbol{S}/\boldsymbol{A} \cong E_9$ in all cases, $[\boldsymbol{S}, \boldsymbol{S}]$ contains A_* with index 3.

Assume we are in the M_{11}^*-case. Thus $|\boldsymbol{A}/A_*| = 9$ (Table 5.4), and hence $A_* < [\boldsymbol{S}, \boldsymbol{S}] < \boldsymbol{A}$. By Lemma 3.19, there is an element $-[i] \in N_{\boldsymbol{\Gamma_0}}(\boldsymbol{T})$, and this extends to $\alpha \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$ by the extension axiom. By the formulas in Lemma 3.23(c), no subgroup of index 3 in $\boldsymbol{A}$ and containing A_* is normalized by α. In particular, $\alpha([\boldsymbol{S}, \boldsymbol{S}]) \ne [\boldsymbol{S}, \boldsymbol{S}]$, which is impossible.

Now assume we are in the A_6- or M_{11}-case. Then $|\boldsymbol{A}/A_*| = 3$ by Table 5.4 again, so $[\boldsymbol{S}, \boldsymbol{S}] = \boldsymbol{A}$, and $\boldsymbol{S}/A_*$ is nonabelian of order 27. Let $x \in \boldsymbol{S} \smallsetminus \boldsymbol{A}$ and $y \in \boldsymbol{S} \smallsetminus \boldsymbol{A}\langle x \rangle$ be arbitrary. Then $\boldsymbol{S} = \boldsymbol{A}\langle x, y \rangle$ and $[x, y] \in \boldsymbol{A} \smallsetminus A_*$. So $x^3 \ne ({}^y x)^3 = {}^y(x^3)$ by (a). In particular, $x^3 \ne 1$, and since x was arbitrary, no element of $\boldsymbol{S} \smallsetminus \boldsymbol{A}$ has order 3.

Assume $R \in \mathbf{E}_{\boldsymbol{\mathcal{F}}}$. Then $\boldsymbol{A} \cap R = \Omega_1(R)$ is characteristic in R. For each $a \in N_{\boldsymbol{A}}(R) \smallsetminus R$, we have $[a, R] \le R \cap \boldsymbol{A}$ and $[a, R \cap \boldsymbol{A}] = 1$, contradicting Lemma B.9. Thus $N_{\boldsymbol{A}}(R) \le R$, so

$N_{\boldsymbol{A}R}(R) = R$, and hence $\boldsymbol{A} \le R$. Thus each $\boldsymbol{\mathcal{F}}$-essential subgroup contains $\boldsymbol{A}$, contradicting the assumption that $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$. □

In Notation 5.3, we identified $O^{3'}(\boldsymbol{\Gamma}) = O^{3'}(\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}})$ (for $\ell = 10$ or 11). In fact, this extends to an inclusion $\boldsymbol{\Gamma} \le \widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$.

Lemma 5.7. *Assume Hypotheses 5.1 and Notation 5.3. Then for $\ell = 10, 11$, $\boldsymbol{N}^{(\boldsymbol{\ell})} = N_{\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}}(\boldsymbol{T})$ and is a maximal subgroup of $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$. Also, as subgroups of $\mathrm{Aut}(\boldsymbol{A})$, we have*

- $\widehat{\boldsymbol{M}}_{\boldsymbol{10}} = N_{\mathrm{Aut}(\boldsymbol{A})}(\boldsymbol{\Gamma}_{\boldsymbol{0}}) \ge \boldsymbol{\Gamma}$ *if* $\boldsymbol{\Gamma}_{\boldsymbol{0}} = \widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{10}} \cong A_6$, *and*
- $\widehat{\boldsymbol{M}}_{\boldsymbol{11}} = N_{\mathrm{Aut}(\boldsymbol{A})}(\boldsymbol{\Gamma}_{\boldsymbol{0}}) \ge \boldsymbol{\Gamma}$ *if* $\boldsymbol{\Gamma}_{\boldsymbol{0}} = \widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{11}} \cong M_{11}$.

Proof. For $\ell = 10, 11$,

$$\boldsymbol{N}^{(\boldsymbol{\ell})} = \boldsymbol{N} \cap \widehat{\boldsymbol{M}}_{\boldsymbol{\ell}} = N_{\widehat{\boldsymbol{M}}_{\boldsymbol{12}}}(\boldsymbol{T}) \cap \widehat{\boldsymbol{M}}_{\boldsymbol{\ell}} = N_{\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}}(\boldsymbol{T}),$$

where the second equality holds by Lemma 3.7. The maximality of $\boldsymbol{N}^{(\boldsymbol{\ell})}$ in $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$ is well known in both cases, but we note the following very simple argument. If $\boldsymbol{N}^{(\boldsymbol{\ell})}$ is not maximal in $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$, then since it has index 10 or 55 when $\ell = 10$ or 11, respectively, there is $\boldsymbol{N}^{(\boldsymbol{\ell})} < H < \widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$ where $[H : \boldsymbol{N}^{(\boldsymbol{\ell})}] = n$ for $n \in \{2, 5, 11\}$. But then H has exactly n Sylow 3-subgroups where $n \equiv 2 \pmod 3$, contradicting the Sylow theorems.

Now let $\ell \in \{10, 11\}$ be such that $\boldsymbol{\Gamma}_{\boldsymbol{0}} = \widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}}$. Since $\boldsymbol{A}$ is absolutely irreducible as an $\mathbb{F}_3\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}}$-module by Lemma 5.2, we have $C_{\mathrm{Aut}(\boldsymbol{A})}(\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}}) = \{\pm\mathrm{Id}\}$, and hence

$$|\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}/\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}}| \le |N_{\mathrm{Aut}(\boldsymbol{A})}(\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}})/\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}}| \le 2 \cdot |\mathrm{Out}(\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}})|.$$

These inequalities are equalities by Table 3.20 (and since $|\mathrm{Out}(A_6)| = 4$ and $|\mathrm{Out}(M_{11})| = 1$), so $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}} = N_{\mathrm{Aut}(\boldsymbol{A})}(\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{\ell}}) \ge \boldsymbol{\Gamma}$. □

We can now begin to apply some of the lemmas in Section 2.

Proposition 5.8. *Assume Hypotheses 5.1 and Notation 5.3. Then $\boldsymbol{\mathcal{F}} = \langle C_{\boldsymbol{\mathcal{F}}}(Z), N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})\rangle$.*

Proof. Assume otherwise, and recall that $\boldsymbol{A} \trianglelefteq \boldsymbol{S}$ by Lemma 5.5. By Proposition 2.3, there are subgroups $X \in Z^{\boldsymbol{\mathcal{F}}}$ and $R \in \mathbf{E}_{\boldsymbol{\mathcal{F}}}$ such that $X \nleq \boldsymbol{A}$, $R = C_{\boldsymbol{S}}(X) = N_{\boldsymbol{S}}(X)$, and $Z = \alpha(X)$ for some $\alpha \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R)$. Set $R_0 = R \cap \boldsymbol{A}$.

Fix $x \in X \smallsetminus \boldsymbol{A}$. Then $|x| = 3$, since $x \in X \cong Z$ and $Z \le \boldsymbol{A}$ has exponent 3. Also, $R_0 = C_{\boldsymbol{A}}(X) = C_{\boldsymbol{A}}(x)$: since either $|X| = |Z| = 3$ and hence $X = \langle x\rangle$, or else we are in the M_{11}-case and $C_{\boldsymbol{A}}(x) = Z = C_{\boldsymbol{A}}(\boldsymbol{S})$. Since x acts on $\boldsymbol{A}$ in all cases with two Jordan blocks (Table 5.4), we have $|R_0| = |C_{\boldsymbol{A}}(x)| = 9$.

Case 1: Assume first that $|R\boldsymbol{A}/\boldsymbol{A}| = 3$. Then $R = R_0\langle x\rangle$, and hence $|R| = 27$.

If we are in the A_6-case, then each member of the $\boldsymbol{S}$-conjugacy class of R has the form $C_{\boldsymbol{A}}(y)\langle y\rangle = R_0\langle y\rangle$ for some $y \in x\boldsymbol{A}$, and $y \in xA_*$ by Lemma 5.6(a) and since $y^3 = 1 = x^3$. Since $C_{\boldsymbol{A}}(x)$ has index 3 in A_*, there are at most three such subgroups, so $|N_{\boldsymbol{S}}(R)/R| \ge \frac{1}{3}[S : R] = 9$, contradicting Lemma B.6(b).

In the M_{11}- and M_{11}^*-cases, $|N_{R\boldsymbol{A}}(R)/R| = |C_{\boldsymbol{A}/R_0}(x)| = 9$, since x acts on $\boldsymbol{A}$ with Jordan blocks of length 3 and 2 (Table 5.4). Thus $|\mathrm{Out}_{\boldsymbol{A}}(R)| = 9$. Since $\mathrm{Out}_{\boldsymbol{A}}(R)$ acts trivially on R_0, and $|R/R_0| = 3$, this contradicts Lemma B.7.

Case 2: Now assume that $|R\boldsymbol{A}/\boldsymbol{A}| \cong E_9$. Thus $R\boldsymbol{A} = \boldsymbol{S}$ and $|R| = 81$.

Assume first we are in the A_6- or M_{11}^*-case. Then $|Z|=3$ and $Z=C_{\boldsymbol{A}}(R)<C_{\boldsymbol{A}}(x)$. So there are $y\in R\smallsetminus \boldsymbol{A}\langle x\rangle$ and $a\in C_{\boldsymbol{A}}(x)\smallsetminus C_{\boldsymbol{A}}(R)$ such that $1\ne[y,a]\in Z$, and hence $Z\le[R,C_{\boldsymbol{A}}(x)]\le[R,R]$. Since $X\nleq[R,R]$, no automorphism of R sends X to Z.

Now assume we are in the M_{11}-case. Then $R_0=Z$ and $N_{\boldsymbol{S}}(R)=RA_*$, so $|N_{\boldsymbol{S}}(R)/R|=|A_*/Z|=9$, and hence $R\cong E_{81}$ by Lemma B.6(b). Each element of order 3 in $\mathrm{Aut}_{\boldsymbol{S}}(R)$ acts on R with Jordan blocks of length at most 2, so by Proposition B.10, $O^{3'}(\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R))\cong SL_2(9)$ with the natural action on R. Also, each element of order 8 in $N_{O^{3'}(\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(R))}(\mathrm{Aut}_{\boldsymbol{S}}(R))$ restricts to an element $\alpha\in\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(Z)$ of order 8 (note that $Z=[N_{\boldsymbol{S}}(R),R]$), and this in turn extends to some $\beta\in\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$ and hence to $\beta|_{\boldsymbol{A}}\in\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ since $\boldsymbol{A}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ by Lemma 5.5. But $\widehat{\boldsymbol{M}}{}_{\boldsymbol{11}}^{\boldsymbol{0}}\le\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})\le\widehat{\boldsymbol{M}}_{\boldsymbol{11}}\cong M_{11}\times C_2$ by Lemma 5.7, so $\mathbb{F}_9^\times\langle\phi\rangle$ or its product with $\{\pm\mathrm{Id}\}$ is a Sylow 2-subgroup of $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$, and by Lemma 3.23(b), the subgroups of order 8 in these groups do not act faithfully on Z. So this case is impossible. □

5.2. The subgroup $\boldsymbol{Q}\trianglelefteq C_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Z})$.

So far, we have shown that $\boldsymbol{\mathcal{F}}=\langle N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}),C_{\boldsymbol{\mathcal{F}}}(Z)\rangle$ in all cases where Hypotheses 5.1 hold. Our next step in studying these fusion systems is to prove that $C_{\boldsymbol{\mathcal{F}}}(Z)$ is constrained by constructing a normal centric subgroup $\boldsymbol{Q}\trianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)$; and proving (as one consequence) that $\boldsymbol{S}$ splits over $\boldsymbol{A}$.

Proposition 5.9. *Assume Hypotheses 5.1 where $\boldsymbol{A}\ntrianglelefteq\boldsymbol{\mathcal{F}}$. Then there is a unique special subgroup $\boldsymbol{Q}\trianglelefteq\boldsymbol{S}$ of exponent 3 such that $Z(\boldsymbol{Q})=Z$, $\boldsymbol{Q}\cap\boldsymbol{A}=A_*$, and $\boldsymbol{Q}/Z\cong E_{81}$, and $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}=\{\boldsymbol{Q}\}$. In particular, $\boldsymbol{Q}\trianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)$, and $\boldsymbol{Q}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{F}}$-centric.*

Proof. Assume Notation 5.3. Define

$$\mathcal{Q}=\{Q\le\boldsymbol{S}\,|\,Q\cap\boldsymbol{A}=A_*,\ Q/Z\text{ abelian of order }3^4\}$$
$$\mathcal{Q}_0=\{Q\in\mathcal{Q}\,|\,Q\text{ of exponent }3\}.$$

Recall that $[\boldsymbol{S},\boldsymbol{S}]=A_*$ by Lemma 5.6(b). Also, $\boldsymbol{S}/A_*$ is elementary abelian by Lemma A.1(a), applied to the group $\boldsymbol{S}/Z$ with center A_*/Z.

We will prove that

$$\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}\subseteq\mathcal{Q}_0\quad\text{and}\quad|\mathcal{Q}_0|\le1.\tag{5.10}$$

Since $\boldsymbol{\mathcal{F}}=\langle C_{\boldsymbol{\mathcal{F}}}(Z),N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})\rangle$ by Proposition 5.8, and since $\boldsymbol{\mathcal{F}}\ne N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ (recall $\boldsymbol{A}\ntrianglelefteq\boldsymbol{\mathcal{F}}$ by assumption), we have $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}\ne\varnothing$. So (5.10) implies that $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}=\mathcal{Q}_0$ has order 1, and for $Q\in\mathcal{Q}_0$, $Q\trianglelefteq C_{\boldsymbol{\mathcal{F}}}(Z)$ and Q is weakly closed in $\mathcal{F}$. By construction, $C_{\boldsymbol{S}}(Q)=C_{\boldsymbol{S}}(\boldsymbol{T})=Z$, so Q is also $\mathcal{F}$-centric.

It thus remains to prove (5.10). Set $\overline{\boldsymbol{S}}=\boldsymbol{S}/Z$ and similarly for subgroups and elements of $\boldsymbol{S}$. In all cases, $Z(\overline{\boldsymbol{S}})=\overline{A}_*\cong E_9$.

Let $\rho\colon Q/A_*\longrightarrow Z$ be the homomorphism of Lemma A.1(b) that sends gA_* to g^3. (Note that ρ is defined on $\overline{Q}=Q/Z$ in the lemma, but factors through Q/A_* since A_* is elementary abelian.)

$\boldsymbol{A_6}$- and $\boldsymbol{M_{11}}$-cases: Here, $|\boldsymbol{A}/A_*|=3$, so $|\mathcal{Q}_0|\le|\mathcal{Q}|=1$ by Lemma A.1(c), applied with $\overline{\boldsymbol{S}}$ and $\overline{A_*}$ in the role of S and Z. Let $\boldsymbol{Q}\in\mathcal{Q}$ be the unique element. Then $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)/Z}\subseteq\{\overline{\boldsymbol{Q}}\}$ by [O1, Lemma 2.3(a)] and since $\overline{\boldsymbol{Q}}$ is the unique abelian subgroup of index 3 in $\overline{\boldsymbol{S}}$, and so $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}\subseteq\{\boldsymbol{Q}\}$ by Lemma 1.19.

Since $\boldsymbol{Q}$ is the only member of $\mathcal{Q}$, it is normalized by $\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$. By Table 3.20, the element

$$\beta_0=\begin{cases}-[i]\in\boldsymbol{N}^{(\boldsymbol{10})}\cap\widehat{\boldsymbol{M}}{}_{\boldsymbol{10}}^{\boldsymbol{0}}\le\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) & \text{in the }A_6\text{-case}\\ -[\zeta]\in\boldsymbol{N}^{(\boldsymbol{11})}\cap\widehat{\boldsymbol{M}}{}_{\boldsymbol{11}}^{\boldsymbol{0}}\le\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}) & \text{in the }M_{11}\text{-case}\end{cases}$$

normalizes $\mathrm{Aut}_{\boldsymbol{S}}(\boldsymbol{A})$, and hence extends to some $\beta \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$. Also, by construction of $\boldsymbol{N}^{(10)} = \boldsymbol{N}^{(11)}$, β permutes the cosets gA_* for $g \in \boldsymbol{Q} \smallsetminus A_*$ — in two orbits of length 4 in the A_6-case, or one orbit of length 8 in the M_{11}-case — and ρ is constant on each of these orbits.

In the A_6-case, where $|Z| = 3$, this implies that $\rho = 1$ and hence $\boldsymbol{Q} \in \mathcal{Q}_0$. In the M_{11}-case, where $|Z| = 9$, it implies that either $\boldsymbol{Q} \in \mathcal{Q}_0$, or all elements of $\boldsymbol{Q} \smallsetminus A_*$ have order 9 and hence A_* is characteristic in $\boldsymbol{Q}$. But in that case, $\boldsymbol{Q} \notin \mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}$ by Lemma B.9, since for $a \in \boldsymbol{A} \smallsetminus A_*$, we have $[a, \boldsymbol{Q}] \leq A_*$ and $[a, A_*] = 1$. We conclude that $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)} \subseteq \mathcal{Q}_0$ in either case, finishing the proof of (5.10).

$\boldsymbol{M_{11}^*}$-case: Now, $|\boldsymbol{A}/A_*| = 9$. Assume $R \in \mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}$. Then $R \geq Z$ and $\overline{R} \in \mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)/Z}$ by Lemma 1.19, and hence $\overline{R} \geq Z(\overline{\boldsymbol{S}}) = \overline{A_*}$. If $\overline{R}$ is not abelian, then $Z(\overline{R}) = \overline{A_*}$, so $\overline{A_*}$ is characteristic in $\overline{R}$, contradicting Lemma B.9 since $[x, \overline{R}] \leq \overline{A_*}$ and $[x, \overline{A_*}] = 1$ for each $x \in \overline{\boldsymbol{S}} \smallsetminus \overline{R}$. Thus $\overline{R}$ is abelian, and is maximal abelian since it is $\mathcal{F}/Z$-centric. So $R \in \mathcal{Q} \cup \{\boldsymbol{A}\}$ by Lemma A.1(d), and $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)} \subseteq \mathcal{Q} \cup \{\boldsymbol{A}\}$.

Since $\boldsymbol{N}^{(11)} \cong (E_9 \rtimes SD_{16}) \times C_2$ is a maximal subgroup of $\widehat{\boldsymbol{M}}_{\boldsymbol{11}}$ by Lemma 5.7 and normalizes Z by Lemma 3.23(c), we see that $\mathrm{Aut}_{N_{\boldsymbol{\mathcal{F}}}(Z)}(\boldsymbol{A}) = C_{\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(Z)$ has index 2 in $\boldsymbol{N}^{(11)}$ and hence contains $\boldsymbol{T}$ as a normal subgroup. So $\boldsymbol{A} \notin \mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}$, and $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)} \subseteq \mathcal{Q}$.

Assume R is not of exponent 3, and set $R_0 = \Omega_1(R)$. Then R_0 has index 3 in R by Lemma A.1(b), and so $R_0/Z(R_0) \cong E_9$ where $Z(R_0) \leq A_*$. Since $|\boldsymbol{A}/A_*| = 9$ and $9 \nmid |\mathrm{Aut}(R_0/Z(R_0))|$, there is $x \in \boldsymbol{A} \smallsetminus A_*$ such that $[x, R_0] \leq Z(R_0)$. Also, $[x, R] \leq A_* \leq R_0$ and $[x, Z(R_0)] = 1$, and by Lemma B.9, this contradicts the assumption that $R \in \mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}$. Thus $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)} \subseteq \mathcal{Q}_0$.

It remains to show that $|\mathcal{Q}_0| \leq 1$. Assume otherwise: assume Q_1 and Q_2 are both in $\mathcal{Q}_0$. Define $\psi\colon \boldsymbol{S}/\boldsymbol{A} \longrightarrow \boldsymbol{A}/A_*$ by setting, for each $g\boldsymbol{A} \in \boldsymbol{S}/\boldsymbol{A}$, $\psi(g\boldsymbol{A}) = (g\boldsymbol{A} \cap Q_1)^{-1}(g\boldsymbol{A} \cap Q_2) \in \boldsymbol{A}/A_*$. (Note that $g\boldsymbol{A} \cap Q_i \in \boldsymbol{S}/A_*$ for $i = 1, 2$.) Since $(g_1)^3 = 1 = (g_2)^3$ for $g_i \in g\boldsymbol{A} \cap Q_i$, and $g_2 \in g_1\psi(g\boldsymbol{A})$, we have $[g, [g, \psi(g\boldsymbol{A})]] = 1$ by Lemma A.5. Using the formulas in Lemma 3.23(c), we identify $\boldsymbol{S}/\boldsymbol{A}$ and $\boldsymbol{A}/A_*$ with $\mathbb{F}_9$, and through that identify ψ with an additive homomorphism $\widehat{\psi}\colon \mathbb{F}_9 \longrightarrow \mathbb{F}_9$ such that

$$0 = [\,(\!(x)\!), [\,(\!(x)\!), [\![\widehat{\psi}(x), 0, 0]\!]\,]\,] = [\![0, 0, \mathrm{Tr}(x^2\widehat{\psi}(x)]\!]$$

for each $x \in \mathbb{F}_9$. Thus $x^2\widehat{\psi}(x) \in i\mathbb{F}_3$, and

$$\widehat{\psi}(x) \in \begin{cases} i\mathbb{F}_3 & \text{if } x = \pm 1, \pm i \\ \mathbb{F}_3 & \text{if } x = \pm\zeta, \pm\zeta^3. \end{cases}$$

Hence $\widehat{\psi}$ is not onto, and either $\widehat{\psi}(1) = \widehat{\psi}(i) = 0$ or $\widehat{\psi}(\zeta) = \widehat{\psi}(\zeta^3) = 0$. This proves that $\widehat{\psi} = 0$ and hence $Q_1 = Q_2$, and finishes the proof of (5.10). ☐

We list some of the properties of these subgroups $\boldsymbol{Q} \trianglelefteq \boldsymbol{S}$ in the Table 5.11 for easy reference. They follow immediately from the descriptions in Lemma 3.23 and Proposition 5.9.

One easy consequence of Proposition 5.9 is that $\boldsymbol{S} \cong \boldsymbol{A} \rtimes \boldsymbol{T}$.

Corollary 5.12. *Assume Hypotheses 5.1 where $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$, and let M be a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ (see Proposition 1.12). Then $\boldsymbol{S}$ and M split over $\boldsymbol{A}$.*

Proof. Let $\boldsymbol{Q} \trianglelefteq \boldsymbol{S}$ be the special subgroup of exponent 3 of Proposition 5.9. To prove that $\boldsymbol{S}$ splits over $\boldsymbol{A}$, it suffices to show that $\boldsymbol{Q}$ splits over $\boldsymbol{Q} \cap \boldsymbol{A} = A_*$. If $|Z| = 9$ (i.e., in the

	$\boldsymbol{\varGamma_0} \cong$	$\operatorname{rk}(\boldsymbol{A})$	$\operatorname{rk}(Z)$	$\lvert\boldsymbol{S}\rvert$	$\boldsymbol{Q} \cong$	$\lvert\operatorname{Out}_{\boldsymbol{S}}(\boldsymbol{Q})\rvert$
A_6-case	A_6	4	1	3^6	3^{1+4}_+	3
M_{11}-case	M_{11}	5	2	3^7	3^{2+4}	3
M_{11}^*-case	M_{11}	5	1	3^7	3^{1+4}_+	9

TABLE 5.11.

M_{11}-case), then we are in the situation of Lemma A.1(d), so there is $B \le \boldsymbol{Q}$ abelian of index 9 such that $B \cap A_* = Z$, and any complement in B to Z is a splitting of $\boldsymbol{Q}$ over A_*.

If $|Z| = 3$, then consider the space $\overline{\boldsymbol{Q}} = \boldsymbol{Q}/Z$, with symplectic form $\mathfrak{b}$ defined by $\mathfrak{b}(xZ, yZ) = \sigma([x, y])$ for some $\sigma\colon Z \xrightarrow{\cong} \mathbb{F}_3$. Following the standard procedure for constructing a symplectic basis for $\overline{\boldsymbol{Q}}$, we fix a basis $\{a_1, a_2\}$ for A_*/Z, choose $b_1 \in \overline{\boldsymbol{Q}} \smallsetminus a_1^\perp$, and choose $b_2 \in \langle a_1, b_1\rangle^\perp \smallsetminus \langle a_2\rangle$. Then $\{a_1, b_1, a_2, b_2\}$ is a basis for $\overline{\boldsymbol{Q}}$, and $\langle b_1, b_2\rangle \le \overline{\boldsymbol{Q}}$ is totally isotropic and lifts to a splitting of $\boldsymbol{Q}$ over A_*.

Since $\boldsymbol{S}$ splits over $\boldsymbol{A}$, it follows from Gaschütz's theorem (see [A1, (10.4)]) that M also splits over $\boldsymbol{A}$. $\square$

Recall that for $\ell = 10, 11$, we set $\boldsymbol{T} = O_3(\boldsymbol{N^{(\ell)}}) \cong E_9$, a Sylow 3-subgroup of $\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$, and set $\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}} = O^{3'}(\widehat{\boldsymbol{M}}_{\boldsymbol{\ell}})$. Also, $\boldsymbol{\varGamma}$ was chosen so that $\boldsymbol{\varGamma_0} = \widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}}$ (see Notation 5.3), and then $\boldsymbol{\varGamma} \le \widehat{\boldsymbol{M}}_{\boldsymbol{\ell}}$ by Lemma 5.7.

Notation 5.13. *Assume Hypotheses 5.1 and Notation 3.18 and 5.3. Let M be a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$, and set $M_0 = O^{3'}(M)$. Then M splits over $\boldsymbol{A}$ by Corollary 5.12, and we identify*

$$M = \boldsymbol{A} \rtimes \boldsymbol{\varGamma} \le \boldsymbol{A} \rtimes \widehat{\boldsymbol{M}}_{\boldsymbol{\ell}} \qquad \text{and} \qquad M_0 = \boldsymbol{A} \rtimes \boldsymbol{\varGamma_0} = \boldsymbol{A} \rtimes \widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{\ell}},$$

where $\ell = 10$ if $\boldsymbol{\varGamma_0} \cong A_6$ and $\ell = 11$ if $\boldsymbol{\varGamma_0} \cong M_{11}$. Thus $\boldsymbol{S} = \boldsymbol{A} \rtimes \boldsymbol{T} \in \operatorname{Syl}_3(M)$ and $\boldsymbol{Q} = A_* \rtimes \boldsymbol{T} \le \boldsymbol{S}$.

One easily sees that $\boldsymbol{Q}$ is special with $Z(\boldsymbol{Q}) = Z$ and $\boldsymbol{Q}/Z \cong E_{81}$. Also, $\boldsymbol{Q}$ has exponent 3 by Lemma 5.6(a), and hence is the subgroup described in Proposition 5.9. In particular, $\boldsymbol{Q} \trianglelefteq C_{\mathcal{F}}(Z)$, and $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)} = \{\boldsymbol{Q}\}$.

Recall Notation 3.18 and Lemma 3.19: $\boldsymbol{T} = \{\langle\!\langle x\rangle\!\rangle \mid x \in \mathbb{F}_9\}$, and

$$\boldsymbol{N^{(10)}} = \boldsymbol{N^{(11)}} = \bigl\langle \langle\!\langle x\rangle\!\rangle, [u], [\phi], -\mathrm{Id} \bigm| x \in \mathbb{F}_9,\ u \in \mathbb{F}_9^\times \bigr\rangle \cong (E_9 \rtimes SD_{16}) \times \{\pm\mathrm{Id}\}.$$

Lemma 5.14. *Assume Hypotheses 5.1 and Notation 5.13, and also that $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$. Then conditions* (i)–(iii) *in Hypotheses 2.12 hold for $\boldsymbol{\mathcal{F}}$, $\boldsymbol{S}$, $\boldsymbol{A}$, and $\boldsymbol{Q}$.*

Proof. Since M is a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$, we have $\boldsymbol{S} \in \operatorname{Syl}_3(M)$ and $M/\boldsymbol{A} \cong \boldsymbol{\varGamma} = \operatorname{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$. Each pair of distinct Sylow 3-subgroups of $\boldsymbol{\varGamma_0} = O^{3'}(\boldsymbol{\varGamma}) \cong A_6$ or M_{11} intersects trivially. Hence for each subgroup R such that $\boldsymbol{A} < R < \boldsymbol{S}$, $\boldsymbol{S}$ is the unique Sylow 3-subgroup of M that contains R. So $1 \ne \operatorname{Out}_{\boldsymbol{S}}(R) \trianglelefteq \operatorname{Out}_M(R) = \operatorname{Out}_{\boldsymbol{\mathcal{F}}}(R)$, and hence $\operatorname{Out}_{\boldsymbol{\mathcal{F}}}(R) = \operatorname{Out}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})}(R)$ does not have a strongly 3-embedded subgroup. Thus no such R can be $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$-essential, proving that $\mathbf{E}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})} \subseteq \{\boldsymbol{A}\}$.

By Proposition 5.8, $\boldsymbol{\mathcal{F}} = \bigl\langle N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}), C_{\boldsymbol{\mathcal{F}}}(Z)\bigr\rangle$. Hence $\mathbf{E}_{\boldsymbol{\mathcal{F}}} \subseteq \mathbf{E}_{N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})} \cup \mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)}$ by Proposition 1.6, while $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)} \subseteq \{\boldsymbol{Q}\}$ by Proposition 5.9. So $\mathbf{E}_{\boldsymbol{\mathcal{F}}} \subseteq \{\boldsymbol{A}, \boldsymbol{Q}\}$. Also, $\boldsymbol{A} \in \mathbf{E}_{\boldsymbol{\mathcal{F}}}$ by Lemma B.1 and since $\boldsymbol{\varGamma_0} = O^{3'}(\operatorname{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})) \cong A_6$ or M_{11} and hence has a strongly embedded subgroup, and $\boldsymbol{Q} \in \mathbf{E}_{\boldsymbol{\mathcal{F}}}$ since otherwise $\boldsymbol{A}$ would be normal in $\boldsymbol{\mathcal{F}}$. Thus $\mathbf{E}_{\boldsymbol{\mathcal{F}}} = \{\boldsymbol{A}, \boldsymbol{Q}\}$, proving 2.12(i).

Recall that $\boldsymbol{Q} = A_*\boldsymbol{T}$. So $\boldsymbol{S} = \boldsymbol{A}\boldsymbol{Q}$, and $C_{\boldsymbol{S}}(\boldsymbol{Q}\cap\boldsymbol{A}) = C_{\boldsymbol{S}}(A_*) = \boldsymbol{A}$ by the relations in Lemma 3.23. This proves 2.12(ii).

By Lemma 5.2, $\boldsymbol{A}$ is absolutely irreducible as an $\mathbb{F}_3\boldsymbol{\Gamma_0}$-module, where $\boldsymbol{\Gamma_0} = O^{3'}(\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}))$ as earlier. Thus the centralizer in $\mathrm{Aut}(\boldsymbol{A})$ of $\boldsymbol{\Gamma_0}$ is $\{\pm\mathrm{Id}\}$. Since $\mathrm{Out}(A_6)$ and $\mathrm{Out}(M_{11})$ are 2-groups, $N_{\mathrm{Aut}(\boldsymbol{A})}(O^{p'}(\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})))/O^{p'}(\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}))$ is also a 2-group, and so 2.12(iii) holds. □

The following notation for elements in $\boldsymbol{Q}$ will be useful.

Notation 5.15. *For $a,b\in\mathbb{F}_9$, and $z\in\mathbb{F}_9$ (in the M_{11}-case) or $z\in\mathbb{F}_3$ (in the A_6- or M_{11}^*-case), set*

$$\langle\!\langle a,b,z\rangle\!\rangle = [\![0,a,z]\!]\,(\!(b)\!) \in A_*\boldsymbol{T} = \boldsymbol{Q}.$$

Thus each element of $\boldsymbol{Q}$ is represented by a unique triple $\langle\!\langle a,b,z\rangle\!\rangle$, for $a,b\in\mathbb{F}_9$ and $z\in\mathbb{F}_3$ or $\mathbb{F}_9$. We sometimes write $\langle\!\langle a,b,*\rangle\!\rangle \in \boldsymbol{Q}/Z$ to denote the class of $\langle\!\langle a,b,z\rangle\!\rangle$ for arbitrary z.

We list in Table 5.16 some of the relations among such triples: all of these are immediate consequences of the definition in Notation 5.15 and the relations in Lemma 3.23.

	A_6-case	M_{11}-case	M_{11}^*-case
	$\langle\!\langle a,b,z\rangle\!\rangle\cdot\langle\!\langle c,d,y\rangle\!\rangle = \langle\!\langle a+c,b+d,z+y+\mu(b,c)\rangle\!\rangle$ where		
	$\mu(b,c)=\mathrm{Tr}(\bar{b}c)$	$\mu(b,c)=bc$	$\mu(b,c)=-\mathrm{Tr}(bc)$
${}^{[\![r,0,0]\!]}\langle\!\langle a,b,z\rangle\!\rangle$	$\langle\!\langle a-br,b,z+rN(b)\rangle\!\rangle$	$\langle\!\langle a-br,b,z+rb^2\rangle\!\rangle$	$\langle\!\langle a+br,b,z+\mathrm{Tr}(rb^2)\rangle\!\rangle$
${}^{[u]}\langle\!\langle a,b,z\rangle\!\rangle$	$\langle\!\langle ua,ub,N(u)z\rangle\!\rangle$	$\langle\!\langle ua,ub,u^2z\rangle\!\rangle$	$\langle\!\langle u^{-1}a,ub,z\rangle\!\rangle$
${}^{[\phi]}\langle\!\langle a,b,z\rangle\!\rangle$	$\langle\!\langle\bar{a},\bar{b},z\rangle\!\rangle$	$\langle\!\langle\bar{a},\bar{b},\bar{z}\rangle\!\rangle$	$\langle\!\langle\bar{a},\bar{b},z\rangle\!\rangle$
${}^{-\mathrm{Id}}\langle\!\langle a,b,z\rangle\!\rangle$	$\langle\!\langle -a,b,-z\rangle\!\rangle$	$\langle\!\langle -a,b,-z\rangle\!\rangle$	$\langle\!\langle -a,b,-z\rangle\!\rangle$

TABLE 5.16. Here, $a,b,c,d\in\mathbb{F}_9$ and $u\in\mathbb{F}_9^\times$ in all cases, $z,y\in\mathbb{F}_3$ in the A_6- and M_{11}^*-cases, and $z,y\in\mathbb{F}_9$ in the M_{11}-case. Also, $r\in\mathbb{F}_3$ in the A_6- and M_{11}-cases, and $r\in\mathbb{F}_9$ in the M_{11}^*-case.

The next two lemmas give more information about $\mathrm{Out}(\boldsymbol{Q})$ and $\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$. We start with the case where $\boldsymbol{\Gamma_0}\cong A_6$.

Lemma 5.17. *Assume Hypotheses 5.1, and Notation 5.3 and 5.13, with $\boldsymbol{\Gamma_0}\cong A_6$. Thus $M_0 = \boldsymbol{A}\rtimes\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{10}} \cong E_{81}\rtimes A_6$. Then each $\alpha\in N_{\mathrm{Aut}(\boldsymbol{Q})}(\mathrm{Aut}_{\boldsymbol{S}}(\boldsymbol{Q}))$ extends to some $\overline{\alpha}\in\mathrm{Aut}(M_0)$.*

Proof. Since $\boldsymbol{N^{(10)}} = N_{\widehat{\boldsymbol{M}}_{\boldsymbol{10}}}(\boldsymbol{T})$ by Lemma 5.7, we have

$$N_{M_0}(\boldsymbol{S}) = \boldsymbol{A}\rtimes(\boldsymbol{N^{(10)}}\cap\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{10}}) = \boldsymbol{S}\langle\beta\rangle \quad\text{where}\quad \beta = -[i]\in\boldsymbol{N^{(10)}}, \tag{5.18}$$

by Lemma 3.19, and β acts on $\boldsymbol{S}$ via

$${}^{\beta}\big([\![a,b,c]\!]\,(\!(x)\!)\big) = [\![-a,-ib,-c]\!]\,(\!(ix)\!)\,. \tag{5.19}$$

For calculations in $\mathrm{Out}(\boldsymbol{Q})$, we use Notation 5.15, and the ordered basis

$$\mathcal{B} = \big\{\langle\!\langle 1,0,*\rangle\!\rangle, \langle\!\langle i,0,*\rangle\!\rangle, \langle\!\langle 0,1,*\rangle\!\rangle, \langle\!\langle 0,i,*\rangle\!\rangle\big\}$$

for $\boldsymbol{Q}/Z$. With respect to $\mathcal{B}$, the symplectic form $\mathfrak{b}$ defined by commutators has matrix $\pm\left(\begin{smallmatrix}0&I\\-I&0\end{smallmatrix}\right)$, and conjugation by $[\![1,0,0]\!]$ (a generator of $\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})$) has matrix $\left(\begin{smallmatrix}I&-I\\0&I\end{smallmatrix}\right)$ by Table 5.16.

We identify $\mathrm{Out}(\boldsymbol{Q})$ with $\mathrm{Aut}(\boldsymbol{Q}/Z,\pm\mathfrak{b})$: the group of automorphisms of $\boldsymbol{Q}/Z$ that preserve $\mathfrak{b}$ up to sign. We have

$$N_{\mathrm{Aut}(\boldsymbol{Q}/Z)}(\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})) = N_{GL_4(3)}\bigl(\langle\left(\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right)\rangle\bigr) = \bigl\{\left(\begin{smallmatrix} A & X \\ 0 & \pm A \end{smallmatrix}\right) \bigm| A \in GL_2(3),\ X \in M_2(\mathbb{F}_3)\bigr\},$$

and hence

$$\begin{aligned} N_{\mathrm{Out}(\boldsymbol{Q})}(\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})) &= \bigl\langle \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right), \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right), \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right) \bigm| A, X \in M_2(\mathbb{F}_3),\ X = X^t,\ AA^t = \pm I \bigr\rangle \\ &= \bigl\langle \left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right),\ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right),\ \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right) \bigm| X = X^t,\ A \in \{\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\} \bigr\rangle \\ &\cong E_{27} \rtimes (SD_{16} \times C_2). \end{aligned} \tag{5.20}$$

Here, each element of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & \pm A \end{smallmatrix}\right)$ in $N_{\mathrm{Out}(\boldsymbol{Q})}(\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q}))$ is conjugation by some element of $\boldsymbol{N}^{(\mathbf{10})}$, and hence extends to an automorphism of M_0.

It remains to prove the lemma for automorphisms of the form $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ when $X = X^t$. Define $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{Aut}(\boldsymbol{S})$ as follows. In each case, $\alpha_i|_{\boldsymbol{A}} = \mathrm{Id}$, and $\omega_i\colon \boldsymbol{T} \longrightarrow \boldsymbol{A}$ is such that $\alpha_i(g) = \omega_i(g)g$ for all $g \in \boldsymbol{T}$:

$$\begin{aligned} &\alpha_1\bigl(\llbracket a,b,c\rrbracket\,(\!(x)\!)\bigr) = \llbracket a, b+x, c+N(x)\rrbracket\,(\!(x)\!) && \omega_1(\,(\!(x)\!)\,) = \llbracket 0, x, N(x)\rrbracket \\ &\alpha_2\bigl(\llbracket a,b,c\rrbracket\,(\!(x)\!)\bigr) = \llbracket a, b+\overline{x}, c-\mathrm{Tr}(x^2)\rrbracket\,(\!(x)\!) && \omega_2(\,(\!(x)\!)\,) = \llbracket 0, \overline{x}, -\mathrm{Tr}(x^2)\rrbracket \\ &\alpha_3\bigl(\llbracket a,b,c\rrbracket\,(\!(x)\!)\bigr) = \llbracket a, b+i\overline{x}, c+\mathrm{Tr}(ix^2)\rrbracket\,(\!(x)\!) && \omega_3(\,(\!(x)\!)\,) = \llbracket 0, i\overline{x}, \mathrm{Tr}(ix^2)\rrbracket\,. \end{aligned}$$

Each of the α_i is seen to be an automorphism of $\boldsymbol{S}$ by checking the cocycle condition

$$\omega_i(\,(\!(x+y)\!)\,) = \omega_i(\,(\!(x)\!)\,) + {}^{(\!(x)\!)}\omega_i(\,(\!(y)\!)\,)$$

on ω_i. (Note the relation $N(x+y) = (x+y)(\overline{x}+\overline{y}) = N(x)+N(y)+\mathrm{Tr}(\overline{x}y)$.) The class of $\alpha_i|_{\boldsymbol{Q}}$ as an automorphism of $\boldsymbol{Q}/Z$ has matrix $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ for $X = I$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, respectively, and thus the classes $[\alpha_i|_{\boldsymbol{Q}}]$ generate $O_3(N_{\mathrm{Out}(\boldsymbol{Q})}(\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})))$ by (5.20). Since α_1 is conjuation by $\llbracket 1,0,0\rrbracket$, it extends to M_0. For $i = 2,3$, the automorphism α_i extends to $\boldsymbol{S}\langle\beta\rangle$ since $[\alpha_i, c_\beta] = 1$ in $\mathrm{Aut}(\boldsymbol{S})$: this follows upon checking the relation ${}^{\beta}\omega_i(\,(\!(x)\!)\,) = \omega_i(\,(\!({}^{\beta}x)\!)\,)$ using (5.19).

Recall that $\boldsymbol{\varGamma_0} = \widehat{\boldsymbol{M}^{\mathbf{0}}_{\mathbf{10}}} \cong A_6$. Then $N_{\boldsymbol{\varGamma_0}}(\boldsymbol{T}) = \boldsymbol{T}\langle\beta\rangle$ (see (5.18)), and the cohomology elements $[\omega_1], [\omega_2], [\omega_3] \in H^1(\boldsymbol{T};\boldsymbol{A})$ are all stable under the action of β. Since $\boldsymbol{T} \in \mathrm{Syl}_3(\boldsymbol{\varGamma_0})$ is abelian, fusion in $\boldsymbol{\varGamma_0} \cong A_6$ among subgroups of $\boldsymbol{T}$ is controlled by $N_{\boldsymbol{\varGamma_0}}(\boldsymbol{T}) = \boldsymbol{T}\langle\beta\rangle$, and hence the $[\omega_i]$ are stable under all fusion in $\boldsymbol{\varGamma_0}$. So they are restrictions of elements of $H^1(\boldsymbol{\varGamma_0};\boldsymbol{A})$ by the stable elements theorem (see [CE, Theorem XII.10.1] or [Br, Theorem III.10.3]), and each α_i extends to an automorphism $\overline{\alpha}_i$ of $M_0 = \boldsymbol{A} \rtimes \boldsymbol{\varGamma_0}$ that is the identity on $\boldsymbol{A}$. □

The next lemma is needed to handle the cases where $\boldsymbol{\varGamma_0} \cong M_{11}$.

Lemma 5.21. *Assume Hypotheses 5.1 and Notation 5.13, where $\boldsymbol{\varGamma_0} \cong M_{11}$. Let $\boldsymbol{Q} \trianglelefteq \boldsymbol{S}$ be as in Proposition 5.9, set $\Delta = \mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$ and $\Delta_0 = O^{3'}(\Delta)$.*

(a) *If we are in the M_{11}-case (i.e., if $|Z(\boldsymbol{S})| = 9$), then there is $\overline{\gamma} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$ of order 2 that acts on $\boldsymbol{Q}/Z$ via $(x \mapsto x^{-1})$. For each such $\overline{\gamma}$, if we set $\gamma = [\overline{\gamma}|_{\boldsymbol{Q}}] \in \mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$, then*

$$\Delta \le C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma) \cong \varGamma L_2(9).$$

If, furthermore, $1 \ne U_0 < U \in \mathrm{Syl}_3(C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma))$, and if $\xi \in C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma)$ has 2-power order and acts on U by $(x \mapsto x^{-1})$, then for $H \cong 2A_4$ or $H \cong 2A_5$, there is a unique subgroup $X \le C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma)$ isomorphic to H, containing U_0, and normalized by ξ.

(b) *If we are in the M_{11}^*-case (i.e., if $|Z(\boldsymbol{S})| = 3$), then there is $\overline{\gamma} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$ of order 4 such that $[\overline{\gamma}|_{\boldsymbol{Q}}] \in \mathrm{Out}_{\mathcal{F}}(\boldsymbol{Q})$ centralizes $\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})$. For each such $\overline{\gamma}$,*

$$\Delta_0 = O^{3'}(C_{\mathrm{Out}(\boldsymbol{Q})}(\overline{\gamma}|_{\boldsymbol{Q}})) \cong SL_2(9).$$

Proof. Recall that $M = \boldsymbol{A} \rtimes \boldsymbol{\Gamma}$ is a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$, and $M_0 = O^{3'}(M) = \boldsymbol{A} \rtimes \boldsymbol{\Gamma_0}$.

(a) Assume we are in the M_{11}-case. By Lemma 3.19 and Table 5.16, the element $[-1] \in \boldsymbol{N^{(11)}} \cap \widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{11}} \le M$ acts on $\boldsymbol{Q}/Z$ via $(x \mapsto x^{-1})$. Set $\overline{\gamma} = c_{[-1]} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$; thus $\overline{\gamma}$ has order 2 and inverts $\boldsymbol{Q}/Z$.

Now let $\overline{\gamma} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$ be an arbitrary element of order 2 that acts on $\boldsymbol{Q}/Z$ via $(x \mapsto x^{-1})$, and set $\gamma = [\overline{\gamma}|_{\boldsymbol{Q}}] \in \Delta = \mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$. Since $\boldsymbol{Q} \cong UT_3(9)$ by the relations in Lemma 3.23(b), we can apply Lemma A.2 to the group $\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q}) \le \mathrm{Out}(\boldsymbol{Q})$. By Lemma A.2(a,c) and since $\gamma \in \Delta$ has order 2 and inverts all elements of $\boldsymbol{Q}/Z$, we have $C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma) \cong \varGamma L_2(9)$. By the same lemma and since $O_3(\Delta) = 1$, Δ is sent isomorphically into $\mathrm{Aut}(\boldsymbol{Q}/Z)$, and hence (since γ is sent to $Z(\mathrm{Aut}(\boldsymbol{Q}/Z))$) we have $\gamma \in Z(\Delta)$. So $\Delta \le C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma)$.

Now fix subgroups $1 \ne U_0 < U \in \mathrm{Syl}_3(C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma))$, and an element $\xi \in C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma)$ of 2-power order that acts on U by $(x \mapsto x^{-1})$. In particular, $|U| = 9$ and $|U_0| = 3$. Since $O^{3'}(C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma)) \cong SL_2(9) \cong 2A_6$, there is a surjective homomorphism $\Psi\colon O^{3'}(C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma)) \longrightarrow A_6$ with kernel of order 2 such that $\Psi(U_0)$ is generated by a 3-cycle. (Recall that A_6 has an outer automorphism that exchanges the two classes of elements of order 3.) Also, c_ξ induces (via Ψ) an automorphism ξ' of A_6. Since ξ' has 2-power order and inverts all elements in $\Psi(U)$, it must be inner, and conjugation by a product of two disjoint transpositions. So there is a unique subgroup $\overline{X} \le A_6$ that contains $\Psi(U_0)$, is normalized by ξ', and is isomorphic to $H/Z(H)$ (i.e., to A_4 or A_5). Thus $X = \Psi^{-1}(\overline{X})$ is the unique subgroup satisfying the corresponding conditions in $\mathrm{Out}(\boldsymbol{Q})$.

(b) Assume we are in the M_{11}^*-case. By Lemma 5.7 (and Notation 5.3),

$$\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{11}} = \boldsymbol{\Gamma_0} \le \boldsymbol{\Gamma} \le \widehat{\boldsymbol{M}}_{\boldsymbol{11}},$$

where $[\widehat{\boldsymbol{M}}_{\boldsymbol{11}}{:}\widehat{\boldsymbol{M}}^{\boldsymbol{0}}_{\boldsymbol{11}}] = 2$ by Table 3.20. By Table 3.20 and Lemma 5.7, $N_{\widehat{\boldsymbol{M}}_{\boldsymbol{11}}}(\boldsymbol{T})/\boldsymbol{T} = \boldsymbol{N^{(11)}}/\boldsymbol{T} \cong SD_{16} \times C_2$, and hence this group has two subgroups of order 8, generated by $[\zeta]$ and $-[\zeta]$, of which only the subgroup $\langle -[\zeta] \rangle$ lies in $\boldsymbol{\Gamma_0}$. By Table 5.16, these elements act on $\boldsymbol{Q}/Z$ as follows:

$$^{[\zeta]}\llangle a, b, * \rrangle = \llangle \zeta^{-1}a, \zeta b, * \rrangle \qquad \text{and} \qquad ^{-[\zeta]}\llangle a, b, * \rrangle = \llangle \zeta^{3}a, \zeta b, * \rrangle. \tag{5.22}$$

By comparing characteristic polynomials or traces for the actions of the ζ^i on $\mathbb{F}_9$, we see that $\boldsymbol{Q}/Z$ splits as a sum of two nonisomorphic irreducible $\mathbb{F}_3C_8$-modules under the action of $\langle [\zeta] \rangle$, while the two summands under the action of $\langle -[\zeta] \rangle$ are isomorphic.

Set $U = \mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q}) = \mathrm{Out}_{\boldsymbol{A}}(\boldsymbol{Q}) \in \mathrm{Syl}_3(\Delta)$. Since $U \cong E_9$ and all elements of order 3 in U are in class **3C** or **3D** (see Table 5.16), we have

$$\Delta_0 \cong 2A_6 \cong SL_2(9)$$

by Lemma A.4. In particular, there is an element $\gamma_0 \in N_{\Delta_0}(U)$ of order 8 that acts on $\boldsymbol{Q}/Z$, as an $\mathbb{F}_3C_8$-module, with two irreducible summands not isomorphic to each other. By the extension axiom, γ_0 extends to $\overline{\gamma}_0 \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$, and $\overline{\gamma}_0|_{\boldsymbol{A}} \in N_{\boldsymbol{\Gamma}}(\boldsymbol{T})$ has order 8. By comparison with the formulas in (5.22), we see that $\overline{\gamma}_0|_{\boldsymbol{A}}$ must be conjugate to $[\zeta]$, and hence does not lie in $\boldsymbol{\Gamma_0}$. Thus $\boldsymbol{\Gamma} > \boldsymbol{\Gamma_0}$, and hence $\boldsymbol{\Gamma} = \widehat{\boldsymbol{M}}_{\boldsymbol{11}} \cong M_{11} \times C_2$. So $c^{\boldsymbol{S}}_{-[i]} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$, it has order 4 and acts on $\boldsymbol{A}$ by $^{-[i]}\llbracket r, s, t \rrbracket = \llbracket r, is, -t \rrbracket$ (see Lemma 3.23(c)), and hence centralizes $U = \mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q}) \cong \boldsymbol{A}/A_*$.

Now let $\overline{\gamma} \in \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})$ be an arbitrary automorphism of order 4 that centralizes $U = \mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})$. Since $\mathrm{Aut}(\Delta_0) \cong \mathrm{Aut}(2A_6) \cong \mathrm{Aut}(A_6)$ where $\mathrm{Out}(A_6) \cong E_4$, and since each outer automorphism of $\varSigma_6$ exchanges 3-cycles with products of disjoint 3-cycles, we have

$C_{\mathrm{Aut}(\Delta_0)}(U) \cong C_{\Sigma_6}(V) = V$ for $V \in \mathrm{Syl}_3(\Sigma_6)$. Since $\overline{\gamma}|_{\boldsymbol{Q}} \in \Delta$ acts on Δ_0 and centralizes U (and since $\overline{\gamma}$ has order prime to 3), we conclude that $c_{\overline{\gamma}}^{\Delta_0} = \mathrm{Id}_{\Delta_0}$ and hence $\Delta_0 \le C_{\mathrm{Out}(\boldsymbol{Q})}(\overline{\gamma})$.

From the list in [Di2] of subgroups of $PSp_4(3)$, we see that $\Delta_0 \cong SL_2(9) \cong 2A_6$ has index 2 in a maximal subgroup of $Sp_4(3)$, and hence index 4 in a maximal subgroup of $\mathrm{Out}(\boldsymbol{Q}) \cong Sp_4^*(3)$. So $\Delta_0 = O^{3'}(C_{\mathrm{Out}(\boldsymbol{Q})}(\overline{\gamma}))$. □

5.3. Fusion systems involving the Todd modules for M_{10} and M_{11}.

We are now ready to state and prove our main theorem on fusion systems satisfying Hypotheses 5.1.

Theorem 5.23. *Let $\boldsymbol{\mathcal{F}}$ be a saturated fusion system over a finite 3-group $\boldsymbol{S}$, with a subgroup $\boldsymbol{A} \le \boldsymbol{S}$. Set $\boldsymbol{\Gamma_0} = O^{3'}(\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}))$, and assume that either*

(i) *$\boldsymbol{A} \cong E_{3^4}$ and $\boldsymbol{\Gamma_0} \cong A_6$; or*

(ii) *$\boldsymbol{A} \cong E_{3^5}$ and $\boldsymbol{\Gamma_0} \cong M_{11}$.*

Assume also that $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$. Then $\boldsymbol{A} \trianglelefteq \boldsymbol{S}$, $\boldsymbol{S}$ splits over $\boldsymbol{A}$, $\boldsymbol{\mathcal{F}}$ is almost simple, and either

(a) *$\boldsymbol{\Gamma_0} \cong A_6$ and $O^{3'}(\boldsymbol{\mathcal{F}})$ is isomorphic to the 3-fusion system of one of the groups $U_4(3)$, $U_6(2)$, McL, or Co_2; or*

(b) *$\boldsymbol{\Gamma_0} \cong M_{11}$, $|Z(\boldsymbol{S})| = 9$, and $O^{3'}(\boldsymbol{\mathcal{F}})$ is isomorphic to the 3-fusion system of Suz or Ly; or*

(c) *$\boldsymbol{\Gamma_0} \cong M_{11}$, $|Z(\boldsymbol{S})| = 3$, and $\boldsymbol{\mathcal{F}}$ is isomorphic to the 3-fusion system of Co_3.*

(Note that (a), (b), and (c) correspond to the A_6-, M_{11}-, and M_{11}^*-cases, respectively.)

Proof. By Lemma 5.5, $\boldsymbol{A} \trianglelefteq \boldsymbol{S}$ and is weakly closed in $\boldsymbol{\mathcal{F}}$. By the same lemma, $\boldsymbol{A}$ is the unique 4-dimensional $\mathbb{F}_3A_6$-module if $\boldsymbol{\Gamma_0} \cong A_6$, and $\boldsymbol{A}$ is the Todd module or its dual if $\boldsymbol{\Gamma_0} \cong M_{11}$. Also, $\boldsymbol{S}$ splits over $\boldsymbol{A}$ by Corollary 5.12 and since $\boldsymbol{A} \ntrianglelefteq \boldsymbol{\mathcal{F}}$. So we are in the situation of Notation 5.3 and 5.13, and can use the terminology listed there.

By Proposition 5.9, there is a unique special subgroup $\boldsymbol{Q} \trianglelefteq \boldsymbol{S}$ of exponent 3 such that $Z(\boldsymbol{Q}) = Z = Z(\boldsymbol{S})$, $\boldsymbol{Q} \cap \boldsymbol{A} = A_*$, and $\boldsymbol{Q}/Z \cong E_{81}$. Also, $\mathbf{E}_{C_{\boldsymbol{\mathcal{F}}}(Z)} = \{\boldsymbol{Q}\}$, so $\boldsymbol{Q} \trianglelefteq C_{\mathcal{F}}(Z)$. Set $\boldsymbol{\Gamma} = \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$, $\Delta = \mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$, $\boldsymbol{\Gamma_0} = O^{3'}(\boldsymbol{\Gamma})$, and $\Delta_0 = O^{3'}(\Delta)$ for short.

If $|Z| = 3$ (i.e., if we are in the A_6- or M_{11}^*-case), then $\boldsymbol{Q} \cong 3_+^{1+4}$, and by Table 5.11, $\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q}) \cong \boldsymbol{S}/\boldsymbol{Q}$ has order 3 (if $\boldsymbol{\Gamma_0} \cong A_6$) or 9 (if $\boldsymbol{\Gamma_0} \cong M_{11}$). Also, all elements of order 3 in $\boldsymbol{\Gamma_0}$ act on $\boldsymbol{Q}/Z$ with two Jordan blocks of length 2 (see Table 5.16), and hence they have class **3C** or **3D** in $O^{3'}(\mathrm{Out}(\boldsymbol{Q})) \cong Sp_4(3)$ by Lemma A.3. So by Lemma A.4, Δ_0 is isomorphic to $2A_4$, $2A_5$, $(Q_8 \times Q_8) \rtimes C_3$, or $2_-^{1+4}.A_5$ if $\boldsymbol{\Gamma_0} \cong A_6$, while $\Delta_0 \cong 2A_6$ if $\boldsymbol{\Gamma_0} \cong M_{11}$.

If $|Z| = 9$, then $\boldsymbol{\Gamma_0} \cong M_{11}$ and $\boldsymbol{A}$ is its Todd module. Also, $\boldsymbol{Q} \cong UT_3(9)$ by the relations in Lemma 3.23(b). So $\mathrm{Aut}(\boldsymbol{Q})/O_3(\mathrm{Aut}(\boldsymbol{Q})) \cong \varGamma L_2(9)$ by Lemma A.2(a,b). Since $O_3(\Delta_0) = 1$ (recall $\boldsymbol{Q} \in \mathbf{E}_{\boldsymbol{\mathcal{F}}}$ and hence $\mathrm{Out}(\boldsymbol{Q})$ has a strongly 3-embedded subgroup), Δ_0 is isomorphic to a subgroup of $SL_2(9)$. The subgroups of $SL_2(9)$ are well known, and since $\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q}) \cong \boldsymbol{S}/\boldsymbol{Q}$ has order 3, we have $\Delta_0 \cong 2A_4$ or $2A_5$.

Thus in all cases, $(\boldsymbol{\Gamma_0}, \Delta_0)$ is one of the pairs listed in the first two rows of Table 5.24. Let G^* be the finite simple group listed in the table corresponding to the pair $(\boldsymbol{\Gamma_0}, \Delta_0)$, and fix $S^* \in \mathrm{Syl}_3(G^*)$. If $G^* \cong U_4(3)$, then it has maximal parabolic subgroups of the form $E_{81} \rtimes A_6$ and $3_+^{1+4}.2\Sigma_4$, so $\mathcal{F}_{S^*}(G^*)$ satisfies Hypotheses 5.1, and there are subgroups $A^*, Q^* \trianglelefteq S^*$ such that $A^* \cong \boldsymbol{A}$, $Q^* \cong \boldsymbol{Q}$, $O^{3'}(\mathrm{Aut}_{G^*}(A^*)) \cong A_6$, and $O^{3'}(\mathrm{Aut}_{G^*}(Q^*)) \cong 2A_4$. In all of the other cases, we refer to the tables in [A3, pp. 7–40], which show that $\mathcal{F}_{S^*}(G^*)$ also satisfies

$\boldsymbol{\Gamma_0}$	A_6				M_{11}		
Δ_0	$2A_4$	$2A_5$	$(Q_8\times Q_8)\rtimes C_3$	$2^{1+4}_-.A_5$	$2A_4$	$2A_5$	$2A_6$
G^*	$U_4(3)$	McL	$U_6(2)$	Co_2	Suz	Ly	Co_3

Table 5.24.

Hypotheses 5.1 with subgroups $A^*\cong\boldsymbol{A}$ and $Q^*\cong\boldsymbol{Q}$ such that $O^{3'}(\mathrm{Aut}_{G^*}(A^*))\cong\boldsymbol{\Gamma_0}$ and $O^{3'}(\mathrm{Aut}_{G^*}(Q^*))\cong\Delta_0$.

Let M be a model for $N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$ (see Proposition 1.12), and set $M^*=N_{G^*}(A^*)$. By Corollary 5.12, applied to $\boldsymbol{\mathcal{F}}$ and to $\mathcal{F}_{S^*}(G^*)$, we have $O^{3'}(M)\cong\boldsymbol{A}\rtimes\boldsymbol{\Gamma_0}\cong O^{3'}(M^*)$. Choose an isomorphism $\varphi\colon O^{3'}(M^*)\xrightarrow{\cong}O^{3'}(M)$ such that $\varphi(A^*)=\boldsymbol{A}$ and $\varphi(S^*)=\boldsymbol{S}$, and set $\mathcal{F}^*={}^{\varphi}(\mathcal{F}_{S^*}(G^*))$. Then $\mathcal{F}^*$ is a fusion system over $\boldsymbol{S}$ isomorphic to $\mathcal{F}_{\boldsymbol{S}^*}(G^*)$, and we will apply Proposition 2.13 to show that $\mathcal{F}^*=O^{3'}(\boldsymbol{\mathcal{F}})$.

The fusion system $\mathcal{F}_{S^*}(G^*)$ is simple in all cases by Proposition 4.1(b), Proposition 4.5(a), or Table 4.1 in [OR2]. (See also (16.3) and (16.10) in [A4], which cover almost all cases.) So $\mathcal{F}^*=O^{3'}(\mathcal{F}^*)$. By construction, $O^{3'}(N_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A}))=O^{3'}(N_{\mathcal{F}^*}(\boldsymbol{A}))$. By Lemma 5.14, the fusion systems $\boldsymbol{\mathcal{F}}$ and $\mathcal{F}^*$ both satisfy Hypotheses 2.12 with respect to $\boldsymbol{A},\boldsymbol{Q}\trianglelefteq\boldsymbol{S}$. So by Proposition 2.13, to show that $O^{3'}(\boldsymbol{\mathcal{F}})=\mathcal{F}^*$, it remains to show that $O^{3'}(\mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q}))=O^{3'}(\mathrm{Out}_{\mathcal{F}^*}(\boldsymbol{Q}))$, and this will be shown by considering the three cases separately. Set

$$\Gamma^*=\mathrm{Aut}_{\mathcal{F}^*}(\boldsymbol{A}),\quad \Delta^*=\mathrm{Out}_{\mathcal{F}^*}(\boldsymbol{Q}),\quad \Gamma_0^*=O^{3'}(\Gamma^*),\quad\text{and}\quad \Delta_0^*=O^{3'}(\Delta^*),$$

and note that $\Delta_0\cong\Delta_0^*$ in all cases by the choice of G^*.

The $\boldsymbol{A_6}$-case: Since $\Delta_0\cong\Delta_0^*$ are both subgroups of $\mathrm{Out}(\boldsymbol{Q})$ with the same Sylow 3-subgroup $\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})$, Lemma A.4 applies to show that they are conjugate in $\mathrm{Out}(\boldsymbol{Q})$, and hence $\Delta_0={}^{\gamma_0}\Delta_0^*$ for some $\gamma_0\in N_{\mathrm{Aut}(\boldsymbol{Q})}(\mathrm{Aut}_{\boldsymbol{S}}(\boldsymbol{Q}))$. By Lemma 5.17, γ_0 extends to some $\gamma\in\mathrm{Aut}(H_0)$, and $\gamma(\boldsymbol{S})=\boldsymbol{S}$ since $\boldsymbol{S}=\boldsymbol{Q}\boldsymbol{A}$. So upon replacing $\mathcal{F}^*$ by ${}^{(\gamma|_{\boldsymbol{S}})}\mathcal{F}^*$, we can arrange that $\Delta_0^*=\Delta_0$ without changing Γ_0^*.

The $\boldsymbol{M_{11}}$-case: Let $\gamma\in\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})=\mathrm{Aut}_{\mathcal{F}^*}(\boldsymbol{S})$ be as in Lemma 5.21(a): γ has order 2, and $\gamma|_{\boldsymbol{Q}}$ acts on $\boldsymbol{Q}/Z$ by inverting all elements. Then $\Delta_0,\Delta_0^*\le O^{3'}(C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma|_{\boldsymbol{Q}}))\cong SL_2(9)\cong 2A_6$ by that lemma.

Set $U_0=\mathrm{Out}_{\boldsymbol{S}}(\boldsymbol{Q})\cong C_3$, and let $U\in\mathrm{Syl}_3(C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma|_{\boldsymbol{Q}}))$ be the (unique) Sylow 3-subgroup that contains U_0. Set $h=-[\zeta]\in\boldsymbol{N^{(11)}}\cap\widehat{\boldsymbol{M}}{}^{\boldsymbol{0}}_{\boldsymbol{11}}<M_0$ (see Lemma 3.19), and set $\overline{\xi}=c_h^{\boldsymbol{S}}\in\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})=\mathrm{Aut}_{\mathcal{F}^*}(\boldsymbol{S})$. Since $\boldsymbol{Q}$ is weakly closed in $\boldsymbol{\mathcal{F}}$ and in $\mathcal{F}^*$ by Proposition 5.9, we have $\xi\overset{\text{def}}{=}[\overline{\xi}|_{\boldsymbol{Q}}]\in\Delta\cap\Delta^*$. So $\Delta_0\cong\Delta_0^*$ both contain U_0 and are normalized by ξ, and they are both isomorphic to $2A_4$ or $2A_5$. Hence $\Delta_0=\Delta_0^*$ by the last statement in Lemma 5.21(a).

The $\boldsymbol{M_{11}^*}$-case: By Lemma 5.21(b), applied to either fusion system $\boldsymbol{\mathcal{F}}$ or $\mathcal{F}^*$, there is $\overline{\gamma}\in\mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{S})=\mathrm{Aut}_{\mathcal{F}^*}(\boldsymbol{S})$ of order 4 such that $\overline{\gamma}|_{\boldsymbol{Q}}$ commutes with $\mathrm{Aut}_{\boldsymbol{S}}(\boldsymbol{Q})$. By the same lemma, for any such γ, we have $\Delta_0=C_{\mathrm{Out}(\boldsymbol{Q})}(\gamma)=\Delta_0^*$. Also, in this case, since $G^*\cong Co_3$, we have $\mathrm{Out}(\mathcal{F}^*)\cong\mathrm{Out}(G^*)=1$ by [O2, Proposition 3.2], and hence $\boldsymbol{\mathcal{F}}=O^{3'}(\boldsymbol{\mathcal{F}})$. □

The automizers of the subgroups $\boldsymbol{A}$ and $\boldsymbol{Q}$ in each case of Theorem 5.23 are described more explicitly in Table 5.25. We refer again to [A3, pp. 7–40] in all cases except that of $U_4(3)$.

	$\boldsymbol{A}$	$\boldsymbol{\varGamma_0}$	$\boldsymbol{Q}$	$\boldsymbol{\varGamma} = \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$	$\Delta = \mathrm{Out}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{Q})$	G
A_6-case	E_{3^4}	A_6	3^{1+4}_+	A_6	$2\Sigma_4$	$U_4(3)$
				Σ_6	$(Q_8 \times Q_8) \rtimes \Sigma_3$	$U_6(2)$
				M_{10}	$2\Sigma_5$	McL
				$(A_6 \times C_2).E_4$	$2^{1+4}_-.\Sigma_5$	Co_2
M_{11}-case	E_{3^5}	M_{11}	3^{2+4}	M_{11}	$(2A_4 \circ D_8).C_2$	Suz
				$M_{11} \times C_2$	$(2A_5 \circ C_8).C_2$	Ly
M_{11}^*-case	E_{3^5}	M_{11}	3^{1+4}_+	$2 \times M_{11}$	$(2A_6 \circ C_4).C_2$	Co_3

Table 5.25. In all cases, $\boldsymbol{\mathcal{F}}$ is a fusion system over $\boldsymbol{S} = \boldsymbol{A} \rtimes \boldsymbol{T}$, and is realized by the group G. Also, $\boldsymbol{A} \trianglelefteq \boldsymbol{S}$ is abelian with $C_{\boldsymbol{S}}(\boldsymbol{A}) = \boldsymbol{A}$ and $Z = Z(\boldsymbol{S})$, $\boldsymbol{\varGamma} = \mathrm{Aut}_{\boldsymbol{\mathcal{F}}}(\boldsymbol{A})$, and $\boldsymbol{\varGamma_0} = O^{3'}(\boldsymbol{\varGamma})$. The subsystem $C_G(Z)$ is constrained with $\boldsymbol{Q} = O_3(C_G(Z))$ and $Z = Z(\boldsymbol{Q})$.

Note that by [BMO, Theorem A(a,d)], the 3-fusion system of $U_6(2)$ is isomorphic to those of $U_6(q)$ for each $q \equiv 2, 5 \pmod 9$, and to those of $L_6(q)$ for each $q \equiv 4, 7 \pmod 9$. Thus $U_6(2)$ could be replaced by any of these other groups in the statement of Theorem 5.23.

6. Some 3-local characterizations of the Conway groups

We finish with some new 3-local characterizations of the three Conway groups, $U_6(2)$, and McLaughlin's group. In each case, the new result is obtained by combining an earlier characterization of the some group with the classifications of fusion systems in Theorem 4.16 or 5.23. It seems likely that one could get stronger results with a little more work, but we prove here only ones that follow easily from Theorems 4.16 and 5.23 together with the earlier characterizations.

We first combine Theorem 4.16 with the 3-local characterization of Co_1 shown by Salarian [Sa], to get the following slightly simpler characterization.

Theorem 6.1. *Let G be a finite group. Assume $A \le S \in \mathrm{Syl}_3(G)$ are such that*

(1) *$A \cong E_{3^6}$, $C_G(A) = A$, and $N_G(A)/A \cong 2M_{12}$;*

(2) *A is not strongly closed in S with respect to G; and*

(3) *$O_{3'}(C_G(Z(S))) = 1$ and $|O_3(C_G(Z(S)))| > 3$.*

Then $G \cong Co_1$.

Proof. By Salarian's theorem [Sa, Theorem 1.1], to show that $G \cong Co_1$, it suffices to find subgroups $H_1, H_2 \ge S \in \mathrm{Syl}_3(G)$ that satisfy the following three conditions:

(i) $H_1 = N_G(Z(O_3(H_1)))$, $O_3(H_1) \cong 3^{1+4}_{\pm}$, $H_1/O_3(H_1) \cong Sp_4(3) \rtimes C_2$, and $C_{H_1}(O_3(H_1)) = Z(O_3(H_1))$;

(ii) $O_3(H_2) \cong E_{3^6}$ and $H_2/O_3(H_2) \cong 2M_{12}$; and

(iii) $(H_1 \cap H_2)/O_3(H_2)$ is an extension of an elementary abelian group of order 9 by $GL_2(3) \times C_2$.

Set $Z = Z(S)$, $H_1 = N_G(Z)$ and $H_2 = N_G(A)$. Since $H_1, H_2 \ge S$ (recall $A \trianglelefteq S$ by assumption), it suffices to prove (i)–(iii).

Set $\mathcal{F} = \mathcal{F}_S(G)$. Then $A \ntrianglelefteq \mathcal{F}$ by (2), and hence $\mathcal{F}$ is isomorphic to the fusion system of Co_1 by (1) and Theorem 4.16. In particular, S is isomorphic to the 3-group $\boldsymbol{S}$ of Notation 4.4 and 4.5, so we can identify S with $\boldsymbol{S}$ and use the notation defined there for subgroups of $\boldsymbol{S}$.

Condition (ii) holds by (1). Also, $(H_1 \cap H_2)/O_3(H_2) = N_{H_2}(Z)/A \cong N_{\mathrm{Aut}_{\mathcal{F}}(A)}(Z)$ where $N_{\mathrm{Aut}_{\mathcal{F}}(A)}(Z) \cong (E_9 \rtimes GL_2(3)) \times C_2$ by Lemma 4.10(c), so (iii) holds.

Set $P = O_3(C_G(Z))$. Then $|P| > 3$ by (3), so $P > Z$. Also, $P \trianglelefteq C_{\mathcal{F}}(Z)$, so $P \le O_3(C_{\mathcal{F}}(Z)) = Q_0 \cong 3^{1+4}_+$ by Lemma 4.13. The action of $\mathrm{Out}_{C_{\mathcal{F}}(Z)}(Q_0) \cong Sp_4(3)$ on $Q_0/Z \cong E_{81}$ is irreducible, and hence $P = Q_0$. Thus $Q_0 = O_3(C_G(Z)) = O_3(H_1)$ since $C_G(Z)$ is normal of index at most 2 in $H_1 = N_G(Z)$.

Now, Q_0 is $\mathcal{F}$-centric by Lemma 4.8, so $Z = Z(Q_0) \in \mathrm{Syl}_3(C_G(Q_0))$, and hence $C_G(Q_0) = K \times Z(Q_0) = K \times Z$ for some K of order prime to 3. Also, $K \trianglelefteq C_G(Z)$ since $Q_0 \trianglelefteq C_G(Z)$, so $K \le O_{3'}(C_G(Z)) = 1$ by (3). Thus $C_{H_1}(Q_0) = Z = Z(Q_0)$, and hence $H_1/Q_0 \cong \mathrm{Out}_{\mathcal{F}}(Q_0)$. Since $\mathrm{Out}_{\mathcal{F}}(Q_0) \cong Sp_4(3){:}2$ by Lemma 4.15, this finishes the proof of (i), and hence the proof of the theorem. □

The following 3-local characterization of Co_3 simplifies slightly that shown by Korchagina, Parker, and Rowley.

Theorem 6.2. *Let G be a finite group. Assume $A \le S \in \mathrm{Syl}_3(G)$ are such that*

(1) *$A \cong E_{3^5}$, $C_G(A) = A$, $|Z(S)| = 3$, and $O^{3'}(N_G(A)/A) \cong M_{11}$;*

(2) *A is not strongly closed in S with respect to G; and*

(3) *$O_{3'}(C_G(Z(S))) = 1$ and $|O_3(C_G(Z(S)))| > 3$.*

Then $G \cong Co_3$.

Proof. By the theorem of Korchagina, Parker, and Rowley [KPR, Theorem 1.1], to show that $G \cong Co_3$, it suffices to find subgroups $M_1, M_2 \le G$ and $A \le S$ that satisfy the following two conditions:

(i) $M_1 = N_G(Z(S))$ is of the form $3^{1+4}_+.C_2.C_2.PSL_2(9).C_2$; and

(ii) $M_2 = N_G(A)$ is of the form $E_{3^5} \rtimes (C_2 \times M_{11})$.

Set $Z = Z(S)$, $M_1 = N_G(Z)$ and $M_2 = N_G(A)$; we claim that (i) and (ii) hold for this choice of subgroups.

Set $\mathcal{F} = \mathcal{F}_S(G)$. Then $A \ntrianglelefteq \mathcal{F}$ by (2). By Table 5.4 and since $|Z| = 3$ by (1), A is the dual Todd module for $O^{3'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cong M_{11}$. Hence $\mathcal{F}$ is isomorphic to the fusion system of Co_3 by Theorem 5.23(c). In particular, S is isomorphic to the 3-group $\boldsymbol{S}$ of Notation 5.3, so we can identify S with $\boldsymbol{S}$ and use the notation defined there for subgroups of $\boldsymbol{S}$.

Condition (ii) holds by (1), and since $N_G(A)/A \cong \mathrm{Aut}_{\mathcal{F}}(A) \cong M_{11} \times C_2$ by Table 5.25.

Set $P = O_3(C_G(Z))$. Then $|P| > 3$ by (3), so $P > Z$. Also, $P \trianglelefteq C_{\mathcal{F}}(Z)$, so $P \le O_3(C_{\mathcal{F}}(Z)) = Q \cong 3^{1+4}_+$ by Proposition 5.9. Since $5 \bigm| |SL_2(9)|$, the action of $\mathrm{Out}_{C_{\mathcal{F}}(Z)}(Q) \cong SL_2(9)$ on $Q/Z \cong E_{81}$ is irreducible, and hence $P = Q$. Thus $Q = O_3(C_G(Z)) = O_3(M_1)$ since $C_G(Z)$ is normal of index at most 2 in $M_1 = N_G(Z)$.

Now, Q is $\mathcal{F}$-centric by Proposition 5.9, so $Z = Z(Q) \in \mathrm{Syl}_3(C_G(Q))$, and hence $C_G(Q) = K \times Z(Q) = K \times Z$ for some K of order prime to 3. Also, $K \trianglelefteq C_G(Z)$ since $Q \trianglelefteq C_G(Z)$, so $K \le O_{3'}(C_G(Z)) = 1$ by (3). Thus $C_{M_1}(Q) = Z = Z(Q)$, and hence $M_1/Q \cong \mathrm{Out}_{\mathcal{F}}(Q)$. Since $\mathcal{F}$ is the fusion system of Co_3, and since $\mathrm{Out}_{\mathcal{F}}(Q) \cong 2(A_6 \times C_2).C_2$ by Table 5.25, this finishes the proof of (i), and hence the proof of the theorem. □

Finally, we combine Theorem 5.23 with results of Parker, Rowley, and Stroth, to get some new 3-local characterizations of McL and $U_6(2)$ as well as of Co_2.

Theorem 6.3. *Let G be a finite group, fix $S \in \mathrm{Syl}_3(G)$, and set $Z = Z(S)$. Assume $A \le S$ is such that*

(1) $A \cong E_{3^4}$, $C_G(A) = A$, *and* $O^{3'}(N_G(A)/A) \cong A_6$*;*

(2) *A is not strongly closed in S with respect to G; and*

(3) $O_{3'}(C_G(Z)) = 1$ *and* $|O_3(C_G(Z))| > 3$.

Then $O_3(N_G(Z)) \cong 3^{1+4}_+$ and $C_G(O_3(C_G(Z))) = Z$. Also, the following hold, where k denotes the index of $O^{3'}(N_G(A)/A)$ in $N_G(A)/A$:

(a) *If $5 \bigm| |C_G(Z)|$, then G is isomorphic to McL, $\mathrm{Aut}(McL)$, or Co_2, depending on whether $k = 2$, 4, or 8, respectively.*

(b) *If $5 \nmid |C_G(Z)|$, $|O_2(C_G(Z)/O_3(C_G(Z)))| \ge 2^6$, and $k \le 4$, then $G \cong U_6(2)$ or $U_6(2) \rtimes C_2$ when $k = 2$ or 4, respectively.*

Proof. Set $\mathcal{F} = \mathcal{F}_S(G)$. Then $A \ntrianglelefteq \mathcal{F}$ by (2). So by (1) and Theorem 5.23(a), $O^{3'}(\mathcal{F})$ is isomorphic to the fusion system of Co_2, $U_4(3)$, McL, or $U_6(2)$.

Set $Q = O_3(C_{\mathcal{F}}(Z))$: an extraspecial group of order 3^5 with $Z(Q) = Z$ by Proposition 5.9. We claim that Q/Z is a simple $\mathbb{F}_3\mathrm{Out}_{\mathcal{F}}(Q)$-module. Assume otherwise, and consider the elements $a = [\![1,0,0]\!] \in S$ and $\beta = [c_{-[i]}] \in \mathrm{Out}_{\mathcal{F}}(Q)$ in the notation of Tables 3.20 and 5.16. Assume $0 \ne V < Q/Z$ is a proper nontrivial submodule, and choose $0 \ne x \in V$. If $x \notin C_{Q/Z}(a)$, then the elements $[a,x]$, $\beta([a,x])$, x, $\beta(x)$ all lie in V and generate Q/Z (see Table 5.16), contradicting the assumption that $V < Q/Z$. Thus $V \le C_{Q/Z}(a)$, with equality since $V \ge \langle x, \beta(x)\rangle = C_{Q/Z}(a)$. But if $C_{Q/Z}(a)$ were a submodule, then by Lemma B.9, Q would not be $\mathcal{F}$-essential, contradicting Proposition 5.9.

Set $P = O_3(C_G(Z))$. Then $P > Z$ by (3), and $P \le Q$ since $P \trianglelefteq C_{\mathcal{F}}(Z)$. Also, P/Z is an $\mathbb{F}_3\mathrm{Out}_{\mathcal{F}}(Q)$-submodule of Q/Z, so $P = Q \cong 3^{1+4}_+$ since Q/Z is simple.

Now, Q is $\mathcal{F}$-centric by Proposition 5.9, so $Z = Z(Q) \in \mathrm{Syl}_3(C_G(Q))$, and hence $C_G(Q) = K \times Z(Q) = K \times Z$ for some K of order prime to 3. Also, $K \trianglelefteq C_G(Z)$ since $Q \trianglelefteq C_G(Z)$, so $K \le O_{3'}(C_G(Z)) = 1$ by (3). Thus $C_G(Q) = Z = Z(Q)$, and hence $C_G(Z)/Q \cong \mathrm{Out}_{\mathcal{F}}(Q)$.

If $5 \bigm| |C_G(Z)/Q| = |\mathrm{Out}_{\mathcal{F}}(Q)|$, then by Table 5.25 again, $O^{3'}(\mathcal{F})$ is the fusion system of McL or Co_2. In the former case, $O^{3'}(N_G(Z)) \cong 3^{1+4}_+.2A_5$ and $C_G(O_3(C_G(Z))) = C_G(Q) \le Q$, so conditions (i)–(iii) in [PSt2, Theorem 1.1] all hold, and $G \cong McL$ or $\mathrm{Aut}(McL)$ by that theorem (with $k = 2$ or 4).

If $O^{3'}(\mathcal{F})$ is the fusion system of Co_2, then by Table 5.25,

(i) $Q = O_3(C_G(Z))$ is extraspecial of order 3^5, $O_2(C_G(Z)/Q)$ is extraspecial of order 2^5, and $C_G(Z)/O_{3,2}(C_G(Z)) \cong A_5$; and

(ii) Z is not weakly closed in S with respect to G.

So $G \cong Co_2$ by a theorem of Parker and Rowley [PR2, Theorem 1.1]. Also, $k = 8$ in this case.

If $5 \nmid |C_G(Z)|$, $|O_2(C_G(Z)/Q)| \ge 2^6$, and $k \le 4$, then by Table 5.25, $C_G(Z)/Q$ contains $2A_4$ with index k or $(Q_8 \times Q_8) \rtimes C_3$ with index $k/2$, and the first would imply $|O_2(C_G(Z)/Q)| \le 2^5$. So $O^{3'}(\mathcal{F})$ is the fusion system of $U_6(2)$, and $C_G(Z)/Q$ contains a normal subgroup isomorphic to $(Q_8 \times Q_8) \rtimes C_3$. Hence $C_G(Z)$ is "similar to a 3-centralizer in a group of type $PSU_6(2)$ or $F_4(2)$" in the sense of Parker and Stroth [PSt1, Definition 1.1], and $F^*(G) \cong U_6(2)$ or

$F_4(2)$ by [PSt1, Theorem 1.3]. The group $F_4(2)$ does contain subgroups isomorphic to E_{81} (a maximal torus and the Thompson subgroup of a Sylow 3-subgroup), but all such subgroups have automiser the Weyl group of F_4, and so we conclude that $G \cong U_6(2)$ or $U_6(2) \rtimes C_2$. □

Appendix A. Some special p-groups

In this section, we give a few elementary results on special or extraspecial p-groups and their automorphism groups. Most of them involve p-groups of the form p^{2+4} or p_+^{1+4}, but we start with the following, slightly more general lemma.

Lemma A.1. *Fix a prime p, and let Q be a finite nonabelian p-group such that $Z(Q) = [Q,Q]$ and is elementary abelian. Set $Z = Z(Q)$ and $\overline{Q} = Q/Z$ for short. Then the following hold.*

(a) *The quotient group $\overline{Q}$ is elementary abelian, and hence Q is a special p-group.*

(b) *If p is odd, then there is a homomorphism $\rho\colon \overline{Q} \longrightarrow Z$ such that $g^p = \rho(gZ)$ for each $g \in Q$.*

(c) *Assume $\overline{Q} \cong E_{p^3}$ and $Z \cong E_{p^2}$. Then there is a unique abelian subgroup $A \le Q$ of order p^4 and index p.*

(d) *Assume $|\overline{Q}| = p^4$, and $|Z| \le p^2$. Then for each $g \in Q \smallsetminus Z$, there is an abelian subgroup $A \le Q$ of index p^2 such that $g \in A$, and A is unique if $[g,Q] = Z \cong E_{p^2}$. If $|Z| = p^2$ and $[g,Q] = Z$ for each $g \in Q \smallsetminus Z$, then there are exactly p^2+1 abelian subgroups of index p^2 in Q, any two of which intersect in Z.*

Proof. Set $\overline{P} = PZ/Z$ and $\overline{g} = gZ \in Q/Z$ for each $H \le Q$ and $g \in Q$. Since $[Q,Q] \le Z(Q)$, the commutator map $\overline{Q} \times \overline{Q} \longrightarrow Z$ is bilinear.

(a) For each $g,h \in Q$, we have $[g,h] \in Z$ and $[g,h]^p = 1$ by assumption. Hence $[g^p,h] = 1$ for all $h \in Q$, so $g^p \in Z(Q) = Z$, and $\overline{Q} = Q/Z$ is elementary abelian.

(b) For each $g,h \in Q$, since $[h,g] \in Z(Q)$, we have $(gh)^n = g^n h^n [h,g]^{n(n-1)/2}$ for each $n \ge 1$. (Recall that $[h,g] = hgh^{-1}g^{-1}$ here.) So if p is odd, then $(gh)^p = g^p h^p$ for each $g,h \in Q$.

(c) Assume $|Q| = p^5$ and $|Z| = p^2$. Since $|[Q,Q]| > p$, there is at most one abelian subgroup of index p in Q (see [O1, Lemma 1.9]).

Fix $a,b,c \in Q$ such that $\{\overline{a},\overline{b},\overline{c}\}$ is a basis for $\overline{Q} \cong E_{p^3}$, and consider the three commutators $[a,b]$, $[a,c]$, and $[b,c]$. Since $\mathrm{rk}(Z) = 2$, one of these is in the subgroup generated by the other two, and without loss of generality, we can assume there are $i,j \in \mathbb{Z}$ such that $[a,b] = [a,c]^i[b,c]^j = [a,c^i][b,c^j]$ (recall $[Q,Q] \le Z(Q)$). Then $[ac^j, bc^{-i}] = 1$, and hence $Z\langle ac^j, bc^{-i}\rangle$ is abelian of index p in Q.

(d) Assume $\overline{Q} \cong E_{p^4}$ and $|Z| \le p^2$, and fix $g \in Q \smallsetminus Z$. Then commutator with g defines a homomorphism $\chi\colon Q/Z\langle g\rangle \longrightarrow Z$, and this is not injective since $\mathrm{rk}(Q/Z\langle g\rangle) > \mathrm{rk}(Z)$. So there is $h \in Q \smallsetminus Z\langle g\rangle$ such that $[g,h] = 1$ and $Z\langle g,h\rangle$ is abelian. If $[g,Q] = Z \cong E_{p^2}$, then χ is surjective, $\mathrm{Ker}(\chi)$ is generated by the class of h, and hence $Z\langle g,h\rangle$ is the only abelian subgroup of index p^2 in Q containing g.

Now assume $[g,Q] = Z \cong E_{p^2}$ for each $g \in Q \smallsetminus Z$, and let $\mathcal{A}$ be the set of abelian subgroups of index p^2 in Q. Then each $\overline{P} \le \overline{Q}$ of order p is contained in $\overline{A}$ for some unique $A \in \mathcal{A}$, and each such $\overline{A}$ has p^2-1 subgroups of order p. So $|\mathcal{A}| = (p^4-1)/(p^2-1) = p^2+1$. □

In the rest of the section, we prove some more specialized results on certain special p-groups. Recall that for each prime power q and each $n \ge 2$, we let $UT_n(q)$ denote the group of upper triangular $(n \times n)$ matrices with 1's on the diagonal. The groups $UT_3(q)$ are a special case of what Beisiegel calls "semi-extraspecial p-groups" in [Bei].

Lemma A.2. *Let p be an odd prime, and set $q = p^m$ for some $m \geq 1$. Set $Q = UT_3(q)$ and $Z = Z(Q)$, and let*

$$\Psi\colon \mathrm{Aut}(Q) \longrightarrow \mathrm{Aut}(Q/Z)$$

be the natural homomorphism. We regard Q/Z as a 2-dimensional $\mathbb{F}_q$-vector space in the canonical way.

(a) *The image $\Psi(\mathrm{Aut}(Q))$ is the group of all $\mathbb{F}_q$-semilinear automorphisms of Q/Z, hence isomorphic to $\Gamma L_2(q)$. For $\alpha \in \mathrm{Aut}(Q)$, we have $\alpha|_Z = \mathrm{Id}$ if and only if $\Psi(\alpha)$ is linear of determinant 1.*

(b) *We have $\mathrm{Ker}(\Psi) = O_p(\mathrm{Aut}(Q)) \cong \mathrm{Hom}(Q/Z, Z) \cong E_{p^n}$ where $n = 2m^2$.*

(c) *Let $\gamma \in \mathrm{Aut}(Q)$ be any automorphism such that $\Psi(\gamma) = -\mathrm{Id}_{Q/Z}$. Then*

$$C_{\mathrm{Aut}(Q)}(\gamma) \cong C_{\mathrm{Out}(Q)}(\gamma) \cong \Psi(\mathrm{Aut}(Q)).$$

More precisely, each $\overline{\alpha} \in \Psi(\mathrm{Aut}(Q))$ is the image under Ψ of a unique element in $C_{\mathrm{Aut}(Q)}(\gamma)$ and of a unique class in $C_{\mathrm{Out}(Q)}(\gamma)$, and hence

$$\mathrm{Aut}(Q) = O_p(\mathrm{Aut}(Q)) \rtimes C_{\mathrm{Aut}(Q)}(\gamma) \quad \text{and} \quad \mathrm{Out}(Q) = O_p(\mathrm{Out}(Q)) \rtimes C_{\mathrm{Out}(Q)}(\gamma).$$

Proof. **(a,b)** See [PR1, Proposition 5.3].

(c) Set $U = \mathrm{Ker}(\Psi) = O_p(\mathrm{Aut}(Q))$ for short. Fix $\gamma \in \mathrm{Aut}(Q)$ such that $\Psi(\gamma) = -\mathrm{Id}$. Then $\gamma|_Z = \mathrm{Id}_Z$ since $Z = [Q,Q]$. Each $\beta \in U$ has the form $\beta(g) = g\chi(g)$ for some $\chi \in \mathrm{Hom}(Q,Z)$ with $Z \leq \mathrm{Ker}(\chi)$, and

$$({}^{\gamma}\beta)(g) = \gamma\beta(\gamma^{-1}(g)) = \gamma(\gamma^{-1}(g)\chi(g^{-1})) = g\chi(g)^{-1} = \beta^{-1}(g).$$

Thus c_γ sends each element of U to its inverse, and since $\gamma \in \alpha_{-I}U$ (where $\alpha_{-I} \in \mathrm{Aut}(Q)$ is defined as in the proof of (a)), we have $\gamma^2 = (\alpha_{-I})^2 = \mathrm{Id}$. Note also that $C_{\mathrm{Aut}(Q)}(\gamma) \cap U = 1$.

Fix $\alpha \in \mathrm{Aut}(Q)$. Then $[\alpha, \gamma] \in U$ since $\Psi(\gamma) \in Z(\mathrm{Aut}(Q/Z))$, so c_γ sends the coset αU to itself. Since $\gamma^2 = 1$ and $|\alpha U| = |U|$ is odd (a power of p), there is some $\alpha' \in \alpha U \cap C_{\mathrm{Aut}(Q)}(\gamma)$. Since $C_{\mathrm{Aut}(Q)}(\gamma) \cap U = 1$, there is at most one such element $\alpha' \in \alpha U$ centralized by γ.

A similar argument shows that each $[\alpha] \in \mathrm{Out}(Q)$ is congruent modulo $U/\mathrm{Inn}(Q)$ to a unique class of automorphisms that centralizes the class of γ in $\mathrm{Out}(Q)$. □

When working with automorphisms of extraspecial groups 3^{1+4}_+, we will need to know the conjugacy classes of elements of order 3 in $Sp_4(3)$.

Lemma A.3. *Let V be a 4-dimensional $\mathbb{F}_3$-vector space with nondegenerate symplectic form $\mathfrak{b}$. Thus $\mathrm{Aut}(V, \mathfrak{b}) \cong Sp_4(3)$. There are four conjugacy classes of elements of order 3 in $\mathrm{Aut}(V, \mathfrak{b})$.*

(a) *The elements $g \in \mathrm{Aut}(V, \mathfrak{b})$ in class* **3A** *or* **3B** *are those that act on V with one Jordan block of length 2 and two of length 1. Also, $g \in$* **3A** *implies $g^{-1} \in$* **3B**.

(b) *The elements $g \in \mathrm{Aut}(V, \mathfrak{b})$ in class* **3C** *or* **3D** *are those that act on V with two Jordan blocks of length 2. If $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ is a basis for V with respect to which the form $\mathfrak{b}$ has matrix $\pm\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, and if g has matrix $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ with respect to $\mathcal{B}$, then $g \in$* **3C** *if $\det(X) = 1$ and $g \in$* **3D** *if $\det(X) = -1$.*

Proof. The conjugacy classes of elements of order 3 in $Sp_4(3)$ were first determined by Dickson, in [Di1, p. 138].

Fix $g \in \mathrm{Aut}(V, \mathfrak{b})$ of order 3. Its Jordan blocks have length at most 3, so there must be at least two of them. Thus $\dim(C_V(g)) \geq 2$ and $C_V(g) \cap [g, V] \neq 0$, so there are $v, w \in V$ such that $\{gv - v, w\}$ are linearly independent and lie in $C_V(g)$. Also, $(gv - v) \perp w$ since g preserves $\mathfrak{b}$, and so $W = \langle gv - v, w\rangle \leq C_V(g)$ is totally isotropic.

Fix a basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ such that $W = \langle v_1, v_2\rangle$, and with respect to which $\mathfrak{b}$ has matrix $\pm\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Then g has matrix $\left(\begin{smallmatrix} I & X \\ 0 & B \end{smallmatrix}\right)$ with respect to $\mathcal{B}$, and $B = I$ and $X = X^t$ since g preserves $\mathfrak{b}$. Such a matrix $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ has Jordan blocks of length $2 + 2$ if $\det(X) \neq 0$, or of length $2 + 1 + 1$ if $\det(X) = 0$, showing that such elements lie in at least two different conjugacy classes of subgroups.

If g and h have matrices $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} I & Y \\ 0 & I \end{smallmatrix}\right)$, respectively, where X and Y are invertible, then $W = C_V(g) = C_V(h)$. So if they are conjugate in $\mathrm{Aut}(V, \mathfrak{b})$, they are conjugate by a matrix of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{smallmatrix}\right)$, and hence $Y = AXA^t$ and $\det(Y) = \det(X)\det(A)^2 = \det(X)$. Thus there are at least three conjugacy classes of subgroups of order 3, and since there are exactly three by [Di1], they are distinguished by $\det(X)$ when there is a generator of the form $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$.

There are 40 maximal isotropic subspaces, each of which is fixed by three subgroups of the form $\left\langle\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)\right\rangle$ for $\det(X) = 1$, and six of that form with $\det(X) = -1$. Also, there are 40 3-dimensional subspaces, each of which is fixed by exactly one subgroup of the form $\left\langle\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)\right\rangle$ with $\det(X) = 0$. Hence there are 120, 240, and 40 subgroups conjugate to $\langle\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)\rangle$ for $\det(X) = 1$, -1, and 0, respectively. Since they are named in order of occurrence in the group, they correspond to the classes **3C**, **3D**, and **3AB**, respectively. □

Finally, we consider certain subgroups of extraspecial groups of order 3^5.

Lemma A.4. *Assume Q is extraspecial of order 3^5 and exponent 3. Let $1 \neq P \leq \mathrm{Out}(Q)$ be such that $O_3(P) = 1$, $O^{3'}(P) = P$, and each element of order 3 in P is of type* **3C** *or* **3D**. *Then either*

(a) *P is isomorphic to $2A_4$, $2A_5$, or $(Q_8 \times Q_8) \rtimes C_3$, in each of which cases there is one $Sp_4(3)$-conjugacy class containing elements of type* **3C** *and one containing elements of type* **3D***; or*

(b) *$P \cong 2^{1+4}_{-}.A_5$ or $2A_6$, in each of which cases there is just one conjugacy class.*

Proof. Set $Z = Z(Q)$ and $V = Q/Z$, and let $\mathfrak{b}$ be the symplectic form on V defined by taking commutators in Q. Thus V is a 4-dimensional vector space over $\mathbb{F}_3$, and $O^{3'}(\mathrm{Out}(Q)) \cong \mathrm{Aut}(V, \mathfrak{b}) \cong Sp_4(3)$. Let $R \leq O^{3'}(\mathrm{Out}(Q))$ be a maximal subgroup that contains P. By a theorem of Dickson [Di2, § 71] (see also [Mi, Theorem 10]), R must lie in one of five conjugacy classes.

- If R is in one of the two classes of maximal parabolic subgroups, then $O^{3'}(R)/O_3(R) \cong SL_2(3) \cong 2A_4$. Since $O_3(P) = 1$, it follows that $P \cong 2A_4$.
- If $R \cong Sp_2(3) \wr C_2 \cong 2A_4 \wr C_2$, then $P \leq O^{3'}(R) \cong 2A_4 \times 2A_4$, and V splits as a direct sum of 2-dimensional $\mathbb{F}_3P$-submodules. Each $g \in P$ of order 3 is in class **3C** or **3D** and hence acts on V with two Jordan blocks of length 2, and thus acts nontrivially on each of the two direct summands. In other words, each such g acts diagonally on $O_2(R) \cong Q_8 \times Q_8$, and so $P \leq (Q_8 \times Q_8) \rtimes C_3$. Hence either $P = (Q_8 \times Q_8) \rtimes C_3$, or $P \cong 2A_4$ diagonally embedded in $2A_4 \times 2A_4$.
- If $O^2(R) \cong Sp_2(9) \cong 2A_6$, then from a list of subgroups of $2A_6 \cong SL_2(9)$ (see [GLS3, Theorem 6.5.1]), we see that $P \cong 2A_4$, $2A_5$, or $2A_6$.

- Assume $R \cong 2^{1+4}_{-}.A_5$, and let $\widehat{P}$ be the image of P in $R/O_2(R) \cong A_5$. Then $\widehat{P} \cong C_3$, A_4, or A_5: these are up to conjugacy the only nontrivial subgroups of A_5 generated by elements of order 3. Also, P acts faithfully on $O_2(R)/Z(R) \cong E_{16}$. Since $O_3(R) = 1$, P must be isomorphic to one of the following groups:

$$\begin{aligned}
\widehat{P} \cong C_3 \quad &\Longrightarrow \quad P \cong Q_8 \rtimes C_3;\\
\widehat{P} \cong A_4 \quad &\Longrightarrow \quad P \cong A_4,\ 2A_4,\ \text{or } 2^{1+3}.A_4 \cong (Q_8 \times Q_8) \rtimes C_3;\\
\widehat{P} \cong A_5 \quad &\Longrightarrow \quad P \cong A_5,\ 2A_5,\ \text{or } 2^{1+4}_{-}.A_5.
\end{aligned}$$

The groups A_4 and A_5 cannot occur as subgroups of $Sp_4(3)$, since an element of order 3 would have to permute three distinct eigenspaces for the action of $O_2(A_4) \cong E_4$, hence have a Jordan block of length 3, which contradicts Lemma A.3.

Thus P is isomorphic to $2A_4$, $2A_5$, $(Q_8 \times Q_8) \rtimes C_3$, $2^{1+4}_{-}.A_5$, or $2A_6$. By §11 and §46 in [Di2], there are two conjugacy classes of subgroups isomorphic to $2A_4$ and two of subgroups isomorphic to $2A_5$. Since $2A_6 \cong SL_2(9) < Sp_4(3)$ has elements of both types **3C** and **3D** (the elements $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \zeta \\ 0 & 1 \end{smallmatrix}\right)$ are in different classes by the criterion in Lemma A.3), the two classes in each case are distinguished by having elements of type **3C** or **3D**. Likewise, by [Di2, § 49], there are two classes of subgroups of the form $(Q_8 \times Q_8) \rtimes C_3$ (and not isomorphic to $2A_4 \times Q_8$), and they are also distinguished by having elements of type **3C** or **3D**. Finally, by [Di2, § 61 & § 68], there is just one conjugacy class of subgroups isomorphic to $2A_6$ and one of subgroups isomorphic to $2^{1+4}_{-}.A_5$. □

We finish the section with the following well known and elementary lemma.

Lemma A.5. *Fix a prime p. Let G be a finite p-group, let $A \trianglelefteq G$ be a normal elementary abelian p-subgroup, and assume $x \in G \smallsetminus A$ is such that $x^p \in A$. Let $\Phi_x \in \mathrm{End}(A)$ be the homomorphism $\Phi_x(a) = [x, a] = {}^x a \cdot a^{-1}$. Then for each $a \in A$, $(ax)^p = x^p$ if and only if $(\Phi_x)^{p-1}(a) = 1$.*

Proof. Set $U = A\langle x\rangle/A \cong C_p$ and $u = xA \in U$, and regard A as an $\mathbb{F}_pU$-module. Then

$$(ax)^p = a \cdot {}^x a \cdots {}^{x^{p-1}} a \cdot x^p = \big((1 + u + \cdots + u^{p-1})a\big) \cdot x^p = (u-1)^{p-1}a \cdot x^p = \Phi_x^{p-1}(a) \cdot x^p$$

(in additive notation). So $(ax)^p = x^p$ if and only if $\Phi_x^{p-1}(a) = 0$. □

Appendix B. Strongly p-embedded subgroups

We collect here some of the basic properties, especially for odd primes p, of finite groups with strongly p-embedded subgroups. All of the results here are proven independently of the classification of finite simple groups (but see remarks in the proof of Proposition B.10).

Lemma B.1. *Let G be a finite group, and let $G_0 \trianglelefteq G$ be normal of index prime to p. Then G_0 has a strongly p-embedded subgroup if and only if G does.*

Proof. Recall (see [HB3, Theorem X.4.11(b)]) that G has a strongly p-embedded subgroup if and only if there is a partition $\mathrm{Syl}_p(G) = X_1 \amalg X_2$, with $X_1, X_2 \neq \varnothing$, such that for each $S_1 \in X_1$ and $S_2 \in X_2$, we have $S_1 \cap S_2 = 1$ (G is "p-isolated" in the terminology of [HB3]). Since $\mathrm{Syl}_p(G_0) = \mathrm{Syl}_p(G)$, the lemma follows immediately. □

Lemma B.2. *Let G be a finite group with a strongly p-embedded subgroup $H < G$.*

(a) *Each proper subgroup $\widehat{H} < G$ that contains H is also strongly p-embedded in G.*

(b) *For each normal subgroup $K \trianglelefteq G$, either HK/K is strongly p-embedded in G/K, or $HK = G$, or $p \nmid |G/K|$.*

Proof. **(a)** Assume $H \le \widehat{H} < G$. If $g \in G \smallsetminus \widehat{H}$ is such that $p \mid |\widehat{H} \cap {}^g\widehat{H}|$, then there is $x \in \widehat{H} \cap {}^g\widehat{H}$ of order p. Since H contains a Sylow p-subgroup of $\widehat{H}$, there are $a, b \in \widehat{H}$ such that $x \in {}^aH$ and $x \in {}^{gb}H$. Thus $p \mid |{}^aH \cap {}^{gb}H| = |H \cap {}^{a^{-1}gb}H|$, so $a^{-1}gb \in H$ since H is strongly p-embedded. Hence $g \in \widehat{H}$ since $a, b \in \widehat{H}$. So $\widehat{H}$ is also strongly p-embedded in G.

(b) If $K \trianglelefteq G$ and $HK < G$, then HK is strongly p-embedded in G by (a). Hence HK/K is strongly p-embedded in G/K if $p \mid |HK/K|$; equivalently, if $p \mid |G/K|$. □

The next few lemmas provides different ways of showing that certain groups do not have strongly p-embedded subgroups.

Lemma B.3. *Fix a finite group G containing a strongly p-embedded subgroup. Let $\{K_i\}_{i\in I}$ be a finite set of normal subgroups, set $K_{I_0} = \bigcap_{i\in I_0} K_i$ for each $I_0 \subseteq I$, and assume $K_I = 1$. Let $J \subseteq I$ be the set of those $i \in I$ such that $p \nmid |K_i|$. Then the following hold.*

(a) *In all cases, $J \ne \varnothing$ and G/K_J has a strongly p-embedded subgroup.*

(b) *If $p^2 \nmid |G|$, or (more generally) if there is a p-subgroup $T \le G$ such that $N_G(T)$ is strongly p-embedded in G, then there is $j \in J$ such that G/K_j has a strongly p-embedded subgroup.*

Proof. Fix $S \in \mathrm{Syl}_p(G)$, and let $H < G$ be the minimal strongly p-embedded subgroup that contains S.

(a) We show this by induction on $|I \smallsetminus J|$. If $I = J$, there is nothing to prove, so assume $I \supsetneqq J$, fix $i_0 \in I \smallsetminus J$, and set $I_0 = I \smallsetminus \{i_0\}$. Then $p \mid |K_{i_0}|$ and $K_{i_0} \cap K_{I_0} = 1$, so $I_0 \ne \varnothing$ and $[K_{i_0}, K_{I_0}] = 1$. For each $g \in K_{I_0}$, we have $H \cap K_{i_0} \le C_H(g) \le H \cap {}^gH$, and $p \mid |H \cap K_{i_0}|$ since S contains some Sylow p-subgroup of K_{i_0}. Thus $g \in H$, and so $K_{I_0} \le H$. So $p \nmid |K_{I_0}|$, and H/K_{I_0} is strongly p-embedded in G/K_{I_0} by Lemma B.2(b). Since $|I_0 \smallsetminus J| < |I \smallsetminus J|$, we now conclude by the induction hypothesis (applied to the group G/K_{I_0} and the subgroups $\{K_i/K_{I_0}\}_{i\in I_0}$) that $J \ne \varnothing$, and that G/K_J has a strongly p-embedded subgroup.

(b) Assume $T \le S$ is such that $H = N_G(T)$ is strongly p-embedded in G. In particular, if $|S| = p$, this holds for $T = S$. We must show that G/K_j has a strongly p-embedded subgroup for some $j \in J$, and it suffices to do this when $I = J$ and $|J| = 2$; e.g., when $I = J = \{1, 2\}$. Thus $K_1 \cap K_2 = 1$, and $p \nmid |K_i|$ for $i = 1, 2$. Set $K = K_1K_2$.

Assume neither G/K_1 nor G/K_2 contains a strongly p-embedded subgroup. Then $G = HK_1 = HK_2$ by Lemma B.2(b). Also,

$$[H \cap K, T] = [N_K(T), T] \le T \cap K = 1,$$

and $H \cap K = N_K(T) = C_K(T)$. So for $i = 1, 2$, we have $K = (H \cap K)K_i = C_K(T)K_i$ since $G = HK_i$, and hence $[K, T] = [K_i, T] \le K_i$.

Thus $[K, T] \le K_1 \cap K_2 = 1$. But then K and H both normalize T, so $G = HK$ normalizes T, contradicting the assumption that $H = N_G(T) < G$. □

The next lemma is an easy consequence of the well known list of subgroups of $PSL_3(p)$.

Lemma B.4. *Fix a prime p and $n \ge 2$. Let $G \le GL_n(p)$ be a subgroup such that $G \ngeq SL_n(p)$, $p^2 \mid |G|$, and G acts irreducibly on $\mathbb{F}_p^n$. Then $n \ge 4$.*

Proof. Since $p^2 \nmid |GL_2(p)|$, we have $n \geq 3$. From the list of maximal subgroups of $PSL_3(p)$ (see [GLS3, Theorem 6.5.3]), we see that there is no proper subgroup $G < SL_3(p)$ (hence none in $GL_3(p)$) such that G is irreducible on $\mathbb{F}_p^3$ and $p^2 \bigm| |H|$. So $n \geq 4$. □

In the next few lemmas, $\Phi(P)$ denotes the Frattini subgroup of a finite p-group P.

Lemma B.5. *Let P be a finite p-group, and let $P_0 \leq P_1 \leq \cdots \leq P_m = P$ be a sequence of subgroups, all normal in P, and such that $P_0 \leq \Phi(P)$. Let $\alpha \in \mathrm{Aut}(P)$ be such that $[\alpha, P_i] \leq P_{i-1}$ for all $1 \leq i \leq m$. Then α has p-power order.*

Proof. For each such α, $\alpha/P_0 \in \mathrm{Aut}(P/P_0)$ has p-power order by [Go, Theorem 5.3.2], and hence α has p-power order by [Go, Theorem 5.1.4]. □

Lemma B.6. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $P \in \mathbf{E}_{\mathcal{F}}$. Let $P_0 \leq P_1 \leq \cdots \leq P_m = P$ be a sequence of subgroups such that $P_0 \leq \Phi(P)$, and such that P_i is normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$ for each $0 \leq i \leq m$. Assume also that $[P, P_i] \leq P_{i-1}$ for each $1 \leq i \leq m$.*

(a) *If $|N_S(P)/P| = p$, then there is at least one index $i = 1, \ldots, m$ such that $\mathrm{rk}(P_i/P_{i-1}) \geq 2$, and such that the image of $\mathrm{Aut}_{\mathcal{F}}(P)$ in $\mathrm{Aut}(P_i/P_{i-1})$ has a strongly p-embedded subgroup.*

(b) *If $|N_S(P)/P| \geq p^2$, then there is at least one index $i = 1, \ldots, m$ such that $\mathrm{rk}(P_i/P_{i-1}) \geq 4$. If there is a unique such index i, then the image of $\mathrm{Aut}_{\mathcal{F}}(P)$ in $\mathrm{Aut}(P_i/P_{i-1})$ has a strongly p-embedded subgroup.*

Proof. Fix $i = 1, \ldots, m$. Since $[P, P_i] \leq P_{i-1}$, the homomorphism $\mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}(P_i/P_{i-1})$ induced by restriction to P_i contains $\mathrm{Inn}(P)$ in its kernel, and hence factors through a homomorphism $\varphi_i\colon \mathrm{Out}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}(P_i/P_{i-1})$. Set $K_i = \mathrm{Ker}(\varphi_i) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(P)$.

Assume that $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ is such that its class $[\alpha] \in \mathrm{Out}_{\mathcal{F}}(P)$ lies in $\bigcap_{i=1}^m K_i$. Thus $[\alpha, P_i] \leq P_{i-1}$ for each i, so α has p-power order by Lemma B.5 and since $P_0 \leq \Phi(P)$. So $\bigcap_{i=1}^m K_i$ is a normal p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$. Since $\mathrm{Out}_{\mathcal{F}}(P)$ has a strongly p-embedded subgroup (recall $P \in \mathbf{E}_{\mathcal{F}}$), we have $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ (recall $O_p(-)$ is contained in all Sylow p-subgroups), and hence $\bigcap_{i=1}^m K_i = 1$. We are thus in the situation of Lemma B.3.

Recall that $N_S(P)/P \cong \mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$. As in Lemma B.3, let J be the set of all $i = 1, \ldots, m$ such that $|K_i|$ is prime to p, and set $K_J = \bigcap_{j\in J} K_j$. By Lemma B.3(a), $J \neq \varnothing$ and $\mathrm{Out}_{\mathcal{F}}(P)/K_J$ contains a strongly p-embedded subgroup.

Without loss of generality, in both points (a) and (b), we can assume that the filtration by the P_i is maximal. Thus each quotient P_i/P_{i-1} is elementary abelian, and the action of $\mathrm{Out}_{\mathcal{F}}(P)$ on it is irreducible.

(a) If $|\mathrm{Out}_S(P)| = p$, then by Lemma B.3(b), there is $j \in J$ such that $\mathrm{Im}(\varphi_j) \cong \mathrm{Out}_{\mathcal{F}}(P)/K_j$ contains a strongly p-embedded subgroup.

(b) Now assume $|\mathrm{Out}_S(P)| \geq p^2$. Recall that the action of $\mathrm{Out}_{\mathcal{F}}(P)$ on P_j/P_{j-1} is irreducible for each $j \in J$. So $\mathrm{rk}(P_j/P_{j-1}) \geq 4$ for each $j \in J$ by Lemma B.4. In particular, if there is a unique i such that $\mathrm{rk}(P_i/P_{i-1}) \geq 4$, then $|J| = 1$, and $\mathrm{Out}_{\mathcal{F}}(P)/K_j$ has a strongly p-embedded subgroup for $j \in J$. □

The next lemma provides another way to show that certain subgroups of a p-group S cannot be essential in any fusion system over S.

Lemma B.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Assume $P < S$ and $T \leq \mathrm{Aut}_S(P)$ are such that $|T/(T \cap \mathrm{Inn}(P))| \geq p^2$ and $[P : C_P(T)] = p$. Then $P \notin \mathbf{E}_{\mathcal{F}}$.*

Proof. Assume otherwise: assume P is $\mathcal{F}$-essential. Set $G = \mathrm{Out}_{\mathcal{F}}(P)$, and set $\overline{T} = T{\cdot}\mathrm{Inn}(P)/\mathrm{Inn}(P) \le \mathrm{Out}_S(P)$. Thus $|\overline{T}| \ge p^2$ by assumption. Let $H < G$ be a strongly p-embedded subgroup that contains $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(G)$. Fix $g \in G \smallsetminus H$, and set $K = \langle T, {}^gT\rangle \le \mathrm{Aut}_{\mathcal{F}}(P)$. Since H is strongly p-embedded and $g \notin H$, no p-subgroup of G can intersect nontrivially with both $\overline{T}$ and ${}^g\overline{T}$, and in particular,

$$\text{either} \quad O_p(K) \cap T \le \mathrm{Inn}(P) \quad \text{or} \quad O_p(K) \cap {}^gT \le \mathrm{Inn}(P). \tag{B.8}$$

By assumption, $C_P(T)$ has index p in P, and so does $C_P({}^gT)$. If $C_P(T) = C_P({}^gT)$, then K is an abelian p-group, contradicting (B.8). So $C_P(K) = C_P(T) \cap C_P({}^gT)$ has index p^2 in P, and $P/C_P(K) \cong E_{p^2}$. The group of elements of K that induce the identity on $P/C_P(K)$ is a p-group by Lemma B.5, and hence contained in $O_p(K)$. Since $p^2 \nmid |GL_2(p)|$, we have $[T : O_p(K) \cap T] \le p$, and since $|\overline{T}| \ge p^2$, this implies $O_p(K) \cap T \nleq \mathrm{Inn}(P)$. But $O_p(K) \cap {}^gT \nleq \mathrm{Inn}(P)$ by a similar argument, this again contradicts (B.8), and so P cannot be $\mathcal{F}$-essential. □

The next lemma gives yet another simple criterion for a subgroup not to be essential. Again, $\Phi(-)$ denotes the Frattini subgroup.

Lemma B.9. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and fix $P \le S$. Assume there are subgroups $P_0 \trianglelefteq P_1 \trianglelefteq \cdots \trianglelefteq P_k = P$, all normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$, such that $P_0 \le \Phi(P)$. Assume also there is $x \in N_S(P) \smallsetminus P$ such that $[x, P_i] \le P_{i-1}$ for each $1 \le i \le k$. Then $P \notin \mathbf{E}_{\mathcal{F}}$.*

Proof. By Lemma B.5 and since $P_0 \le \Phi(P)$, the group Γ of all $\alpha \in \mathrm{Aut}(P)$ such that $[\alpha, P_i] \le P_{i-1}$ for $1 \le i \le k$ is a p-subgroup of $\mathrm{Aut}(P)$, and $\Gamma \cap \mathrm{Aut}_{\mathcal{F}}(P)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(P)$ since the P_i are normalized by $\mathrm{Aut}_{\mathcal{F}}(P)$. So $c_x \in O_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and either $c_x \in \mathrm{Inn}(P)$, in which case $x \in PC_S(P) \smallsetminus P$ and hence P is not $\mathcal{F}$-centric, or $O_p(\mathrm{Out}_{\mathcal{F}}(P)) \ne 1$, in which case $\mathrm{Out}_{\mathcal{F}}(P)$ has no strongly p-embedded subgroup (since $O_p(-)$ is contained in all Sylow p-subgroups). In either case, $P \notin \mathbf{E}_{\mathcal{F}}$. □

We finish by listing the subgroups of $SL_4(p)$ that have strongly p-embedded subgroups and order a multiple of p^2. We indicate how to arrange the proof so as to be independent of the classification of finite simple groups.

Proposition B.10. *Fix an odd prime p, let V be a 4-dimensional vector space over $\mathbb{F}_p$, and let $H < G \le \mathrm{Aut}(V)$ be such that $p^2 \mid |G|$ and H is strongly p-embedded in G. Set $G_0 = O^{p'}(G)$. Then either $G_0 \cong SL_2(p^2)$ and V is its natural module, in which case each element of order p in G_0 acts on V with two Jordan blocks of length 2; or $G_0 \cong PSL_2(p^2)$ and V is the natural $\Omega_4^-(p)$-module, in which case each element of order p in G_0 acts on V with Jordan blocks of lengths 1 and 3.*

Proof. By Aschbacher's theorem [A2], applied to the finite simple classical group $PSL_4(p)$, either G is contained in a member of one of the "geometric" classes $\mathscr{C}_i$ $(1 \le i \le 8)$ defined in [A2], or the image of G in $\mathrm{Aut}(V)/Z(\mathrm{Aut}(V)) \cong PGL_4(p)$ is almost simple.

By Lemma B.1, $G_0 = O^{p'}(G)$ also has a strongly p-embedded subgroup.

Case 1: Assume G is contained in a member of Aschbacher's class $\mathscr{C}_k$, for some $1 \le k \le 8$. Since $\mathbb{F}_p$ has no proper subfields, the class $\mathscr{C}_5$ is empty.

If $k = 1$ or $k = 2$, then G_0 acts reducibly on V, contradicting Lemma B.6(b).

If $k = 3$, then G_0 is contained in $SL_2(p^2)$ (where V is the natural module). Since $SL_2(p^2)$ is generated by any two of its Sylow p-subgroups (and since they have order p^2), G_0 cannot be a proper subgroup of $SL_2(p^2)$.

If $k = 4$ or $k = 7$, then the restriction of V to G_0 splits as a tensor product of 2-dimensional representations, and G_0 is isomorphic to a subgroup of $SL_2(p)\circ SL_2(p)$. By Lemma B.2(b), the image of G_0 in $PSL_2(p)\times PSL_2(p)$ has a strongly p-embedded subgroup. But this contradicts Lemma B.3(a), applied with K_i the kernels of the two projections to $PSL_2(p)$.

The class $\mathscr{C}_6$ consists of the normalizers of $K \cong 2^{1+4}_{\pm}$ (if $p \equiv 3 \pmod 4$), or that of $K \cong C_4 \circ 2^{1+4}$ (if $p \equiv 1 \pmod 4$). Thus $\mathrm{Out}(K) \cong \Sigma_3 \wr C_2$, Σ_5, or Σ_6, respectively. If $k = 6$, then since $p^2 \mid |G|$, we have $p = 3$ and $K \cong 2^{1+4}_{+}$, so G_0 is a subgroup of $SL_2(3)\circ SL_2(3)$, and G is contained in a member of $\mathscr{C}_7$.

Assume $k = 8$. The class $\mathscr{C}_8$ consists of the normalizers of $Sp_4(p)$, $\Omega_4^+(p) \cong SL_2(p)\circ SL_2(p)$, and $\Omega_4^-(p) \cong PSL_2(p^2)$. The symplectic group $Sp_4(p)$ is generated by the two parabolic subgroups that contain S, each of which would be contained in a strongly p-embedded subgroup if there were one. So $G \not\cong Sp_4(p)$, and the proper subgroups of this group are eliminated by again applying Aschbacher's theorem using similar arguments. The subgroup $SO_4^+(p)$ is in class $\mathscr{C}_7$. This leaves the case $G_0 \le \Omega_4^-(p) \cong PSL_2(p^2)$ (see [Ar, Théorème 5.21] or [Ta, Corollary 12.43]), with equality since $PSL_2(p^2)$ is generated by any two of its Sylow p-subgroups.

Case 2: It remains to check the cases where the image in $PGL_4(p)$ of G is almost simple, and show that none of them (aside from those already listed) have strongly p-embedded subgroups. By Tables 8.9 and 8.13 in [BHR], the only almost simple groups that could appear in this way as *maximal* subgroups of $SL_4(p)$ are normalizers of $L_2(7)$ or A_7 (if $p \equiv 1, 2, 4 \pmod 7$), or $U_4(2)$ (if $p \equiv 1 \pmod 6$) in $L_4(p)$, or A_6, A_7 (if $p = 7$), $L_2(p)$ (if $p > 7$) in $Sp_4(p)$. None of these subgroups can occur when $p = 3$, which is the only odd prime whose square can divide the order of the subgroup, so they and their subgroups do not come under consideration.

The tables in [BHR] were made using the classification of finite simple groups. But lists of maximal subgroups of $PSL_4(q)$ and $PSp_4(q)$ for odd q, compiled independently of the classification, had already appeared in [Mi] for the symplectic case, and in [Bl, Chapter VII] and the main theorems in [ZS, Su] for the linear case.

Elements of order p: The description of the Jordan blocks for the natural action of $SL_2(p^2)$ is clear. So assume V is the natural module for $G_0 = \Omega_4^-(p) \cong PSL_2(p^2)$. The isomorphism extends to an isomorphism $GO_4^-(p) \cong P\Gamma L_2(p^2)$ between automorphism groups, so all elements of order p in G_0 have similar actions on V. Hence it suffices to describe the action of one element t of order p in $\Omega_3(p) \le \Omega_4^-(p)$. The action of $\Omega_3(p)$ on $\mathbb{F}_p^3$ is induced by the conjugation action of $PSL_2(p)$ on the additive group $M_2^0(\mathbb{F}_p)$ of (2×2)-matrices of trace 0 (see, e.g., [LO, Proposition A.5]), and using this one easily checks that t acts with one Jordan block of length 3. □

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