Around Brouwer's theory of fixed point free planar homeomorphisms

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RESUME: On expose tout d'abord la théorie de Brouwer. De façon générale, cette théorie montre que toute forme de récurrence pour un homéomorphisme préservant l'orientation du plan $\mathbb{R}^2$ implique l'existence d'un point fixe. Nous commencons par montrer que la présence d'une orbite périodique impose l'existence d'un point fixe. Nous prouvons ensuite un résultat plus fort, le théorème de translation plane, affirmant que si h est un homéomorphisme de $\mathbb{R}^2$ sans point fixe et préservant l'orientation alors on peut recouvrir le plan par des ouverts simplement connexes invariants où la dynamique de h est celle d'une simple translation.

Par la suite, nous donnons des résultats analogues pour les homéomorphismes de la sphère $\mathbb{S}^2$ qui renversent l'orientation. Dans ce cadre, l'absence de point 2-périodique interdit à peu près toute forme de récurrence. Plus précisement, si un tel homéomorphisme h est sans point 2-périodique, alors on peut recouvrir le complément des points fixes par des ouverts invariants où la dynamique est celle de $(x, y) \mapsto (x + 1, -y)$ ou de $(x, y) \mapsto \frac{1}{2}(x, -y)$.

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Chapter 1

Brouwer's theory

Notations

We write respectively $Cl(X)$, $Int(X)$ and ∂X for the closure, the interior and the frontier of a subset X of the 2-sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. If $X \subset Y \subsetneqq \mathbb{S}^2$ then $Cl_Y(X)$, $Int_Y(X)$ and $\partial_Y X$ are the closure, the interior and the frontier of X with respect to Y. Finally $\pi_0(X)$ denotes the set of all connected components of X.

1.1 Translations arcs

Definition 1.1 *Let f be a homeomorphism of the 2-sphere $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. An arc $\alpha \subset \mathbb{S}^2$ is said to be* a translation arc for a f if

1. one of its two endpoints, say p, is mapped by f onto the other one,
2. we have furthermore $\alpha \cap f(\alpha) = \{p, f(p)\} \cap \{f(p), f^2(p)\}$.

Note that $\mathrm{Fix}(f)$ is disjoint from $\bigcup_{k\in\mathbb{Z}} f^k(\alpha)$. We make the convention that the arcs $f^k(\alpha)$ are oriented from $f^k(p)$ to $f^{k+1}(p)$ ($k \in \mathbb{Z}$). Of course α could also be thought of as a translation arc for f^{-1} and the arcs $f^k(\alpha)$ would be then oriented from $f^{k+1}(p)$ to $f^k(p)$. This definition can also be used for planar homeomorphisms since a homeomorphism f of $\mathbb{R}^2$ can be extended to $\mathbb{S}^2$ by letting $f(\infty) = \infty$.

Lemma 1.2 *For any orientation preserving homeomorphism h of $\mathbb{R}^2$ and any point $m \in \mathbb{R}^2 \setminus \mathrm{Fix}(h)$ there exists a translation arc $\alpha \subset \mathbb{R}^2$, say with endpoints $p, h(p)$, such that $m \in \alpha \setminus \{p, h(p)\}$.*

Proof. Let U be the connected component of $\mathbb{R}^2 \setminus \mathrm{Fix}(h)$ which contains m. Since h preserves the orientation we have $h(U) = U$ ([BK]) and, since an open connected subset of $\mathbb{S}^2$ is arcwise connected, there exists an arc γ lying in U with endpoints m and $h(m)$. We can slightly enlarge this arc γ and obtain a topological closed disc Δ such that $\gamma \subset Int(\Delta) \subset \Delta \subset U$. For any $R > 0$, let us denote by D_R the closed disc in $\mathbb{R}^2$ with center the origin $o = (0,0)$ and radius R. Up to conjugacy in $\mathbb{R}^2$, we can suppose that $m = o$ and $\Delta = D_1$. Thus there exists $R_1 \in (0,1)$ such that

$$\partial D_{R_1} \cap h(\partial D_{R_1}) = D_{R_1} \cap h(D_{R_1}) \neq \emptyset.$$

Let $p \in \partial D_{R_1}$ be such that $h(p) \in \partial D_{R_1}$. Then any arc α from p to $h(p)$ satisfying $o \in \alpha \setminus \{p, h(p)\} \subset Int(D_{R_1})$ (see Fig. 1.1) is as required, possibly with $p = h^2(p)$. ■

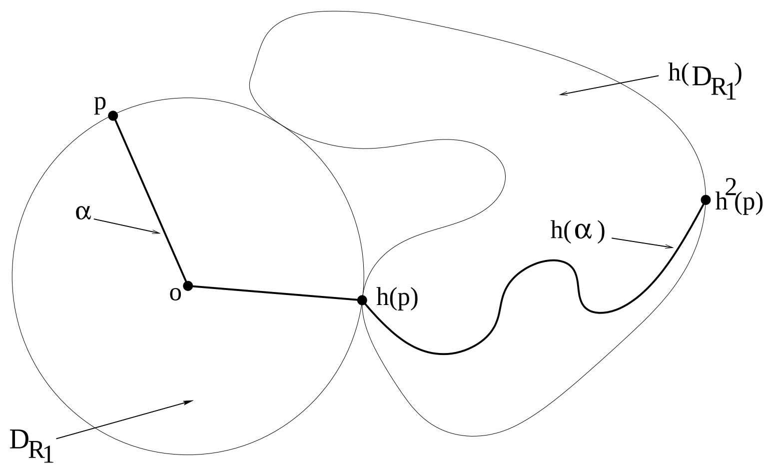

Figure 1.1: The translation arc α

1.2 Brouwer's Lemma

Proposition 1.3 (Brouwer's Lemma) *Let h be an orientation preserving homeomorphism of $\mathbb{R}^2$. Assume that we can find a translation arc α for h such that $\bigcup_{k\in\mathbb{Z}} h^k(\alpha)$ is not a simple curve. Then there exists a Jordan curve $J \subset \mathbb{R}^2$ such that* $\mathrm{Ind}(h, J) = 1$.

The following proof is due to M. Brown ([Brow]). One can also read [Brou, Fa, G1, Ke1].

Proof. We write $p, h(p)$ for the two endpoints of α. It is convenient to define an integer $n \geq 1$ and a point $x \in h^n(\alpha)$ as follows:

- if $p = h^2(p)$ then $n = 1$ and $x = p = h^2(p)$,
- if $p \neq h^2(p)$ then n is the least integer ≥ 2 such that $h^n(\alpha) \cap \alpha \neq \emptyset$ and x is the first point on $h^n(\alpha)$ to meet α.

With these notations the set

$$J = [x, h(p)]_\alpha \cup \big(\bigcup_{i=1}^{n-1} h^i(\alpha) \big) \cup [h^n(p), x]_{h^n(\alpha)}$$

is a Jordan curve (we have simply $J = \alpha \cup h(\alpha)$ is $n = 1$). We write $h_1 \sim h_2$ if and only if h_1, h_2 are two planar homeomorphisms such that $\mathrm{Fix}(h_1) = \mathrm{Fix}(h_2)$ and $\mathrm{Ind}(h_1, J') = \mathrm{Ind}(h_2, J')$ for any Jordan curve J' disjoint from $\mathrm{Fix}(h_1)$. Thus $\sim$ is an equivalence relation and it is enough to prove Proposition 1.3 for a homeomorphism $h_* \sim h$. We first reduce to the situation where $h^n(\alpha)$ meets α in a nice way.

Lemma 1.4 *There exists $h_* \sim h$ possessing $\alpha_* = [x, h(p)]_\alpha$ as a translation arc with $h_*(x) = h(p)$ and such that*

- *for $i \in \{1, \ldots, n-1\}$ $\quad h_*^i(\alpha_*) = h^i(\alpha)$,*
- *$h_*^n(\alpha_*) = [h^n(p), x]_{h^n(\alpha)}$.*

Proof. There is nothing to do if $n = 1$ so we assume $n \geq 2$. The proof is in two steps.
STEP 1: If $x = p$ we rename $g = h$. Otherwise we have $x \neq h(p)$ because of the minimality of n so the arc $[p, x]_\alpha$ has the following properties:

(i) it is disjoint from its image under h,

(ii) it is disjoint from $h^i(\alpha)$ for every $i \in \{1, \ldots, n-1\}$.

On can construct a topological closed disc D_1 which is a neighbourhood of $[p, x]_\alpha$ and so thin that it also satisfies (i)-(ii). There also exists a homeomorphism φ with support in D_1 such that $\varphi(\alpha_*) = \alpha$ and we define $g = h \circ \varphi$. The Alexander trick gives an isotopy $(\varphi)_{0 \leq t \leq 1}$ with support in D_1 from $\varphi_0 = Id_{\mathbb{R}^2}$ to $\varphi_1 = \varphi$. It follows from $D_1 \cap h(D_1) = \emptyset$ that all the homeomorphisms $h \circ \varphi_t$ $(0 \leq t \leq 1)$ have the same fixed point set and consequently $g \sim h$. Moreover we have:

- $g(\alpha_*) = h(\alpha)$ with $g(x) = h(p)$,

- $\forall i \in \{1, \ldots, n\}$ $\quad g^i(\alpha_*) = h^i(\alpha)$ with $g^i(x) = h^i(p)$ because $g = h$ on $\bigcup_{i=1}^{n-1} h^i(\alpha)$.

STEP 2: If $\{x\} = \{g^{n+1}(x)\} = \alpha_* \cap g^n(\alpha_*)$ we just let $h_* = g$. Otherwise we have $x \neq g^n(x) = h^n(p)$ and the arc $[x, g^{n+1}(x)]_{g^n(\alpha_*)}$ satisfies

(iii) it is disjoint from its image under g,

(iv) it is disjoint from $g^i(\alpha_*)$ for every $i = 1, \ldots, n-1$.

Choose a disc neighbourhood D_2 of $[x, g^{n+1}(x)]_{g^n(\alpha_*)}$ which also satisfies (iii)-(iv) and a homeomorphism ψ with support in D_2 such that $\psi(g^n(\alpha_*)) = [g^n(x), x]_{g^n(\alpha_*)}$. As in the first step one check that $h_* = \psi \circ g \sim g$ has the required properties. ■

We can now prove Proposition 1.3. According to Lemma 1.4 there is no loss in supposing $\alpha \cap h^n(\alpha) = \{p\} = \{h^{n+1}(p)\}$, so that $J = \bigcup_{i=0}^{n} h^i(\alpha)$. It is easy to construct an orientation preserving homeomorphism g of $\mathbb{R}^2$ such that

1. $g = h$ on $\bigcup_{i=0}^{n-1}(\alpha)$,
2. g maps $h^n(\alpha)$ on α (hence $g(Cl(int(J))) = Cl(int(J))$).

Thus $g^{-1} \circ h$ is an orientation preserving planar homeomorphism which coincides with $Id_{\mathbb{R}^2}$ on the arc $\bigcup_{i=0}^{n-1} h^i(\alpha)$. A variant of the Alexander trick provides an isotopy $(\varphi_t)_{0 \leq t \leq 1}$ from $\varphi_0 = Id_{\mathbb{R}^2}$ to $\varphi_1 = g^{-1} \circ h$ such that $\varphi_t(z) = z$ for every $t \in [0,1]$ and $z \in \bigcup_{i=0}^{n-1} h^i(\alpha)$. Defining $h_t = g \circ \varphi_t$ $(0 \leq t \leq 1)$ we get an isotopy from $h_0 = g$ to $h_1 = h$ such that $h_t = h$ on $\bigcup_{i=0}^{n-1}(\alpha)$. Consequently all the h_t are fixed point free on $\bigcup_{i=0}^{n-1}(\alpha) \cup h_t\big(\bigcup_{i=0}^{n-1}(\alpha)\big) = J$. Hence we get $\mathrm{Ind}(h, J) = \mathrm{Ind}(g, J)$ and $g(Cl(int(J))) = Cl(int(J))$ implies $\mathrm{Ind}(g, J) = 1$. ■

As a consequence of Lemma 1.2 and of Proposition 1.3 we get

Theorem 1.5 *Let h be an orientation preserving homeomorphism of $\mathbb{R}^2$ possessing a k-periodic point, $k \geq 2$. Then there exists a Jordan curve J such that $\mathrm{Ind}(h, J) = 1$. In particular h has a fixed point in $int(J)$.*

Corollary 1.6 *Let h be an orientation preserving of $\mathbb{R}^2$ without any Jordan curve J such that Ind(h, J) = 1. If a topological closed disc $D \subset \mathbb{R}^2$ satisfies $D \cap h(D) = \emptyset$ then we have $D \cap h^n(D) = \emptyset$ for every $n \neq 0$. In particular the only nonwandering points of h are its fixed points (if any).*

Proof. Suppose that $k \geq 2$ is the smallest positive integer such that $D \cap h^k(D) \neq \emptyset$. Enlarging slightly D if necessary, we can assume $Int(D) \cap h^k(Int(D)) \neq \emptyset$. Pick $z \in Int(D) \cap h^{-k}(Int(D))$ and let φ be a homeomorphism with support in D such that $\varphi(h^k(z)) = z$. Thus z is a k-peridic point of the homeomorphism $g = \varphi \circ h \sim h$, contradicting Theorem 1.5. ■

1.3 Brouwer plane translation theorem

We state two equivalent versions.

Theorem 1.7 (BPTT, version 1) *Let h be a fixed point free orientation preserving homeomorphism of $\mathbb{R}^2$. Every point $m \in \mathbb{R}^2$ belongs to a properly embedded topological line Δ which separates $h^{-1}(\Delta)$ and $h(\Delta)$.*

Theorem 1.8 (BPTT, version 2) *Let h be a fixed point free orientation preserving homeomorphism of $\mathbb{R}^2$. For every point $m \in \mathbb{R}^2$ there exists a topological embedding $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

1. $m \in \varphi(\mathbb{R}^2)$,
2. $\forall x \in \mathbb{R} \quad \varphi(\{x\} \times \mathbb{R})$ *is a closed subset of* $\mathbb{R}^2$,
3. $h \circ \varphi = \varphi \circ \tau$ *where τ is the translation $\tau(x, y) = (x + 1, y)$.*

The proof given below is essentially the one of P. Le Calvez and A. Sauzet ([LS],[S]), based on their notion of *brick decompositions* and on Franks' Lemma. It has been slightly simplified by following an idea of F. Le Roux ([LeR]) which allows to suppress a hypothesis of "transversality" for the brick decompositions appearing in the original work of Le Calvez and Sauzet ([LS]). Others proofs can be found in [Ke1, G1, G3, Fr2].

1.3.1 Franks' Lemma

The following result is due to J. Franks ([Fr1]). It consists roughly in changing a periodic "pseudo-orbit" into a true periodic orbit, without modifying neither the fixed point set nor the Lefchetz index on Jordan curves.

Proposition 1.9 *Let h be an orientation preserving homeomorphism of $\mathbb{R}^2$. Assume that there exists a sequence $D_1, \dots, D_n$ of topological closed discs in $\mathbb{R}^2$ such that*

(i) $\forall i, j \in \{1, \dots, n\} \quad D_i = D_j$ *or* $Int(D_i) \cap Int(D_j) = \emptyset$,

(ii) $\forall i \in \{1, \dots, n\} \quad D_i \cap h(D_i) = \emptyset$,

(iii) $\forall i \in \{1, \dots, n-1\}\, \exists k_i \geq 1$ *such that* $h^{k_i}(D_i) \cap Int(D_{i+1}) \neq \emptyset$ *and* $\exists k_n \geq 1$ *such that* $h^{k_n}(D_n) \cap Int(D_1) \neq \emptyset$.

Then there exits a Jordan curve such that $Ind(h, J) = 1$. In particular $int(J) \cap$ $\mathrm{Fix}(h) \neq \emptyset$.

Remarque: A sequence $D_1, \ldots, D_n$ as above is often called a *periodic chain of discs.*

Proof. Choose a sequence $D_1, \ldots, D_{n_0}$ satisfying (i)-(iii) whose length $n_0 \geq 1$ is minimal among all these sequences. This implies $Int(D_i) \cap Int(D_j) = \emptyset$ for $1 \leq i \neq j \leq n_0$. Remark that if $n_0 = 1$ then $k_1 \geq 2$. We can also assume that the integers $k_1, \ldots, k_{n_0}$ are minimal for property (iii). For simplicity we define $D_{n_0+1} = D_1$ and $U_i = Int(D_i)$ $(1 \leq i \leq n_0 + 1)$. For each $i = 1, \ldots, n_0$ pick a point $x_i \in U_i \cap h^{-k_i}(U_{i+1}) \neq \emptyset$. Because $U_1, \ldots, U_{n_0}$ are pairwise disjoint one can construct a homeomorphism φ with support in $\bigcup_{i=1}^{n_0} D_i$ which preserves setwise each D_i and satisfies $\varphi(h^{k_i}(x_i) = x_{i+1}$ $(i \leq i \leq n_0 - 1)$, $\varphi(h^{k_{n_0}}(x_{n_0}) = x_1$. Moreover for every $i, j \in \{1, \ldots, n_0\}$ we have

$$1 \leq k \leq k_i - 1 \Rightarrow h^k(x_i) \notin D_j.$$

Otherwise this would contradict the minimality of k_1 if $n_0 = 1$ and, if $n_0 \geq 2$, we would get a contradiction with the minimality of n_0 by considering either the subsequence $D_1, \ldots, D_i, D_j, \ldots, D_{n_0}$ or $D_j, \ldots, D_i$. It follows that x_1 is a periodic point of $g = \varphi \circ h$ with period $k_1 + \ldots + k_{n_0} \geq 2$. Finally one deduces from (ii) that $\mathrm{Fix}(h) = \mathrm{Fix}(g)$ and even better, by considering an Alexander isotopy in each D_i, that $g \sim h$ as defined in the proof of Brouwer's Lemma. We conclude with Theorem 1.5. ∎

We will actually use the following version of F. Le Roux ([LeR]).

Lemma 1.10 *If in Proposition 1.9 we replace (iii) with the weaker*

(iii') $\forall i \in \{1, \ldots, n-1\}\, \exists k_i \geq 1$ *such that* $h^{k_i}(D_i) \cap D_{i+1} \neq \emptyset$ *and* $\exists k_n \geq 1$ *such that* $h^{k_n}(D_n) \cap D_1 \neq \emptyset$ *then the conclusion still holds.*

Proof. Let $D_1, \ldots, D_{n_0}$ be a sequence of topological closed discs satisfying (i)-(ii)-(iii') whose lenght $n_0 \geq 1$ is minimal among all these sequences. Writing $D_{n_0+1} = D_1$, let us choose $x_i \in D_i \cap h^{-k_i}(D_{i+1})$ for each $i = 1, \ldots, n_0$. First remark that these points $x_1, \ldots, x_{n_0}$ are distinct because because

$$(1 \leq i, j \leq n_0 \ \text{ and } \ x_i = x_j) \quad \Longrightarrow \quad h^{k_j}(x_i) = h^{k_j}(x_j) \in h^{k_j}(D_i) \cap D_{j+1}$$

and the fact that $D_1, \ldots, D_{n_0}$ has minimal length gives $i = j$.

Now, if we can find $i, j \in \{1, \ldots, n_0\}$ and an integer $k \geq 1$ such that $h^k(x_i) = x_j$ then our lemma is proved. Indeed this implies

$$h^{k_j+k}(x_i) = h^{k_j}(x_j) \in h^{k_j+k}(D_i) \cap D_{j+1}.$$

Again because $D_1, \ldots, D_{n_0}$ has minimal length, this is possible only for $i = j$ and consequently $h^k(x_i) = x_i$. Because of (ii) we have necessarily $k \geq 2$ and we conclude with Theorem 1.5. Thus we can assume without loss of generality

$$(*) \quad \forall i,j \in \{1,\dots,n_0\}\, \forall k \neq 0 \quad h^k(x_i) \neq x_j.$$

For convenience we let $k_0 = k_{n_0}$ and $x_0 = x_{n_0}$. Then we choose for each $i \in \{1,\dots,n_0\}$ an arc γ_i with endpoints x_i and $h^{k_{i-1}}(x_{i-1}) \neq x_i$ lying entirely in $Int(D_i)$ except possibly its endpoints in ∂D_i. Since $\gamma_i \subset D_i$, these arcs possesse the same property (ii) as the discs D_i. Moreover, remembering that the D_i's have disjoint interiors and the x_i's are pairwise distinct ($1 \leq i \leq n_0$), we obtain using $(*)$:

(i') $i \neq j \Longrightarrow \gamma_i \cap \gamma_j = \emptyset$.

By the construction we have also

$$\forall i \in \{1,\dots,n_0-1\} \quad h^{k_i}(\gamma_i) \cap \gamma_{i+1} \neq \emptyset \quad \text{and} \quad h^{k_{n_0}}(\gamma_{n_0}) \cap \gamma_1 \neq \emptyset.$$

One can construct for each $i \in \{1,\dots,n_0\}$ a topological closed disc D'_i neighbourhood of γ_i and so close to γ_i that (i'),(ii) are still true with the D'_i 's in place of the γ_i's. Such a sequence $D'_1,\dots,D'_{n_0}$ then satisfies the conditions (i)-(iii) of Proposition 1.9. ∎

1.3.2 Brick decompositions

This notion is due to P. Le Calvez and A. Sauzet ([LS], [S]). It is also used with some variants in several papers around Brouwer theory of fixed point free planar homeomorphisms (e.g. [Bo],[G2],[LeC1], [LeC2], [LeR]).

Definition 1.11 *A brick decomposition $\mathcal{D}$ of a nonempty open set $U \subset \mathbb{S}^2$ is a collection $\{B_i\}_{i\in I}$ of topological closed discs where I is a finite or countable set and such that*

1. $\bigcup_{i\in I} B_i = U$,
2. *if $i \neq j$ then $B_i \cap B_j$ is either empty or an arc contained in $\partial B_i \cap \partial B_j$,*
3. *for every point $z \in U$, the set $I(z) = \{i \in I \mid z \in B_i\}$ contains at most three elements and $\bigcup_{i\in I(z)} B_i$ is a neighbourhood of z in U.*

The B_i's are called the *bricks* of the decomposition. Of course the set I is finite only for $U = \mathbb{S}^2$ and we will not be concerned with this situation. For a point $z \in U$, the neighbourhood $\bigcup_{i\in I(z)} B_i$ is necessarily of one of the three kinds pictured in Fig. 1.2 (up to a homeomorphism). A brick decomposition of an open set $U \subset \mathbb{S}^2$ can be readily constructed from a triangulation $\mathcal{T}$ of U: for

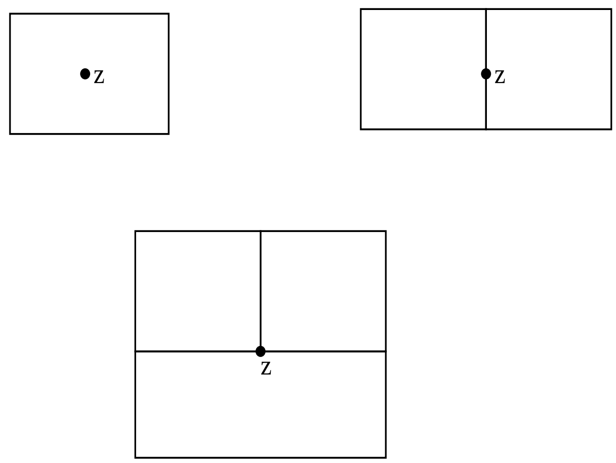

Figure 1.2: The neighbourhood $\bigcup_{i\in I(z)} B_i$ for a point $z \in U$

example, if $\mathcal{T}'$ denotes the barycentric subdivision of $\mathcal{T}$, one can observe that the set

$$\mathcal{D} = \{\mathrm{star}(v, \mathcal{T}') \,|\, v \text{ is a vertex of } \mathcal{T}\}$$

is a brick decomposition of U [recall that $\mathrm{star}(v, \mathcal{T}')$ is the union of the triangles of $\mathcal{T}'$ containing v]. Moreover it is always possible to "subdivide" a brick decomposition in such a way that every brick has diameter less than a given $\epsilon > 0$.

We also have the following property which is one of the main motivation for the use of brick decompositions.

Property 1.12 *Let $\mathcal{D} = \{B_i\}_{i\in I}$ be a brick decomposition of an open set $U \subset \mathbb{S}^2$ and let J be a nonempty subset of I. Then $\bigcup_{i\in J} B_i$ is a closed subset of U. Furthermore $\partial_U(\bigcup_{i\in J} B_i)$ is a 1-dimensional submanifold without boundary of U. In particular, its connected components are homeomorphic either to $\mathbb{S}^1$ or to $\mathbb{R}$.*

Proof. If $z \in Cl_U(\bigcup_{i\in J} B_i)$ it is clear from the definition that $I(z) \cap J \neq \emptyset$ hence $\bigcup_{i\in J} B_i$ is closed in U. Consider now a point $z \in \partial_U(\bigcup_{i\in J} B_i)$. Its neighbourhood $\bigcup_{i\in I(z)} B_i$ contains necessarily two or three bricks, (at least) one of them is in $\{B_i\}_{i\in J}$ and (at least) one of them is not. The result is then obvious with Fig. 1.2. ∎

Attractors and repellers. Let $\mathcal{D} = \{B_i\}_{i\in I}$ be a brick decomposition of an open set $U \subset \mathbb{S}^2$ and let h be a homeomorphism of $\mathbb{S}^2$ such that $h(U) = U$. For a given brick $B_{i_0} \in \mathcal{D}$ we define

$$I_0 = \{i_0\}, \quad \mathcal{A}_0 = \mathcal{R}_0 = \bigcup_{i \in I_0} B_i = B_{i_0}$$

and inductively, for $n \in \mathbb{N}$,

$$I_{n+1} = \{i \in I \,|\, h(\mathcal{A}_n) \cap B_i \neq \emptyset\}, \quad \mathcal{A}_{n+1} = \bigcup_{i \in I_{n+1}} B_i,$$

$$I_{-n-1} = \{i \in I \,|\, h^{-1}(\mathcal{R}_{-n}) \cap B_i \neq \emptyset\}, \quad \mathcal{R}_{-n-1} = \bigcup_{i \in I_{-n-1}} B_i.$$

Definition 1.13 *With the above notations, the two sets*

$$\mathcal{A} = \bigcup_{n \geq 1} \mathcal{A}_n \quad \text{and} \quad \mathcal{R} = \bigcup_{n \geq 1} \mathcal{R}_{-n}$$

are said to be respectively the attractor *and the* repeller *associated to the brick* B_{i_0}.

Note that, according to Property 1.12, $\mathcal{A}$ and $\mathcal{R}$ are closed subsets of U. Furthermore one can check:

Property 1.14 *We have* $h(\mathcal{A} \cup B_{i_0}) \subset Int(\mathcal{A})$. *Consequently* $h^k(\partial_U \mathcal{A}) \cap h^l(\partial_U \mathcal{A}) = \emptyset$ *for any two integers* $k \neq l$ *in* $\mathbb{Z}$.

1.3.3 Proof of BPTT.

We prove the first version (Theorem 1.7). Let α be a translation arc for h with endpoints $p, h(p)$ such that $m \in \alpha \setminus \{p, h(p)\}$. Up to conjugacy in $\mathbb{R}^2$ one can suppose that $h^{-1}(\alpha) = [-1, 0] \times \{0\}$ and $h = \tau$ on $h^{-1}(\alpha) \cup \alpha = [-1, 1] \times \{0\}$ and $m = (3/4, 0)$. For $\epsilon > 0$ we consider the three rectangles (see Fig. 1.3)

$$D_{-1} = \{(x, y) \in \mathbb{R}^2 \,|\, -\frac{1}{4} \leq x \leq \frac{1}{4} \text{ and } -\epsilon \leq y \leq \epsilon\},$$

$$D_0 = \{(x, y) \in \mathbb{R}^2 \,|\, \frac{1}{4} \leq x \leq \frac{3}{4} \text{ and } -\epsilon \leq y \leq \epsilon\},$$

$$D_1 = \{(x, y) \in \mathbb{R}^2 \,|\, \frac{3}{4} \leq x \leq \frac{5}{4} \text{ and } -\epsilon \leq y \leq \epsilon\}.$$

Lemma 1.15 *There exist* $\epsilon > 0$ *and a brick decomposition* $\mathcal{D} = \{B_i\}_{i \in \mathbb{N}}$ *of* $\mathbb{R}^2$ *such that:*

1. D_{-1}, D_0 *and* D_1 *are bricks of* $\mathcal{D}$,

2. *every brick* B_i *satifies* $B_i \cap h(B_i) = \emptyset$.

The details of the construction of $\mathcal{D}$ are omitted. Let $\mathcal{A}, \mathcal{R} \subset \mathbb{R}^2$ be respectively the attractor and the repeller associated to D_0 and h. As a consequence of Lemma 1.10 we have $Int(D_0) \cap \mathcal{A} = \emptyset$ and $Int(\mathcal{R}) \cap \mathcal{A} = \emptyset$. We also have $D_1 \subset \mathcal{A}$ and $D_{-1} \subset \mathcal{R}$ because respectively $h(D_0) \cap D_1 \neq \emptyset$ and $h^{-1}(D_0) \cap D_{-1} \neq \emptyset$. In particular the vertical segment $\{3/4\} \times [-\epsilon, \epsilon]$ is contained in a connected component Δ of $\partial_{\mathbb{R}^2}\mathcal{A}$ which is either a topological line closed in $\mathbb{R}^2$ or a planar Jordan curve (Property 1.12). In both cases Δ separates $\mathbb{R}^2$ in two connected components by Jordan Theorem. It is now enough to see that Δ separates $h^{-1}(\Delta)$ and $h(\Delta)$ and that Δ cannot be a Jordan curve. To do that we give the following

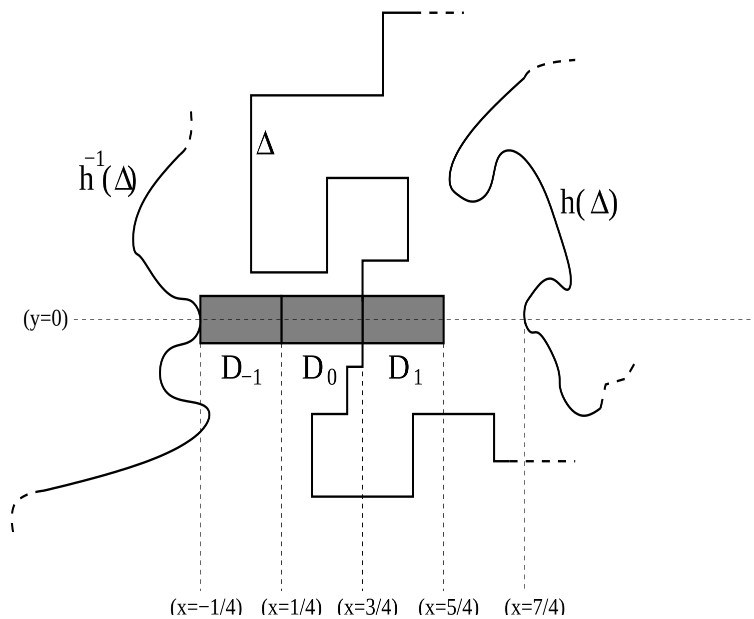

Figure 1.3: The bricks D_0, $D_{\pm 1}$ and Δ, $h^{\pm 1}(\Delta)$ close to these bricks

notations and an elementary but important lemma.

Notations 1.16

$$\gamma_- = \{(x,0) \mid -\frac{1}{4} < x < \frac{3}{4}\},$$
$$\gamma_+ = \{(x,0) \mid \frac{3}{4} < x < \frac{7}{4}\} = h(\gamma_-),$$
$$\gamma = \{(x,0) \mid -\frac{1}{4} < x < \frac{7}{4}\} = \gamma_- \cup \{(\frac{3}{4},0)\} \cup \gamma_+.$$

Lemma 1.17 *The set $h^{-1}(\Delta) \cup \gamma_-$ (resp.$\gamma_+ \cup h(\Delta)$) is connected and contained in $\mathbb{R}^2 \setminus \mathcal{A}$ (resp. in $Int(\mathcal{A})$).*

Proof. For the connectedness, just remark that

$$(-\frac{1}{4},0) \in h^{-1}(\Delta) \cap Cl_{\mathbb{R}^2}(\gamma_-) \quad \text{and} \quad (\frac{7}{4},0) \in Cl_{\mathbb{R}^2}(\gamma_+) \cap h(\Delta).$$

Property 1.14 gives $h(\Delta) \subset h(\mathcal{A}) \subset Int(\mathcal{A})$ and also

$$h^{-1}(\Delta) \cap \mathcal{A} = h^{-1}(\Delta \cap h(\mathcal{A})) \subset h^{-1}(\partial_{\mathbb{R}^2}\mathcal{A} \cap Int(\mathcal{A})) = \emptyset.$$

Moreover we have

$$Int(D_{-1}) \cap \mathcal{A} \subset Int(\mathcal{R}) \cap \mathcal{A} = \emptyset \quad \text{and} \quad Int(D_0) \cap \mathcal{A} = \emptyset$$

hence $\gamma_- \cap \mathcal{A} = \emptyset$.

It remains to check that $\gamma_+ \subset Int(\mathcal{A})$. This follows from

$$\{(x,0) \mid \frac{3}{4} < x < \frac{5}{4}\} \subset Int(D_1) \subset Int(\mathcal{A})$$

and, with Property 1.14, from

$$\{(x,0) \mid \frac{5}{4} \leq x < \frac{7}{4}\} \subset h(D_0) \subset Int(\mathcal{A}).$$

■

We deduce from the last Lemma that Δ separates $h^{-1}(\Delta)$ and $h(\Delta)$ since otherwise the segment γ would intersects Δ tranversely and would meet only one connected component of $\mathbb{R}^2 \setminus \Delta$, which is absurd. Finally we also obtain that Δ is not a Jordan curve since otherwise we would get $h^{\pm 1}(Cl(int(\Delta))) \subset Cl(int(\Delta))$ and h would have a fixed point in $int(\Delta)$. ■

Chapter 2

The case of orientation reversing homeomorphisms

The following results are contained in [Bo].

2.1 Period $k \geq 3$ implies period 2

The aim of this section is to prove the following result, which can be regarded as the counterpart of Theorem 1.5 in the framework of orientation reversing homeomorphisms.

Theorem 2.1 *Let h be an orientation reversing homeomorphism of the sphere $\mathbb{S}^2$ with a point of period at least three. Then h also admits a 2-periodic point. More precisely there exist a Jordan curve $C \subset \mathbb{S}^2 \setminus \mathrm{Fix}(h^2)$ and a point z such that, writing U, U' for the two connected components of $\mathbb{S}^2 \setminus C$, we have:*

$$z = h^2(z) \in U, \quad h(z) \in U' \text{ and } \mathrm{Ind}(h, U) = 0, \quad \mathrm{Ind}(h^2, U) = 1.$$

This result is actually a consequence of Lemma 2.3 and of Propositions 2.6-2.11 below.

2.1.1 Detecting a 2-periodic orbit using Lefschetz index

As is the proof of Theorem 1.5 we will need some "perturbations" of h in order to compute more easily the Lefschetz index on some Jordan domains. The difference is that we deal now with perturbations which do not alter not only the fixed point set of h but also the set of 2-periodic orbits. Thus we introduce a new equivalence relation $\sim$ as follows.

Notations 2.2 *Let f, g be two homeomorphisms of $\mathbb{S}^2$. We write $f \sim g$ if and only if*

1. *they have exactly the same fixed points and the same 2-periodic orbits, i.e. Fix(f^i) = Fix(g^i) for $i = 1, 2$ and $f(z) = g(z)$ for every $z \in \mathrm{Fix}(f^2)$.*
2. *$\forall i = 1, 2 \quad \mathrm{Ind}(f^i, \Omega) = \mathrm{Ind}(g^i, \Omega)$ for any Jordan domain $\Omega \subset \mathbb{S}^2$ such that $\partial\Omega \cap \mathrm{Fix}(f^i) = \emptyset$.*

Clearly Theorem 2.1 will be proved if its conclusion holds for some $g \sim h$. We will show (after replacing h with some suitable $g \sim h$ is necessary) that for a connected component U of $\mathbb{S}^2 \setminus C$, the set $U \cap h(U)$ is a disjoint union of Jordan domains such that $\mathrm{Ind}(h^2, U \cap h(U)) = 0$ (possibly $U \cap h(U) = \emptyset$). In particular $\mathrm{Ind}(h^2, U \cap h(U)) \neq \mathrm{Ind}(h^2, U)$ and the properties of the Lefschetz index then imply $\mathrm{Fix}(h^2) \cap U \cap h(U) \neq \mathrm{Fix}(h^2) \cap U$. In others words there exists a point $z \in U$ such that $h^2(z) = z$ and $h(z) = h^{-1}(z) \in \mathbb{S}^2 \setminus Cl(U)$, as required.

2.1.2 Construction of suitable translation arcs

Lemma 2.3 *Let h be a homeomorphism of $\mathbb{S}^2$ such that $h^2 \neq Id_{\mathbb{S}^2}$ and let m be a point in $\mathbb{S}^2 \setminus Fix(h^2)$. Then at least one of the following two assertions holds:*

A1 : There exists a translation arc α for h, with endpoints p and $h(p)$, such that $\alpha \cap h(\alpha) = \{h(p)\}$, $\alpha \cap h^2(\alpha) = \{p\} \cap \{h^3(p)\}$ and $m \in \alpha \setminus \{p, h(p)\}$,

A2 : There exists a translation arc β for h^2, with endpoints q and $h^2(q)$, such that $\beta \cap h(\beta) = \emptyset$ and $m \in \beta \setminus \{q, h^2(q)\}$.

Proof. Let U be the connected component of $\mathbb{S}^2 \setminus Fix(h^2)$ which contains m. We know that $h^2(U) = U$ ([BK]) hence there exists an arc γ lying in U with endpoints m and $h^2(m)$. We can slightly enlarge this arc γ and obtain a topological closed disc Δ such that $\gamma \subset Int(\Delta) \subset \Delta \subset U$. For any $R > 0$, let us denote by D_R the closed disc in $\mathbb{R}^2$ with center the origin $o = (0, 0)$ and radius R. Up to conjugacy in $\mathbb{S}^2$, we can suppose that $m = o$ and $\Delta = D_1$. Define respectively $R_1 > 0$ and $R_2 > 0$ to be the unique real numbers such that

$$\partial D_{R_1} \cap h(\partial D_{R_1}) = D_{R_1} \cap h(D_{R_1}) \neq \emptyset$$

and

$$\partial D_{R_2} \cap h^2(\partial D_{R_2}) = D_{R_2} \cap h^2(D_{R_2}) \neq \emptyset.$$

Observe that, since $h^2(o) \in Int(D_1) \cap h^2(Int(D_1))$, we have necessarily $R_2 < 1$ so

$$D_{R_2} \subset D_1 \subset \mathbb{S}^2 \setminus \mathrm{Fix}(h^2) \subset \mathbb{S}^2 \setminus \mathrm{Fix}(h).$$

Lemma 2.3 then follows from the comparison of R_1 and R_2:
• If $R_1 \leq R_2$, let us choose a point $p \in \partial D_{R_1}$ such that $h(p) \in \partial D_{R_1}$. Since $D_{R_1} \subset D_{R_2}$, the points $p, h(p), h^2(p)$ are pairwise distinct and any arc α from p to $h(p)$ satisfying $o \in \alpha \setminus \{p, h(p)\} \subset Int(D_{R_1})$ has the properties required in the assertion A1.
• If $R_1 > R_2$, let $q \in \partial D_{R_2}$ such that $h^2(q) \in \partial D_{R_2}$. Choose an arc β from q to $h^2(q) \neq q$ such that $o \in \beta \setminus \{q, h^2(q)\} \subset Int(D_{R_2})$. It is clear that β is an arc as described in the assertion A2 (possibly with $q = h^4(q)$). ■

2.1.3 An index zero lemma

Suppose that $U, V \subset \mathbb{S}^2$ are two Jordan domains such that $V \subset U$, $V \neq U$, $\partial V \cap \partial U$ contains at least two points. Every connected component μ of $\partial V \cap U$ is then an open subarc of ∂V whose endpoints x, y are in ∂U. Then $U \setminus \mu = U' \cup U''$ where U', U'' are two disjoint Jordan domains; moreover we have $\partial U' = \mu \cup \alpha'$ and $\partial U'' = \mu \cup \alpha''$ where α' is one of the two subarcs of ∂U with endpoints x, y and α'' is the other one. Since V is connected and contained in $U \setminus \mu$ we have either $V \subset U'$ or $V \subset U''$.

Notations 2.4 *We write $U_{\mu,V}$ for the connected component of $U \setminus \mu$ which contains V anf μ_* for the subarc of ∂U with endpoints x, y such that $\mu \cup \mu_* = \partial U_{\mu,V}$.*

Then we have

Lemma 2.5 *Let $U, V \subset \mathbb{S}^2$ be two Jordan domains as above and let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous. Assume furthermore that*

(i) f has no fixed point in ∂V,

(ii) $U \cap \partial V \cap f(U) = \emptyset$,

(iii) there exists $\mu \in \pi_0(U \cap \partial V)$ such that $f(\mu_*) \cap U = \emptyset$.

Then we have Ind$(f, V) = 0$.

Proof. Because of (i), Ind(f, V) is defined. Since $\partial U_{\mu,V} = \mu \cup \mu_*$ it is easy to construct a homotopy

$$\begin{array}{ccc} Cl(U_{\mu,V}) \times [0,1] & \to & Cl(U_{\mu,V}) \\ (z,t) & \mapsto & r_t(z) \end{array}$$

with the following properties:

1. r_0 is the identity map of $Cl(U_{\mu,V})$,

2. $r_1(Cl(U_{\mu,V})) = \mu_*$,

3. $\forall t \in [0,1]\, \forall z \in \mu_* \quad r_t(z) = z$,

4. if $0 < t \leq 1$ then $r_t(Cl(U_{\mu,V})) \subset U_{\mu,V} \cup \mu_*$.

Essentially, this simply means that $(r_t)_{0\leq t\leq 1}$ is a strong retracting deformation of $Cl(U_{\mu,V})$ onto μ_*. The additional fourth property ensures that the maps $f \circ r_t$ have no fixed point on ∂V $(0 \leq t \leq 1)$. Indeed there is nothing to prove for $f \circ r_0|_{\partial V} = f|_{\partial V}$ and for $0 < t \leq 1$, $z \in \partial V \subset Cl(U_{\mu,V})$, we have:
- If $z \in \mu_*$ then $f \circ r_t(z) = f(z) \neq z$,
- If $z \in U_{\mu,V} \cup \mu$ then with (4)

$$f \circ r_t(z) \in f(U_{\mu,V}) \cup f(\mu_*)$$

and consequently $z \neq f \circ r_t(z)$ since, using (ii) and (iii),

$$\partial V \cap (U_{\mu,V} \cup \mu) \cap f(U_{\mu,V}) \subset \partial V \cap U \cap f(U) = \emptyset$$

and

$$\partial V \cap (U_{\mu,V} \cup \mu) \cap f(\mu_*) \subset U \cap f(\mu_*) = \emptyset.$$

Moreover we have

$$f \circ r_1(V) \cap V \subset f \circ r_1(Cl(U_{\mu,V})) \cap U = f(\mu_*) \cap U = \emptyset$$

which gives $\mathrm{Ind}(f \circ r_1, V) = 0$. We conclude by using the homotopy invariance property of the Lefschetz index with

$$\begin{array}{ccc} Cl(V) \times [0,1] & \to & \mathbb{S}^2 \\ (z,t) & \mapsto & f \circ r_t(z) \end{array}.$$

$\blacksquare$

2.1.4 A proposition about translation arcs of h

Proposition 2.6 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$. Assume that we can find a translation arc α for h, say with endpoints $p, h(p)$, such that:*

- $\alpha \cap h(\alpha) = \{h(p)\}, \quad \alpha \cap h^2(\alpha) = \{p\} \cap \{h^3(p)\}$,
- *the set $\bigcup_{k\in\mathbb{Z}} h^k(\alpha)$ is not a simple curve.*

Then there exist a Jordan curve C and a point z as announced in Theorem 2.1.

This proposition is a consequence of the following lemmas.

Lemma 2.7 *Let h, α be as in Proposition 2.6. Define n to be the minimun of the set $\{k \geq 2 | \alpha \cap h^k(\alpha) \neq \emptyset\}$ and x to be the first point on $h^n(\alpha)$ to fall in α. Then there exists an orientation reversing homeomorphism $h_* \sim h$ admitting $\alpha_* = [x, h(p)]_\alpha$ as a translation arc such that $h_*(x) = h(p)$ and*

- $\forall i \in \{1, \ldots, n-1\} \quad h_*^i(\alpha_*) = h^i(\alpha)$,
- $h_*^n(\alpha_*) = [h^n(p), x]_{h^n(\alpha)}$.

Proof. Mimic the proof of Lemma 1.4 observing that the support D_1 (resp. D_2) of the homeomorphism φ (resp. ψ) can be choosen such that $h^k(D_1) \cap D_1 = \emptyset$ (resp. $g^k(D_2) \cap D_2 = \emptyset$) for both $k = 1$ and $k = 2$. Using an Alexander isotopy in each D_i, this is easily seen to imply $h \sim g = h \circ \varphi \sim h_* = \psi \circ g$. ■.

Lemma 2.8 *Let h, α, n be as in Lemma 2.7. We assume furthermore that $\alpha \cap h^n(\alpha) = \{p\} = \{h^{n+1}(p)\}$ and we consider the Jordan curve $C = \bigcup_{i=0}^n h^i(\alpha)$. If U is a connected component of $\mathbb{S}^2 \backslash C$, then we have Ind$(h, U)$ = 0 and Ind(h^2, U) = 1.*

Proof. Consider an orientation reversing homeomorphism g of $\mathbb{S}^2$ possessing the following properties:

1. $g = h$ on $\bigcup_{i=0}^{n-1} h^i(\alpha)$,
2. g maps $h^n(\alpha)$ onto α (hence $g(C) = C$),
3. g interchanges the two connected components of $\mathbb{S}^2 \setminus C$.

Thus $g^{-1} \circ h$ is an orientation preserving homeomorphism of the sphere which coincides with the identity map $Id_{\mathbb{S}^2}$ on the arc $\bigcup_{i=0}^{n-1} h^i(\alpha)$. As in the proof of Theorem 1.5, one can find an isotopy $(\varphi_t)_{0 \leq t \leq 1}$ from $\varphi_0 = Id_{\mathbb{S}^2}$ to $\varphi_1 = g^{-1} \circ h$ such that

$$\forall t \in [0,1] \, \forall z \in \bigcup_{i=0}^{n-1} h^i(\alpha) \quad \varphi_t(z) = z.$$

Defining $h_t = g \circ \varphi_t$ $(0 \leq t \leq 1)$, we obtain an isotopy from $h_0 = g$ to $h_1 = h$ such that $h_t = h$ on $\bigcup_{i=0}^{n-1} h^i(\alpha)$ and $(h_t^2)_{0 \leq t \leq 1}$ is then an isotopy from g^2 to h^2 such that $h_t^2 = h^2$ on $\bigcup_{i=0}^{n-2} h^i(\alpha)$. Clearly h^2 has no fixed point on α and then also on $\bigcup_{i \in \mathbb{Z}} h^i(\alpha)$. Consequently, for every $t \in [0,1]$, the homeomorphism h_t^2 (and so h_t) has no fixed point on

$$\bigcup_{i=0}^{n-2} h^i(\alpha) \cup h_t\big(\bigcup_{i=0}^{n-2} h^i(\alpha)\big) \cup h_t^2\big(\bigcup_{i=0}^{n-2} h^i(\alpha)\big) = \bigcup_{i=0}^{n} h^i(\alpha) = C.$$

Hence all the indices $\mathrm{Ind}(h_t, U)$ and $\mathrm{Ind}(h_t^2, U)$ are defined and we deduce $\mathrm{Ind}(g, U) = \mathrm{Ind}(h, U)$, $\mathrm{Ind}(g^2, U) = \mathrm{Ind}(h^2, U)$. Finally $U \cap g(U) = \emptyset$ gives $\mathrm{Ind}(g, U)$=0 and $U = g^2(U)$ implies $\mathrm{Ind}(g^2, U)$=1. ■

Remarks 2.9 *If in Lemma 2.8 we have $n \geq 3$, then $\alpha \cup h(\alpha)$ is a translation arc for the orientation preserving homeomorphism h^2 and Brouwer's lemma gives directly Ind(h^2, U)=1.*

Lemma 2.10 *Let h, α, n and C be as in Lemma 2.8. Then there exists a connected component U of $\mathbb{S}^2 \setminus C$ such that every connected component of $U \cap h(U)$ is a Jordan domain (if any) and* $\mathrm{Ind}(h^2, U \cap h(U)) = 0$.

Proof. Let U_1 and $U_2 = \mathbb{S}^2 \setminus Cl(U_1)$ be the two connected components of $\mathbb{S}^2 \setminus C$. We can assume $U_i \cap h(U_i) \neq \emptyset$ for both $i = 1$ and $i = 2$ since otherwise the result is obvious. Choose for example $U = U_1$. First we remark that every connected component V of $U \cap h(U)$ is a Jordan domain such that $\partial V \subset \partial U \cup h(\partial U)$. This is a straightforward consequence of a result of Kerékjártó: pick any point

$$z_\infty \in U_2 \cap h(U_2) = (\mathbb{S}^2 \setminus Cl(U)) \cap (\mathbb{S}^2 \setminus Cl(h(U))) \neq \emptyset$$

and any homeomorphism of $\mathbb{S}^2$ such that $\varphi(z_\infty) = \infty$ so that we reduce to the situation of Proposition 3.4 by considering the Jordan curves $\varphi(\partial U)$ and $\varphi(h(\partial U))$.

It suffices now to prove that $\mathrm{Ind}(h^2, V) = 0$ for any given $V \in \pi_0(U \cap h(U))$. To do that we check that we are exactly in the situation discribed in Lemma 2.5. Since h reverses the orientation, every point $z \in C \setminus h^n(\alpha)$ admits a neighbourhood N_z such that $h(N_z \cap U) = h(N_z) \cap U_2$ and $h(N_z \cap U_2) = h(N_z) \cap U$. Consequently we have $(C \setminus \alpha) \cap Cl(U \cap h(U)) = \emptyset$. In particular this shows $h^{\pm 1}(U) \not\subset U$ and we obtain the following properties for every $V \in \pi_0(U \cap h(U))$:

(1) $V \subset U$ with $V \neq U$,

(2) V is a Jordan domain such that $\partial V \subset \alpha \cup h^{n+1}(\alpha)$,

(3) $\partial V \cap C$ contains at least two points.

The first one is clear since $U \not\subset h(U)$. We know that V is a Jordan domain such that $\partial V \subset C \cup h(C) = \bigcup_{i=0}^{n+1} h^i(\alpha)$ and, since $Cl(V) \subset Cl(U \cap h(U))$ is disjoint from $C \setminus \alpha$, we obtain more precisely $\partial V \subset \alpha \cup h^{n+1}(\alpha)$. The third property follows since otherwise we would have

$$\partial V = Cl(\partial V \setminus C) \subset h^{n+1}(\alpha)$$

which is absurd because an arc cannot contain a Jordan curve.

Because of $C\setminus\alpha \subset C\setminus Cl(V)$, a point $a \in U$ close enough to $C\setminus\alpha$ is separated from V, inside U, by a connected component μ of $U \cap \partial V \subset h^{n+1}(\alpha)$. Using the notations $U_{\mu,V}$ and μ_* introduced for Lemma 2.5 we have then $\partial U_{\mu,V} = \mu \cup \mu_*$ with $\mu_* \subset \alpha$. We obtain finally $\mathrm{Ind}(h^2, V) = 0$ applying Lemma 2.5 with $f = h^2$ because

$$\begin{aligned} &U \cap \partial V \cap h^2(U) \subset h^{n+1}(\alpha) \cap h^2(U) = h^2(h^{n-1}(\alpha) \cap U) = \emptyset, \\ &h^2(\mu_*) \cap U \subset h^2(\alpha) \cap U = \emptyset. \end{aligned}$$

■

Proof of Proposition 2.6: We consider the integer $n \geq 2$ and the point $x \in h^n(\alpha)$ defined in Lemma 2.7. The set

$$C = [x, h(p)]_\alpha \bigcup_{i=1}^{n-1} h^i(\alpha) \cup [h^n(p), x]_{h^n(\alpha)}$$

is then a Jordan curve. If necessary we can replace h,α with h_*,α_* given by Lemma 2.7 so there is no loss in supposing $x = p = h^{n+1}(p)$ and $C = \bigcup_{i=0}^n h^i(\alpha)$. We complete the proof using Lemmas 2.8 and 2.10. ■

2.1.5 A proposition about translation arcs of h^2

Proposition 2.11 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$. Assume that we can find a translation arc β for h^2, with endpoints $q, h^2(q)$, such that*

- *$\beta \cap h(\beta) = \emptyset$,*
- *the sets $\bigcup_{i\in\mathbb{Z}} h^{2i}(\beta)$ and $\bigcup_{j\in\mathbb{Z}} h^{2j+1}(\beta)$ are not two disjoint simple curves.*

Then there exist a Jordan curve C and a point z as announced in Theorem 2.1.

Beginning of the proof of Proposition 2.11: Define an integer $n \geq 2$ and a point $x \in h^n(\beta)$ as follows:

- if $q = h^4(q)$ then $n = 2$ and $x = q = h^4(q)$,
- if $q \neq h^4(q)$ then n is the minimum of the set $\{k \geq 3 | \beta \cap h^k(\beta) \neq \emptyset\}$ and x is the first point on $h^n(\beta)$ to fall in β.

Let us remark that, because of the minimality of n, we have necessarily $x \notin \{h^2(q), h^n(q)\}$. We also note that h^2 (and so h) has no fixed point on $\bigcup_{k\in\mathbb{Z}} h^k(\beta)$. The proof of Proposition 2.11 depends on the parity of n, as explained below.

2.1.5.1 n is even

We consider the set

$$C = [x, h^2(q)]_\beta \bigcup_{2i=2}^{n-2} h^{2i}(\beta) \cup [h^n(q), x]_{h^n(\beta)}.$$

It is a Jordan curve contained in $\bigcup_{2i=0}^{n} h^{2i}(\beta)$ (we have $C = \beta \cup h^2(\beta)$ if $n = 2$). It follows from the minimality of n that

$$\left(\bigcup_{2i=0}^{n} h^{2i}(\beta)\right) \cap \left(\bigcup_{2j+1=1}^{n-1} h^{2j+1}(\beta)\right) = \emptyset.$$

Hence $\bigcup_{2j+1=1}^{n-1} h^{2j+1}(\beta)$ is disjoint from C and, by connectedness, is contained in one of the two connected components U_1, U_2 of $\mathbb{S}^2 \backslash C$, say in U_2. Thus we have also $\bigcup_{2i=2}^{n} h^{2i}(\beta) \subset h(U_2)$. Observe that this implies $h^{\pm 1}(U_1) \not\subset U_1$ and $U_2 \cap h(U_2) \neq \emptyset$. Since β is a translation arc for h^2, Brouwer's lemma gives $\mathrm{Ind}(h^2, U_1)$=1 and $U_1 \cap \mathrm{Fix}(h^2) \neq \emptyset$. We can suppose $U_1 \cap h(U_1) \neq \emptyset$ since otherwise we have $U_1 \cap \mathrm{Fix}(h) = \emptyset$, hence $\mathrm{Ind}(h, U_1) = 0$, and every fixed point z of h^2 in U_1 satisfies $h(z) \in U_2$. We write simply $U = U_1$. As in the proof of Lemma 2.10 one deduces from Proposition 3.4 that every connected component V of $U \cap h(U)$ is a Jordan domain such that $\partial V \subset C \cup h(C)$. Since $Cl(V) \subset Cl(U) \cap Cl(h(U))$ is disjoint from $\bigcup_{k=1}^{n} h^k(\beta)$ we get in fact

$$\partial V \subset [x, h^2(q)]_\beta \cup [h^{n+1}(q), h(x)]_{h^{n+1}(\beta)}.$$

Thus a point $a \in U$ close enough to $C \cap \left(\bigcup_{2i=2}^{n} h^{2i}(\beta)\right)$ is separated from V, inside U, by a connected component μ of $\partial V \cap U \subset [h^{n+1}(q), h(x)]_{h^{n+1}(\beta)}$. Using the notations preceding Lemma 2.5 we have then $\mu_* \subset [x, h^2(q)]_\beta$ (Fig. 2.1). Furthermore, since

$$U \cap \partial V \cap h(U) \subset \partial(U \cap h(U)) \cap U \cap h(U) = \emptyset,$$
$$h(\mu_*) \cap U \subset h(\beta) \cap U = \emptyset,$$

and

$$U \cap \partial V \cap h^2(U) \subset h^{n+1}(\beta) \cap h^2(U) = h^2(h^{n-1}(\beta) \cap U) = \emptyset,$$
$$h^2(\mu_*) \cap U \subset h^2(\beta) \cap U = \emptyset,$$

one can use Lemma 2.5 with successively $f = h$, $f = h^2$ and thus obtain $\mathrm{Ind}(h, V)$ $= 0 = \mathrm{Ind}(h^2, V)$. Since $\mathrm{Fix}(h) \cap U \cap h(U) = \mathrm{Fix}(h) \cap U$ we get from the properties

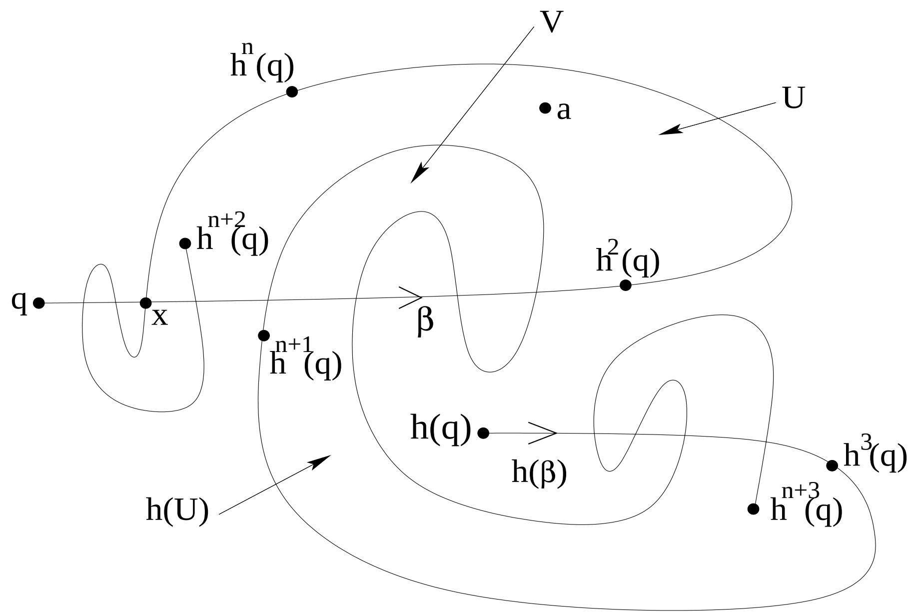

Figure 2.1: The Jordan domains $U, h(U)$ and V

of the Lefschetz index:

$$0 = \sum_{V \in \pi_0(U \cap h(U))} \mathrm{Ind}(h, V) = \mathrm{Ind}(h, U \cap h(U)) = \mathrm{Ind}(h, U),$$

$$0 = \sum_{V \in \pi_0(U \cap h(U))} \mathrm{Ind}(h^2, V) = \mathrm{Ind}(h^2, U \cap h(U)).$$

This proves Proposition 2.11 when n is even.

2.1.5.2 n is odd and $h^{n+1}(\beta) \cap \beta = \emptyset$

We begin with a lemma which plays the same role as Lemma 2.7. Note that the assumption $h^{n+1}(\beta) \cap \beta = \emptyset$ is useless in this proof.

Lemma 2.12 (see Fig. 2.2) *There exists an orientation reversing homeomorphism $h_* \sim h$ such that h_*^2 admits $\beta_* = [x, h^2(q)]_\beta$ as a translation arc with $h_*^2(x) = h^2(q)$ and*

- $h_*(\beta_*) = [h(x), h^3(q)]_{h(\beta)}$,
- $\forall i \in \{2, \ldots, n-1\} \quad h_*^i(\beta_*) = h^i(\beta)$,

- $h_*^n(\beta_*) = [h^n(q), x]_{h^n(\beta)}$,
- $h_*^{n+1}(\beta_*) = [h^{n+1}(q), h(x)]_{h^{n+1}(\beta)}$.

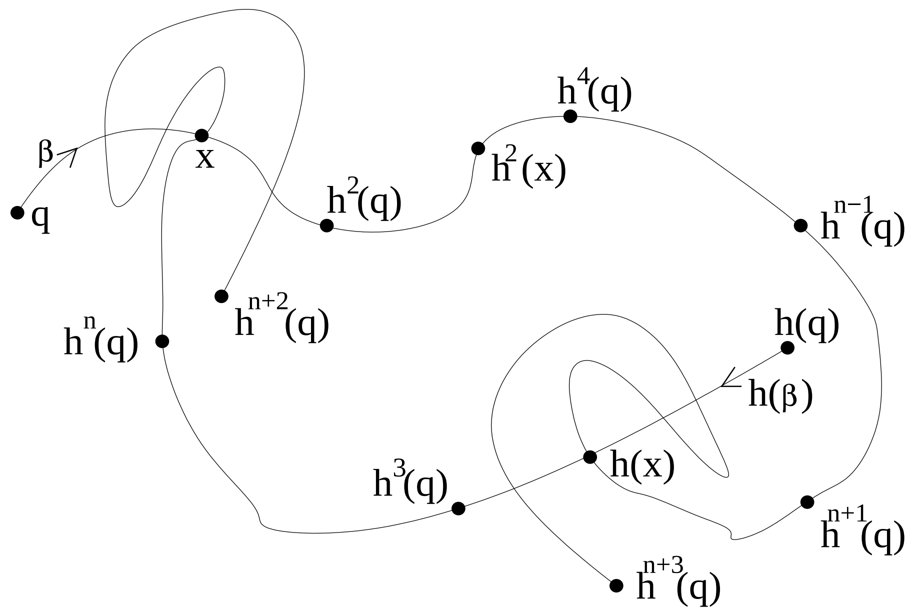

Figure 2.2: The arcs $h^i(\beta)$, $0 \le i \le n+1$

Proof. As in Lemma 2.7 the proof is in two steps;
STEP 1. If $x = q$ we rename $h = g$. Otherwise observe that the arc $[h^2(q), h^2(x)]_{h^2(\beta)}$ has the following properties:

(i) it is disjoint from its images by h and h^2 ,

(ii) it is disjoint from $h^i(\beta)$ for every integer $i \in \{1\} \cup \{3, \dots, n+1\}$.

One can construct a homeomorphism φ of $\mathbb{S}^2$ mapping $[h^2(x), h^4(q)]_{h^2(\beta)}$ onto $h^2(\beta)$ whose support is contained in a topological closed disc D_1 so close to $[h^2(q), h^2(x)]_{h^2(\beta)}$ that it satisfies also (i) and (ii). Defining $g = \varphi \circ h$, we have then $g \sim h$ and $g = h$ on $\beta \cup \bigcup_{i=2}^n h^i(\beta)$, hence $g(\beta_*) = [h(x), h^3(q)]_{h(\beta)}$ with $g(x) = h(x)$ and

$$\forall i \in \{2, \dots, n+1\} \quad g^i(\beta_*) = h^i(\beta) \text{ with } g^i(x) = h^i(p).$$

STEP 2. If $\{x\} = \{g^{n+2}(x)\} = \beta_* \cap g^n(\beta_*)$ it is enough to define $h_* = g$. Otherwise we remark that the arc $[x, g^{n+2}(x)]_{g^n(\beta_*)}$ is disjoint from its images by

g and g^2 and also from the set $\left(\bigcup_{i=1}^{n-1} g^i(\beta_*)\right) \cup g^{n+1}(\beta_*)$. It is possible to have the same for a topological closed disc D_2 containing the support of a homeomorphism ψ of $\mathbb{S}^2$ such that $\psi(g^n(\beta_*)) = [g^n(x), x]_{g^n(\beta_*)}$. Then $h_* = \psi \circ g \sim g$ possesses the announced properties. ■

Continuation of the proof of Proposition 2.11: We consider now the sets

$$\gamma_- = [x, h^2(q)]_\beta \cup \bigcup_{2i=2}^{n-1} h^{2i}(\beta) \cup [h^{n+1}(q), h(x)]_{h^{n+1}(\beta)},$$

$$\gamma_+ = [h(x), h^3(q)]_{h(\beta)} \bigcup_{2j+1=3}^{n-2} h^{2j+1}(\beta) \cup [h^n(q), x]_{h^n(\beta)}$$

and finally $C = \gamma_- \cup \gamma_+$. Keeping in mind that $\beta \cap h^{n+1}(\beta) = \emptyset$, we see that γ_- and γ_+ are two arcs which meet only in their common endpoints $x, h(x)$. Consequently C is a Jordan curve. Replacing h, β with respectively h_*,β_* given by Lemma 2.12, one can suppose that $x = q = h^{n+2}(q)$, that is

$$\gamma_- = \bigcup_{2i=0}^{n+1} h^{2i}(\beta), \quad \gamma_+ = \bigcup_{2j+1=1}^{n} h^{2j+1}(\beta) \text{ and } C = \bigcup_{i=0}^{n+1} h^i(C).$$

Lemma 2.13 *Let U be a connected component of $\mathbb{S}^2 \backslash C$. Then we have Ind($h, U$) = 0 and Ind($h^2, U$) = 1.*

Proof. It is similar to the one of Lemma 2.8. Construct an orientation reversing homeomorphism g of $\mathbb{S}^2$ such that

1. $g = h$ on the set $\bigcup_{i=0}^{n} h^i(\beta)$,
2. g maps $h^{n+1}(\beta)$ onto β (hence $g(C) = C$),
3. g interchanges the two connected components of $\mathbb{S}^2 \setminus C$.

Thus $g^{-1} \circ h$ is an orientation preserving homeomorphism of the sphere which coincides with $Id_{\mathbb{S}^2}$ on the arc $\bigcup_{i=0}^{n} h^i(\beta)$. There exists an isotopy $(\varphi_t)_{0 \le t \le 1}$ from $\varphi_0 = Id_{\mathbb{S}^2}$ to $\varphi_1 = g^{-1} \circ h$ such that

$$\forall t \in [0,1]\, \forall z \in \bigcup_{i=0}^{n} h^i(\beta) \quad \varphi_t(z) = z.$$

Defining $h_t = g \circ \varphi_t$ $(0 \le t \le 1)$, we get an isotopy from g to h such that $h_t = h$ on $\bigcup_{i=0}^{n} h^i(\beta)$ and also an isotopy $(h_t^2)_{0\le t\le 1}$ from g^2 to h^2 such that $h_t^2 = h^2$ on $\bigcup_{i=0}^{n-1} h^i(\beta)$. It follows, for every $t \in [0,1]$, that h_t^2 has no fixed point on

$$\bigcup_{i=0}^{n-1} h^i(\beta) \cup h_t^2\Big(\bigcup_{i=0}^{n-1} h^i(\beta)\Big) = \bigcup_{i=0}^{n+1} h^i(\beta) = C.$$

We get consequently $\mathrm{Ind}(g,U)$=$\mathrm{Ind}(h,U)$ and $\mathrm{Ind}(g^2,U) = \mathrm{Ind}(h^2,U)$. We obtain finally $\mathrm{Ind}(g,U) = 0$ (resp. $\mathrm{Ind}(g^2,U)$=1) because $U \cap g(U) = \emptyset$ (resp. $U = g^2(U)$). ■

Continuation of the proof of Proposition 2.11: Let U_1, U_2 be the two connected components of $\mathbb{S}^2 \setminus C$. According to Lemma 2.13 we have $\mathrm{Ind}(h,U_i)$ $= 0$ and $\mathrm{Ind}(h^2,U_i) = 1$. In particular we have $U_i \cap \mathrm{Fix}(h^2) \neq \emptyset$ $(i \in \{1,2\})$. If one can find $i \in \{1,2\}$ such that $U_i \cap h(U_i) = \emptyset$ then the result is easy. Otherwise we consider for example $U = U_1$. Let V be any connected component of $U \cap h(U)$. Since h reverses the orientation, every point $z \in C \setminus h^{n+1}(\beta)$ possesses a neighbourhood N_z such that $h(N_z \cap U) = h(N_z) \cap U_2$ and $h(N_z \cap U_2) = h(N_z) \cap U$. It follows that $C \setminus \beta$ is disjoint from $Cl(U \cap h(U))$ and in particular from $Cl(V)$. Using one more time Proposition 3.4, we obtain that V is a Jordan domain such that $\partial V \subset \beta \cup h^{n+2}(\beta)$. Hence a point $a \in U$ close to $C \setminus \beta$ is separated from V, inside U, by a connected component μ of $U \cap \partial V \subset h^{n+2}(\beta)$ and the corresponding arc μ_* satisfies $\mu_* \subset \beta$. We have then

$$\partial V \cap U \cap h^2(U) \subset h^{n+2}(\beta) \cap h^2(U) = h^2(h^n(\beta) \cap U) = \emptyset,$$
$$h^2(\mu_*) \cap U \subset h^2(\beta) \cap U = \emptyset,$$

and Lemma 2.5 gives $\mathrm{Ind}(h^2,V) = 0$. Thus we get

$$0 = \sum_{V \in \pi_0(U \cap h(U))} \mathrm{Ind}(h^2,V) = \mathrm{Ind}(h^2, U \cap h(U)).$$

2.1.5.3 n is odd and $h^{n+1}(\beta) \cap \beta \neq \emptyset$

The following remarks allow us to reduce to the two cases studied before. We consider the last point y on β to fall into $h^n(\beta) \cup h^{n+1}(\beta)$. Since $\beta \cap h(\beta) = \emptyset$, y does not belong to $h^n(\beta)$ and $h^{n+1}(\beta)$ simultaneously, and $y \neq q$. We also have $y \neq h^2(q)$ because of the minimality of n. We can then assert:

Lemma 2.14 *There exists an orientation reversing homeomorphism $\hat{h} \sim h$ such that $\hat{h}^2$ admits $\hat{\beta} = [y, h^2(q)]_\beta$ as a translation arc with $\hat{h}^2(y) = h^2(q)$ and*

- $\hat{h}(\hat{\beta}) = [h(y), h^3(q)]_{h(\beta)}$,

- $\forall i \in \{2, \dots, n+1\} \quad \hat{h}^i(\hat{\beta}) = h^i(\beta)$.

Proof. Replace x with y in the construction of the intermediate homeomorphism g in the proof of Lemma 2.12. ■

End of the proof of Proposition 2.11: By the definition of y we have:
- if $y \in h^n(\beta) = \hat{h}^n(\hat{\beta})$ then $\hat{h}^{n+1}(\hat{\beta}) \cap \hat{\beta} = h^{n+1}(\beta) \cap \hat{\beta} = \emptyset$ and we reduce to the situation of Section 2.1.5.2 by replacing h, β with $\hat{h}, \hat{\beta}$.
- if $y \in h^{n+1}(\beta) = \hat{h}^{n+1}(\hat{\beta})$ then $\hat{h}^4(y) = h^4(q) \neq y$ and $n+1$ is the smallest integer $k \in \{3, \dots, n+1\}$ such that $\hat{h}^k(\hat{\beta}) = h^k(\beta)$ intersects $\hat{\beta}$. We reduce to the case treated in Section 2.1.5.1 by replacing h, β and n with $\hat{h}, \hat{\beta}$ and $n+1$. Proposition 2.11 is proved. ■

2.2 An analogue of BPTT

We prove in this section the following result:

Theorem 2.15 *Let h be an orientation reversing homeomorphism of the sphere $\mathbb{S}^2$ without a 2-periodic point. Then for any point $m \in \mathbb{S}^2 \setminus Fix(h)$ there exists a topological embedding $\varphi : \mathcal{O} \to \mathbb{S}^2 \setminus Fix(h)$ such that*

- *$\mathcal{O}$ is either $\mathbb{R}^2$ or $\{(x,y) \in \mathbb{R}^2 | y \neq 0\}$ or $\mathbb{R}^2 \setminus \{(0,0)\}$,*

- *$m \in \varphi(\mathcal{O})$,*

- *if $\mathcal{O} = \mathbb{R}^2$ or $\mathcal{O} = \{(x,y) \in \mathbb{R}^2 | y \neq 0\}$ then*

  (i) $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$ where $G(x,y) = (x+1, -y)$,

  (ii) for every $x \in \mathbb{R}$, $\varphi\big((\{x\} \times \mathbb{R}) \cap \mathcal{O}\big)$ is a closed subset of $\mathbb{S}^2 \setminus Fix(h)$ (it is said that φ is a proper embedding),

- *if $\mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\}$ then*

  (iii) $h \circ \varphi = \varphi \circ H|_{\mathcal{O}}$ where $H(x,y) = \frac{1}{2}(x, -y)$.

Note that, since we are looking for conjugacy outside the fixed point set, the map H in the statement of Theorem 2.15 can be replaced with any map $(x,y) \mapsto \lambda(x, -y)$ where $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$. Moreover, although we use the word *proper* for an embedding φ satisfying the property (ii) above, we do not assert that the backward image $\varphi^{-1}(K)$ of a compact set $K \subset \mathbb{S}^2 \setminus \mathrm{Fix}(h)$ is compact (it is actually not difficult to find an example showing that this property is not true in general). The word "proper" should only be regarded in this text as a convenient abbreviation for "proper on each vertical line $(\{x\} \times \mathbb{R}) \cap \mathcal{O}$".

2.2.1 Some recurrence properties

The next lemma can be regarded as the counterpart of Franks' Lemma in the case of an orientation reversing homeomorphism.

Lemma 2.16 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$. Assume that there exists a finite sequence of topological closed discs $D_1, \ldots, D_n$ satisfying*

(i) $\forall i, j \in \{1, \ldots, n\} \quad D_i = D_j$ *or* $Int(D_i) \cap Int(D_j) = \emptyset$,

(ii) $\forall i \in \{1, \ldots, n\} \quad h(D_i) \cap D_i = \emptyset = h^2(D_i) \cap D_i$,

(iii) $\forall i, j \in \{1, \ldots, n\} \quad D_j$ *meets at most one of the two sets* $h^{-1}(D_i)$ *or* $h(D_i)$,
Equivalently: $h(D_i) \cap D_j \neq \emptyset \Longrightarrow h(D_j) \cap D_i = \emptyset$,

(iv) $\forall i \in \{1, \ldots, n-1\}\ \exists k_i \geq 1$ *such that* $h^{k_i}(D_i) \cap Int(D_{i+1}) \neq \emptyset$ *and* $\exists k_n \geq 1$ *such that* $h^{k_n}(D_n) \cap Int(D_1) \neq \emptyset$.

Then h possesses a 2-periodic point.

Proof. Let us choose a sequence $D_1, \ldots, D_{n_0}$ satisfying (i)-(iv) and whose length n_0 is minimal among all these sequences. Remark that if $n_0 = 1$ then $k_1 \geq 3$. We can also suppose that the integers $k_1, \ldots, k_{n_0}$ are minimal for the property (iv). We also define $D_{n_0+1} = D_1$. We have clearly

$$h^{k_i}(D_i) \cap Int(D_{i+1}) \neq \emptyset \Longleftrightarrow h^{k_i}(Int(D_i)) \cap Int(D_{i+1}) \neq \emptyset$$

so we can choose for every $i \in \{1, \ldots, n_0\}$ a point $x_i \in Int(D_i)$ such that $h^{k_i}(x_i) \in Int(D_{i+1})$. Since the sequence $D_1, \ldots, D_{n_0}$ has minimal length we have

$$1 \leq i \neq j \leq n_0 \Longrightarrow Int(D_i) \cap Int(D_j) = \emptyset$$

so there exists an orientation preserving homeomorphism ψ of $\mathbb{S}^2$ with support in $D_1 \cup \ldots \cup D_{n_0}$ preserving setwise each disc D_i $(1 \leq i \leq n_0)$ and such that

$$\forall i \in \{1, \ldots, n_0 - 1\} \quad \psi(h^{k_i}(x_i)) = x_{i+1}, \qquad \psi(h^{k_{n_0}}(x_{n_0})) = x_1.$$

Furthermore we have for every $i, j \in \{1, \ldots, n_0\}$

$$1 \leq k \leq k_i - 1 \Longrightarrow h^k(x_i) \notin D_j$$

since otherwise the minimality of either k_i or n_0 would be contradicted. Thus the homeomorphism $g = \psi \circ h$ reverses the orientation and possesses x_1 as a periodic point with period $k_1 + \ldots + k_{n_0} \geq 2$. Theorem 2.1 then gives a 2-periodic point for g and it is enough to check that $\mathrm{Fix}(h) = \mathrm{Fix}(g)$ and $\mathrm{Fix}(h^2) = \mathrm{Fix}(g^2)$.

• The first equality follows from the fact that $D_i \cap h(D_i) = \emptyset$ for every $i \in \{1, \ldots, n_0\}$.

• Let us check that $\mathrm{Fix}(g^2) = \mathrm{Fix}(h^2)$.

- First we observe that if $m \in h^{-1}(D_j)$ for an index $j \in \{1, \ldots, n_0\}$ then necessarily $m \neq g^2(m)$: For such a point m we have $g(m) = \psi(h(m)) \in \psi(D_j) = D_j$ so $h(g(m)) \in h(D_j)$. If $h(g(m)) \notin \bigcup_{i=1}^{n_0} D_i$ then $g^2(m) = \psi(h(g(m))) = h(g(m))$ and consequently $m \neq g^2(m)$ since $h^{-1}(D_j) \cap h(D_j) = \emptyset$. If one can find $i \in \{1, \ldots, n_0\}$ such that $h(g(m)) \in D_i$ then we obtain $h(g(m)) \in D_i \cap h(D_j) \neq \emptyset$ and (iii) implies $D_i \cap h^{-1}(D_j) = \emptyset$. Since $g^2(m) = \psi(h(g(m))) \in \psi(D_i) = D_i$, it follows that $g^2(m) \neq m$.

- Secondly we remark that if $m \notin \bigcup_{i=1}^{n_0} h^{-1}(D_i)$ but $m \in h^{-2}(D_j)$ for a $j \in \{1, \ldots, n_0\}$ then we also have $m \neq g^2(m)$. Indeed we have then $g(m) = h(m)$, $g^2(m) = \psi(h^2(m)) \in \psi(D_j) = D_j$ and consequently $m \neq g^2(m)$ since $h^{-2}(D_j) \cap D_j = \emptyset$.

Thus we obtain:

$$m = g^2(m) \Longrightarrow m \notin \left(\bigcup_{i=1}^{n_0} h^{-1}(D_i) \right) \cup \left(\bigcup_{i=1}^{n_0} h^{-2}(D_i) \right) \Longrightarrow g^2(m) = h^2(m).$$

On the other hand, it is easily seen with (ii) that

$$m = h^2(m) \Longrightarrow m \notin \left(\bigcup_{i=1}^{n_0} h^{-1}(D_i) \right) \cup \left(\bigcup_{i=1}^{n_0} h^{-2}(D_i) \right) \Longrightarrow g^2(m) = h^2(m).$$

■

In the same way as Lemma 1.10 improves Proposition 1.9, the following slightly stronger lemma relax the hypothesis (iv) of Lemma 2.16.

Lemma 2.17 *If in Lemma 2.16 we replace the condition (iv) with the weaker*

(iv') $\forall i \in \{1, \ldots, n-1\}$ $\exists k_i \geq 1$ such that $h^{k_i}(D_i) \cap D_{i+1} \neq \emptyset$ and $\exists k_n \geq 1$ such that $h^{k_n}(D_n) \cap D_1 \neq \emptyset$,

then the conclusion still holds.

One also deduces from Lemma 2.16:

Lemma 2.18 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$ without a 2-periodic point and let V be an open connected subset of $\mathbb{S}^2$ such that $V \cap h(V) = \emptyset = V \cap h^2(V)$. Then we have $V \cap h^k(V) = \emptyset$ for any integer $k \neq 0$.*

2.2.2 Proof of Theorem 2.15

Let h and m be as in Theorem 2.15. We define $U = \mathbb{S}^2 \setminus \mathrm{Fix}(h) = \mathbb{S}^2 \setminus \mathrm{Fix}(h^2)$. Of course we have $h(U) = U \neq \emptyset$ and, according to the Lefschetz-Hopf Theorem, $U \neq \mathbb{S}^2$. Let us remark that there is a situation where our result is easily seen. According to a theorem of Epstein, a connected component K of $\mathrm{Fix}(h)$ is either a point or an arc or a Jordan curve and, in the last two cases, h interchanges locally the two sides of K (see [E]). If one can choose K to be a Jordan curve then $\mathbb{S}^2 \setminus K$ has exactly two connected components, say U_1 and U_2 with $m \in U_1$, which are interchanged by h (this also implies $K = \mathrm{Fix}(h)$). Since the U_i's are homeomorphic to $\mathbb{R}^2$ we can use the Brouwer plane translation theorem with $h^2|_{U_1}$ to find a proper topological embedding $\varphi : \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} \to U_1$ such that $\varphi(0,1) = m$ and $h^2 \circ \varphi(x,y) = \varphi \circ \tau(x,y)$ for $y > 0$, where $\tau(x,y) = (x+2,y) = G^2(x,y)$. We obtain a proper topological embedding $\varphi : \mathcal{O} = \{(x,y) \in \mathbb{R}^2 \,|\, y \neq 0\} \to U$ such that $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$ defining

$$\forall\, y < 0 \quad \varphi(x,y) = h \circ \varphi \circ G^{-1}(x,y) \in U_2.$$

Thus we can suppose that $\mathbb{S}^2 \setminus K$ is connected for every connected component K of $\mathrm{Fix}(h)$ and this implies that $U = \mathbb{S}^2 \setminus \mathrm{Fix}(h)$ is connected (see for example [N][Chapter V]). According to Lemma 2.3, Propositions 2.6 and 2.11, at least one of the two following properties is true:

- P1: There exists a translation arc α for h containing the point m and such that $\bigcup_{k\in\mathbb{Z}} h^k(\alpha)$ is a simple curve contained in U.
- P2: There exists a translation arc β for h^2 containing the point m and such that $\bigcup_{k\in\mathbb{Z}} h^{2k}(\beta)$ and $\bigcup_{k\in\mathbb{Z}} h^{2k+1}(\beta)$ are two disjoint simple curves contained in U.

2.2.2.1 Proof when P1 is true

Up to conjugacy in $\mathbb{S}^2$, we can suppose that

$$h^{-1}(\alpha) = [-1, 0] \times \{0\},$$

$$h(x,y) = (x+1, -y) \text{ for every } (x,y) \in h^{-1}(\alpha) \cup \alpha = [-1,1] \times \{0\},$$

$$m = (\tfrac{3}{4}, 0).$$

For $\epsilon > 0$ we consider the three rectangles (see Fig. 1.2)

$$D_{-1} = \{(x,y) \in \mathbb{R}^2 \,|\, -\frac{1}{4} \leq x \leq \frac{1}{4} \text{ and } -\epsilon \leq y \leq \epsilon\},$$

$$D_0 = \{(x,y) \in \mathbb{R}^2 \,|\, \frac{1}{4} \leq x \leq \frac{3}{4} \text{ and } -\epsilon \leq y \leq \epsilon\},$$

$$D_1 = \{(x,y) \in \mathbb{R}^2 \,|\, \frac{3}{4} \leq x \leq \frac{5}{4} \text{ and } -\epsilon \leq y \leq \epsilon\}.$$

One can check:

Lemma 2.19 *There exist $\epsilon > 0$ and a brick decomposition $\mathcal{D} = \{B_i\}_{i\in\mathbb{N}}$ of U such that:*

1. *D_{-1}, D_0 and D_1 are bricks of $\mathcal{D}$,*
2. *for any two bricks $B_i, B_j \in \mathcal{D}$ we have*
   - *$\forall k = 1,2 \quad h^k(B_i) \cap B_i = \emptyset$,*
   - *at most one of the two sets $h^{-1}(B_i) \cap B_j$ or $h(B_i) \cap B_j$ is nonempty.*

Let us consider the attractor and the reppeller $\mathcal{A}, \mathcal{R} \subset U$ associated to $B_{i_0} = D_0$. We remark that $D_1 \subset \mathcal{A}$ and $D_{-1} \subset \mathcal{R}$ since respectively $h(D_0) \cap D_1 \neq \emptyset$ and $h^{-1}(D_0) \cap D_{-1} \neq \emptyset$. Moreover Lemma 2.17 implies $Int(D_0) \cap \mathcal{A} = \emptyset$ and $Int(\mathcal{R}) \cap \mathcal{A} = \emptyset$. So the vertical segment $\{\frac{3}{4}\} \times [-\epsilon, \epsilon]$ is contained in a connected component Δ of $\partial_U \mathcal{A}$ and we know from Property 1.12 that Δ is a closed subset of U homeomorphic to either $\mathbb{S}^1$ or to $\mathbb{R}$. As in the proof of BPTT we use the

Notations 2.20

$$\gamma_- = \{(x,0) \,|\, -\frac{1}{4} < x < \frac{3}{4}\},$$

$$\gamma_+ = \{(x,0) \,|\, \frac{3}{4} < x < \frac{7}{4}\} = h(\gamma_-),$$

$$\gamma = \{(x,0) \,|\, -\frac{1}{4} < x < \frac{7}{4}\} = \gamma_- \cup \{(\frac{3}{4}, 0)\} \cup \gamma_+.$$

The same arguments as for Lemma 1.17 allow one to state:

Lemma 2.21 *The set $h^{-1}(\Delta) \cup \gamma_-$ (resp.$\gamma_+ \cup h(\Delta)$) is connected and contained in $U \setminus \mathcal{A}$(resp. in $Int(\mathcal{A})$).*

CASE 1: The set Δ is a Jordan curve.

CLAIM 1: The set Δ separates $h^{-1}(\Delta)$ and $h(\Delta)$ in $\mathbb{S}^2$.

Proof: Otherwise Lemma 2.21 would show that γ intersects Δ transversely and meets only one connected component of $\mathbb{S}^2 \setminus \Delta$, which is absurd. ■

Let us write V_+ for the connected component of $\mathbb{S}^2 \setminus \Delta$ containing $h(\Delta)$. We have

$\partial h(V_+) = h(\Delta) \subset V_+$ so $h(V_+) \cap V_+ \neq \emptyset$ and actually $h(Cl(V_+)) \subset V_+$ since, according to the above claim,

$$h(V_+) \cap \partial V_+ = h(V_+) \cap \Delta = h(V_+ \cap h^{-1}(\Delta)) = \emptyset.$$

It is now routine to construct topological embedding φ defined on $\mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\}$ and conjugating h and H. We just sketch the construction: defining $\Omega = V_+ \setminus h(Cl(V_+))$, we clearly have $Cl(\Omega) = \Delta \cup \Omega \cup h(\Delta) \subset U$. Let $\varphi : \mathbb{S}^1 \to \Delta$ be a homeomorphism. It can be extended to a homeomorphism

$$\varphi : \mathbb{S}^1 \cup H(\mathbb{S}^1) \to \Delta \cup h(\Delta)$$

by defining $\varphi|_{H(\mathbb{S}^1)} = h \circ \varphi \circ H^{-1}|_{H(\mathbb{S}^1)}$. Using suitably the Schoenflies Theorem, one can extend again φ to a homeomorphism from the compact annulus $A = \{z \in \mathbb{C} \,|\, \frac{1}{2} \leq |z| \leq 1\}$ onto $Cl(\Omega)$. Finally, for any point $z \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exists a unique $k \in \mathbb{Z}$ such that $z \in H^k(A \setminus \partial^- A)$, where $\partial^- A = \{z \in \mathbb{C} \,|\, |z| = \frac{1}{2}\}$, and we define

$$\varphi(z) = h^k \circ \varphi \circ H^{-k}(z) \in h^k(Cl(\Omega)).$$

One can easily check that $\varphi : \mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\} \to U$ is a well-defined one-to-one continuous map such that $h \circ \varphi = \varphi \circ H|_{\mathcal{O}}$ and $\varphi(\mathcal{O}) = \bigcup_{k \in \mathbb{Z}} h^k(Cl(\Omega))$.

CASE 2: The set Δ is homeomorphic to $\mathbb{R}$. Since Δ is a closed subset of U we have $\emptyset \neq Cl(\Delta) \setminus \Delta \subset \mathrm{Fix}(h)$. Moreover, $Cl(\Delta) \setminus \Delta$ has at most two connected components, say L_1 and L_2 with possibly $L_1 = L_2$, and each L_i is contained in a connected component K_i of $\mathrm{Fix}(h)$. It will be convenient to compactify $\mathbb{S}^2 \setminus (K_1 \cup K_2)$ as follows; let us choose a_1 and a_2 in $\mathbb{S}^2$ with the convention that $a_1 = a_2$ if and only if $K_1 = K_2$. Since U has been assumed to be connected, we have the same for $\mathbb{S}^2 \setminus (K_1 \cup K_2)$ and it is then very classical that this latter set is homeomorphic to $\mathbb{S}^2 \setminus \{a_1, a_2\}$ (see for example [N]Chapter VI). Now, if ψ is any homeomorphism from $\mathbb{S}^2 \setminus (K_1 \cup K_2)$ onto $\mathbb{S}^2 \setminus \{a_1, a_2\}$, we define $\hat{h} : \mathbb{S}^2 \to \mathbb{S}^2$ by

$$\hat{h}(z) = \begin{cases} z & \text{if } z \in \{a_1, a_2\}, \\ \psi \circ h \circ \psi^{-1}(z) & \text{if } z \notin \{a_1, a_2\}. \end{cases}$$

One can check that $\hat{h}$ is a homeomorphism and that $Cl(\psi(\Delta)) \setminus \psi(\Delta) = \{a_1, a_2\}$. Furthermore, since we are looking for a (proper) topological embedding φ taking its values in $U \subset \mathbb{S}^2 \setminus (K_1 \cup K_2)$, it is enough to prove our theorem for $\hat{h}$ instead of h. In other words, there is no loss in supposing that K_i (and so L_i) is reduced to one point ($i \in \{1,2\}$). This will be assumed from now on.

CLAIM 2: We have necessarily $K_1 = K_2$.
Proof: Suppose this is not true and define

$$C = Cl(\Delta \cup h(\Delta)) = \Delta \cup h(\Delta) \cup K_1 \cup K_2.$$

Thus C is a Jordan curve. Let us remark that the sets $h^{-1}(\Delta) \cup \gamma_-$ and γ_+ are both connected and contained in $U \setminus (\Delta \cup h(\Delta)) \subset \mathbb{S}^2 \setminus C$; for $h^{-1}(\Delta) \cup \gamma_-$, this is contained in Lemma 2.21 since we know from Property 1.14 that $\Delta \cup h(\Delta) \subset \mathcal{A}$. Lemma 2.21 also gives

$$\gamma_+ \cap \Delta \subset Int(\mathcal{A}) \cap \partial_U \mathcal{A} = \emptyset$$

and

$$\gamma_+ \cap h(\Delta) = h(\gamma_- \cap \Delta) \subset h(\gamma_- \cap \mathcal{A}) = \emptyset.$$

Now, since the segment γ intersects $\Delta \subset C$ transversely, we deduce that the connected components V_-, V_+, of $\mathbb{S}^2 \setminus C$ containing respectively $h^{-1}(\Delta) \cup \gamma_-$ and γ_+ are different. It follows that

$$\partial h^{-1}(V_+) \cap V_+ = h^{-1}(C) \cap V_+ = h^{-1}(\Delta) \cap V_+ = \emptyset$$

so we have either $V_+ \subset h^{-1}(V_+)$ or $V_+ \cap h^{-1}(V_+) = \emptyset$. We remark now that none of these two situations is possible. The first one would imply

$$\gamma_+ \cup \gamma_- = \gamma_+ \cup h^{-1}(\gamma_+) \subset h^{-1}(V_+)$$

which is absurd because the segment γ intersects $\Delta \subset h^{-1}(C)$ transversely. Suppose now that $h^{-1}(V_+) \cap V_+ = \emptyset$. We first remark that we cannot have $h^{-1}(Cl(V_+)) \cup Cl(V_+) = \mathbb{S}^2$ since this would imply $h^{-1}(\Delta) = h(\Delta)$ which contradicts Property 1.14. So the set $h^{-1}(Cl(V_+)) \cup Cl(V_+)$ is contained in the domain of a single chart of $\mathbb{S}^2$ and can be represented as in Fig 2.3. Keeping in mind that K_1, K_2 are fixed points of h, this contradicts the fact that h reverses the orientation. ■

Thus $Cl(\Delta) = \Delta \cup K_1$ is a Jordan curve. Again, γ intersects $\Delta \subset Cl(\Delta)$ transversely so we can write with Lemma 2.21:
CLAIM 3: The set $Cl(\Delta)$ separates $h^{-1}(\Delta)$ and $h(\Delta)$ in $\mathbb{S}^2$.
Now, let V_+ be the connected component of $\mathbb{S}^2 \setminus Cl(\Delta)$ containing $h(\Delta)$. Since $h(\Delta) \subset \partial h(V_+) \cap V_+$ we have $h(V_+) \cap V_+ \neq \emptyset$ and in fact $h(V_+ \cup \Delta) \subset V_+$ because the third claim implies

$$h(V_+) \cap \partial V_+ = h(V_+) \cap Cl(\Delta) = h(V_+) \cap \Delta = h(V_+ \cap h^{-1}(\Delta)) = \emptyset.$$

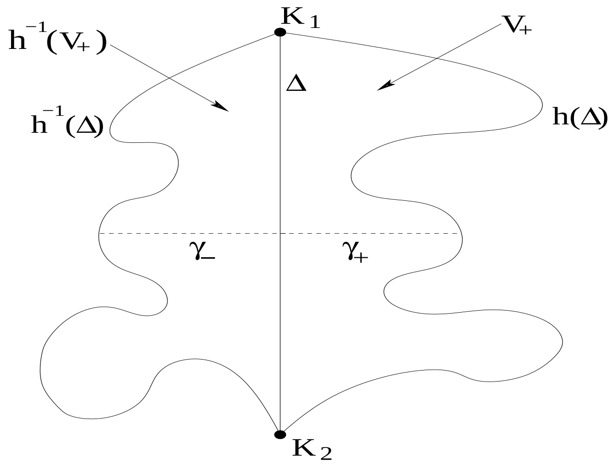

Figure 2.3: $V_+ \cap h^{-1}(V_+) = \emptyset$ is not possible

We conclude as follows. Let us define $\Omega = V_+ \setminus h(Cl(V_+))$. We have obviously $Cl(\Omega)\setminus K_1 = \Delta\cup\Omega\cup h(\Delta) \subset U$. Using the Schoenflies Theorem, one can construct a homeomorphism

$$\varphi : \{(x,y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1\} \cup \{\infty\} \to Cl(\Omega)$$

such that $\varphi(\infty) = K_1$, $\varphi(\{0\} \times \mathbb{R}) = \Delta$ and

$$\forall y \in \mathbb{R} \quad \varphi(1,y) = h \circ \varphi \circ G^{-1}(1,y) \in h(\Delta).$$

Now, if $k \leq x < k+1$ ($k \in \mathbb{Z}$) we let

$$\varphi(x,y) = h^k \circ \varphi \circ G^{-k}(x,y) \in h^k(\Delta \cup \Omega).$$

It is easily seen that $\varphi : \mathcal{O} = \mathbb{R}^2 \to U$ defined in this way is a proper topological embedding, with image $\varphi(\mathcal{O}) = \bigcup_{k\in\mathbb{Z}} h^k(\Delta \cup \Omega)$, such that $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$. This completes the proof of Theorem 2.15 when Property P1 is true.

2.2.2.2 Proof when P2 is true

Up to conjugacy in $\mathbb{S}^2$, we can suppose that

$$h^{-2}(\beta) = [-2,0] \times \{-1\},$$

$$h(x,y) = (x+1,-y) \text{ if } (x,y) \in \bigcup_{k=-2}^{1} h^k(\beta) = [-2,2] \times \{-1\} \cup [-1,3] \times \{1\},$$

$$m = (\frac{3}{2}, -1).$$

For $\epsilon > 0$, let us consider the five rectangles (see Fig. 2.4)

$$D_i = \{(x,y) \mid \frac{i+1}{2} \leq x \leq \frac{i+3}{2} \text{ and } -1-\epsilon \leq y \leq -1+\epsilon\} \quad \text{for } i \in \{0, \pm 2\},$$

$$D_i = \{(x,y) \mid \frac{i+1}{2} \leq x \leq \frac{i+3}{2} \text{ and } 1-\epsilon \leq y \leq 1+\epsilon\} \quad \text{for } i = \pm 1.$$

One can check:

Lemma 2.22 *There exist $\epsilon > 0$ and a brick decomposition $\mathcal{D} = \{B_i\}_{i\in\mathbb{N}}$ of U such that:*

1. *$D_0, D_{\pm 1}$ and $D_{\pm 2}$ are bricks of $\mathcal{D}$,*
2. *for any two bricks $B_i, B_j \in \mathcal{D}$ we have*
   - *$\forall k = 1,2 \quad h^k(B_i) \cap B_i = \emptyset$,*
   - *at most one of the two sets $h^{-1}(B_i) \cap B_j$ or $h(B_i) \cap B_j$ is nonempty.*

Let $\mathcal{A}, \mathcal{R} \subset U$ be respectively the attractor and the repeller associated to the brick $B_{i_0} = D_0$. First we remark that $D_1 \cup D_2 \subset \mathcal{A}$ and $D_{-1} \cup D_{-2} \subset \mathcal{R}$ since, on the one hand, $h(D_0) \cap D_1 \neq \emptyset \neq h(D_1) \cap D_2$, and on the other hand, $h^{-1}(D_0) \cap D_{-1} \neq \emptyset \neq h^{-1}(D_{-1}) \cap D_{-2}$. Using Lemma 2.17 we see that the vertical segment $\{\frac{3}{2}\} \times [-1-\epsilon, -1+\epsilon]$ is contained in a connected component Δ of $\partial_U \mathcal{A}$. We give again some convenient notations and a basic lemma before to study the situation where Δ is homeomorphic to $\mathbb{S}^1$ (resp. to $\mathbb{R}$).

Notations 2.23

$$\gamma_- = \{(x,-1) \mid -\frac{1}{2} < x < \frac{3}{2}\},$$
$$\gamma_+ = \{(x,-1) \mid \frac{3}{2} < x < \frac{7}{2}\} = h^2(\gamma_-),$$
$$\gamma = \{(x,-1) \mid -\frac{1}{2} < x < \frac{7}{2}\} = \gamma_- \cup \{(\frac{3}{2}, -1)\} \cup \gamma_+.$$

As for Lemmas 1.17 and 2.21 one can see

Lemma 2.24 *The set $h^{-2}(\Delta) \cup \gamma_-$ (resp. $\gamma_+ \cup h^2(\Delta)$) is connected and contained in $U \setminus \mathcal{A}$ (resp. in $Int(\mathcal{A})$).*

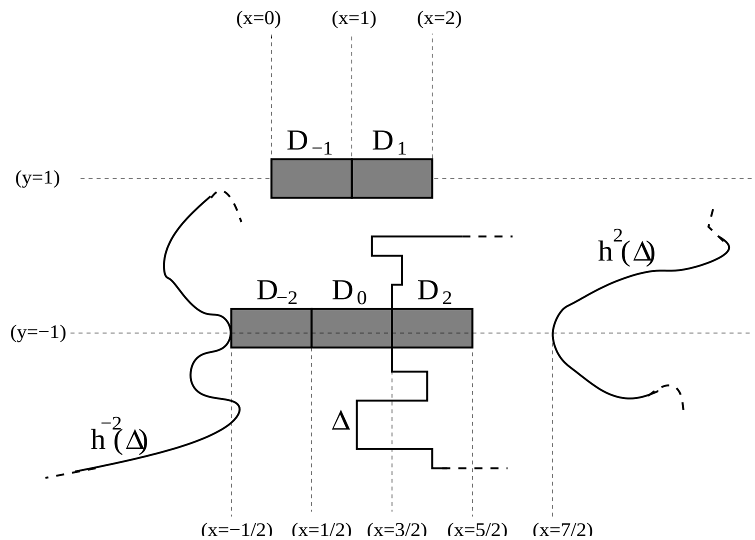

Figure 2.4: The bricks $D_0, D_{\pm 1}, D_{\pm 2}$ and $\Delta, h^{\pm 2}(\Delta)$ close to these bricks

First case: The set Δ is a Jordan curve.

CLAIM 4: The set Δ separates $h^{-1}(\Delta)$ and $h(\Delta)$ in $\mathbb{S}^2$.

Proof: First we remark that Δ separates $h^{-2}(\Delta)$ and $h^2(\Delta)$ in $\mathbb{S}^2$: this follows from Lemma 2.24 and from the fact that γ intersects Δ transversely. Let us denote V_-, V_+ the connected components of $\mathbb{S}^2 \setminus \Delta$ containing respectively $h^{-2}(\Delta)$ and $h^2(\Delta)$. As in Section 2.2.2.1 (with h^2 in the place of h), one can check that $h^2(Cl(V_+)) \subset V_+$ or equivalently $Cl(V_-) \subset h^2(V_-)$. According to the Brouwer fixed point Theorem, h^2 possesses two fixed points $z_- \in V_-$ and $z_+ \in V_+$ and these points are also fixed points of h since h has no 2-periodic point. In particular we have

$$V_+ \cap h(V_+) \neq \emptyset \neq V_- \cap h^{-1}(V_-).$$

We deduce now from $h(\Delta) \cap \Delta = \emptyset$ that $h(\Delta) \subset V_+$: otherwise we would have $h(\Delta) \subset V_-$ and consequently

$$V_+ \cap \partial h(V_+) = V_+ \cap h(\Delta) = \emptyset$$

so $V_+ \subset h(V_+) \subset h^2(V_+)$ which contradicts $h^2(Cl(V_+)) \subset V_+$. We get similarly $h^{-1}(\Delta) \subset V_-$ replacing h,V_+ with h^{-1},V_-. ■

Defining $\Omega = V_+ \setminus h(Cl(V_+))$, we proceed now exactly as in Section 5.3.1 to construct a topological embedding

$$\varphi : \mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\} \to U$$

with image $\varphi(\mathcal{O}) = \bigcup_{k\in\mathbb{Z}} h^k(Cl(\Omega))$ such that $h \circ \varphi = \varphi \circ H|_{\mathcal{O}}$.

Second case: The set Δ is homeomorphic to $\mathbb{R}$.
We denote again L_1, L_2 the connected components of the nonempty set $Cl(\Delta) \setminus \Delta \subset \mathrm{Fix}(h)$, with possibly $L_1 = L_2$. Each L_i is contained in a connected component K_i of $\mathrm{Fix}(h)$ and, as explained in Section 2.2.2.1, there is no loss in supposing that K_i (and so L_i) is reduced to one point.

For convenience we will use the following notations for the two half-planes on both sides of the x-axis:

$$P_+ = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\} \quad \text{and} \quad P_- = \{(x, y) \in \mathbb{R}^2 \,|\, y < 0\}.$$

We first suppose $K_1 = K_2$.

Then $Cl(\Delta) = \Delta \cup K_1$ is a Jordan curve. Using again Lemma 2.24 and since $\gamma \cap \Delta$ is a transverse intersection, one can write:
CLAIM 5: The set $Cl(\Delta)$ separates $h^{-2}(\Delta)$ and $h^2(\Delta)$ in $\mathbb{S}^2$.
We consider now the two connected components V_-, V_+ of $\mathbb{S}^2 \setminus Cl(\Delta)$, with $h^2(\Delta) \subset V_+$ and $h^{-2}(\Delta) \subset V_-$. One can easily derive from the claim above that $h^2(V_+ \cup \Delta) \subset V_+$, i.e. $V_- \cup \Delta \subset h^2(V_-)$.
CLAIM 6: There are three possible situations:

S1: $h(V_+ \cup \Delta) \subset V_+$,

S2: $h(V_+ \cup \Delta) \subset V_-$,

S3: $h(V_- \cup \Delta) \subset V_+$.

Proof: Suppose that we are neither in the situation S1 nor in the situation S2. Then $h(V_+ \cup \Delta)$ meets $\partial V_+ = \partial V_- = Cl(\Delta)$. Since $h(\Delta) \cap \Delta = \emptyset$, this implies $h(V_+) \cap \Delta \neq \emptyset$ and then $\Delta \subset h(V_+)$. Consequently $h(V_- \cup \Delta)$ is a connected subset of $\mathbb{S}^2 \setminus Cl(\Delta)$ and we get either $h(V_- \cup \Delta) \subset V_+$ or $h(V_- \cup \Delta) \subset V_-$. The latter is actually not possible because of $V_- \cup \Delta \subset h^2(V_-)$. ■

We construct now a proper topological embedding $\varphi : \mathcal{O} \to U$ conjugating h and G which will be defined on $\mathcal{O} = \mathbb{R}^2$ in the first situation and on $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 \,|\, y \neq 0\}$ in the last two ones.
• In the situation S1 we proceed exactly as in Section 5.3.1.
• Remark now that

$$h(V_- \cup \Delta) \subset V_+ \iff V_- \cup \Delta \subset h(V_+) \iff h^{-1}(V_- \cup \Delta) \subset V_+$$

which shows that the situation S3 can be reduced to the situation S2 replacing h with h^{-1}. Since it is equivalent to prove Theorem 2.15 for h or for h^{-1}, it suffices to consider S2. In this case, let us denote $\Omega = V_+ \setminus h^2(Cl(V_+))$. We have then $Cl(\Omega) \setminus K_1 = \Delta \cup \Omega \cup h^2(\Delta) \subset U$. We construct the required embedding φ as follows. We consider for example the set $D = \{(x, \frac{1}{x}) \mid x > 0\}$ and we write B for the domain between D and $G^2(D)$ in the upper half-plane P_+. Using the Schoenflies Theorem, one can construct a homeomorphism

$$\varphi : Cl(B) = Cl_{\mathbb{R}^2}(B) \cup \{\infty\} \to Cl(\Omega)$$

such that $\varphi(\infty) = K_1$, $\varphi(D) = \Delta$ and $\varphi \circ G^2|_D = h^2 \circ \varphi|_D$. Then we define the map φ on the half-plane P_+ observing that for every point $z \in P_+$ there exists a unique even integer $2k \in \mathbb{Z}$ such that $z \in G^{2k}(D \cup B)$ and then defining

$$\varphi(z) = h^{2k} \circ \varphi \circ G^{-2k}(z) \in h^{2k}(\Delta \cup \Omega).$$

In particular we have at this stage

$$h^2 \circ \varphi = \varphi \circ G^2|_{P_+}.$$

Afterwards we extend φ on P_- by

$$\forall y < 0 \quad \varphi(x, y) = h \circ \varphi \circ G^{-1}(x, y) \in \bigcup_{k \in \mathbb{Z}} h^{2k+1}(\Delta \cup \Omega).$$

It is easily seen that we have obtained in this way a continuous map

$$\varphi : \mathcal{O} = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\} \to U$$

satisfying $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$ and such that, for every $x \in \mathbb{R}$, $\varphi((\{x\} \times \mathbb{R}) \cap \mathcal{O})$ is a closed subset of U . It is not totally obvious that this map φ is one-to-one (in contrast to the previously constructed embeddings). To check this property, it is enough to see that the sets $h^k(\Delta \cup \Omega)$, $k \in \mathbb{Z}$, are pairwise disjoint. This turns out to be true because $h^k(\Delta) \cap h^l(\Delta) = \emptyset$ for $k \neq l$ (Property 1.14) and because

$$\Omega \cap h(\Omega) \subset V_+ \cap h(V_+) = \emptyset, \qquad \Omega \cap h^2(\Omega) \subset \Omega \cap h^2(V_+) = \emptyset$$

which implies, according to Lemma 2.18, $h^k(\Omega) \cap h^l(\Omega) = \emptyset$ for $k \neq l$.

We suppose now $K_1 \neq K_2$.

Let us define $C = Cl(\Delta \cup h^2(\Delta)) = \Delta \cup h^2(\Delta) \cup K_1 \cup K_2$. Thus C is a Jordan curve.

CLAIM 7: The set C separates $h^{-2}(\Delta)$ and γ_+ in $\mathbb{S}^2$.

Proof: Property 1.14 gives $\Delta \cup h^2(\Delta) \subset \mathcal{A}$ so, with Lemma 2.24, $h^{-2}(\Delta) \cup \gamma_-$ is contained in a connected component V_- of $\mathbb{S}^2 \setminus C$. This lemma also gives $\gamma_+ \cap \Delta \subset \gamma_+ \cap \partial_U \mathcal{A} = \emptyset$ and $\gamma_+ \cap h^2(\Delta) \subset h^2(\gamma_- \cap \mathcal{A}) = \emptyset$ hence γ_+ is also contained in a connected component V_+ of $\mathbb{S}^2 \setminus C$. We have necessarily $V_- \neq V_+$ since the segment γ intersects $\Delta \subset C$ transversely. ∎

We keep the notations V_-, V_+ of the proof above, that is V_- (resp. V_+) is the connected component of $\mathbb{S}^2 \setminus C$ containing $h^{-2}(\Delta)$ (resp. γ_+). In particular we have $\partial V_- = \partial V_+ = C$.

CLAIM 8: We have $h^2(V_+) \cap V_+ = \emptyset = h(V_+) \cap V_+$.

Proof: According to the previous claim we have

$$\partial h^{-2}(V_+) \cap V_+ = (h^{-2}(\Delta) \cup \Delta) \cap V_+ = \emptyset$$

so we have either $h^{-2}(V_+) \cap V_+ = \emptyset$ or $V_+ \subset h^{-2}(V_+)$. The latter would imply that γ is contained in $h^{-2}(V_+)$ except for the point $(\frac{3}{2}, -1)$ which is absurd since this segment intersects $\Delta \subset h^{-2}(C)$ transversely. This proves $h^2(V_+) \cap V_+ = \emptyset$. For the other equality, we first observe that the situations $h^{\pm 1}(V_+) \subset V_+$ are not possible since they contradict $h^2(V_+) \cap V_+ = \emptyset$. Suppose now $V_+ \cap h(V_+) \neq \emptyset$. Then we have

$$h(V_+) \cap C \neq \emptyset \quad \text{and} \quad V_+ \cap h(C) \neq \emptyset,$$

that is

$$h(V_+) \cap (\Delta \cup h^2(\Delta)) \neq \emptyset \quad \text{and} \quad V_+ \cap (h(\Delta) \cup h^3(\Delta)) \neq \emptyset.$$

For convenience we define four sets $E_1, \dots, E_4$ by

$$E_1 = h(V_+) \cap \Delta,\ E_2 = h(V_+) \cap h^2(\Delta),\ E_3 = V_+ \cap h(\Delta),\ E_4 = V_+ \cap h^3(\Delta).$$

Since $h^k(\Delta) \cap h^l(\Delta) = \emptyset$ for $k \neq l$ we see that E_i is either empty or equal, for respectively $i = 1, 2, 3, 4$, to the whole set Δ, $h^2(\Delta), h(\Delta), h^3(\Delta)$.

It turns out that necessarily $E_1 = \emptyset$, hence $E_2 = h^2(\Delta)$. Otherwise we would have $\Delta \subset h(V_+)$, i.e. $h^{-1}(\Delta) \subset V_+$, and $h^{-1}(Cl(\Delta))$ would be a connected set joining K_1 and K_2 in $Cl(V_+)$. Moreover, $Cl(\gamma_+)$ is an arc contained in V_+ except one endpoint on Δ and the other one on $h^2(\Delta)$ so it separates K_1 and K_2 in $Cl(V_+)$. This implies $h^{-1}(\Delta) \cap \gamma_+ \neq \emptyset$. On the other hand, since $\gamma_+ \subset \mathcal{A}$, we get with Property 1.14

$$h^{-1}(\Delta) \cap \gamma_+ = h^{-1}(\Delta \cap h(\gamma_+)) \subset h^{-1}(\partial_U \mathcal{A} \cap Int(\mathcal{A})) = \emptyset,$$

a contradiction.

We also observe that the two sets E_2 and E_4 cannot be simultaneously non-empty since this would give $h^3(\Delta) \subset h^2(V_+) \cap V_+ = \emptyset$. It remains to study the situation $h(\Delta) \subset V_+$, i.e. $h^2(\Delta) \subset h(V_+)$. We first observe that we cannot have $Cl(V_+) \cup h(Cl(V_+)) = \mathbb{S}^2$ because this would imply $\Delta \subset h(V_+)$ and then $h(\Delta) \subset h^2(V_+) \cap V_+ = \emptyset$. Thus the whole set $Cl(V_+) \cup h(Cl(V_+)$ is contained in the domain of a single chart of $\mathbb{S}^2$. In such a chart, the situation is as in Fig. 2.5 and, K_1 and K_2 being fixed points, we obtain a contradiction with the fact that h reverses the orientation. The claim is proved. ■

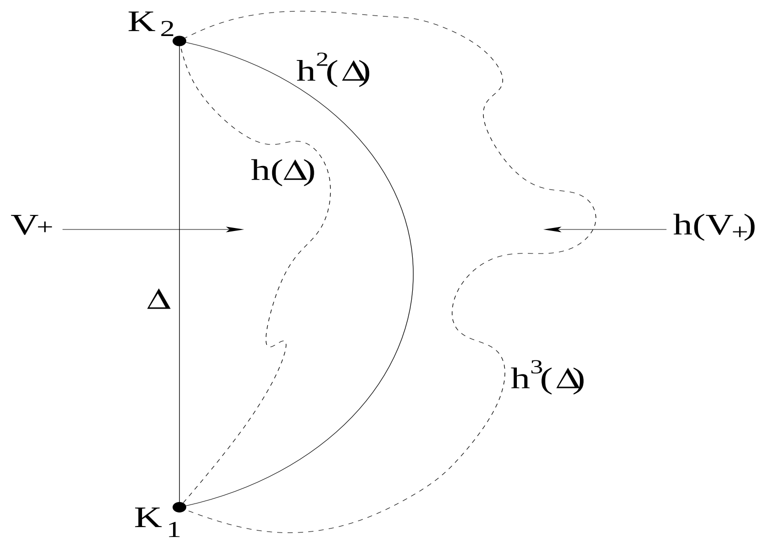

Figure 2.5: The situation $h(\Delta) \subset V_+$ is not possible

We consider now a new "model" homeomorphism G_1 defined by

$$\forall\,(x,y) \in \mathbb{R}^2 \quad G_1(x,y) = (x+|y|, -y).$$

Let $D = \{(0,y) \in \mathbb{R}^2 \,|\, y > 0\}$ and let B be the domain between D and $G_1^2(D)$ in the half-plane P_+. Using again the Schoenflies Theorem, one can construct a homeomorphism $\varphi_1 : Cl(B) \to Cl(V_+)$ such that $\varphi_1(0,0) = K_1$, $\varphi_1(\infty) = K_2$, $\varphi_1(D) = \Delta$ and $\varphi_1 \circ G_1^2|_D = h^2 \circ \varphi_1|_D$. For every point $z \in P_+$ there exists a unique even integer $2k \in \mathbb{Z}$ such that $z \in G_1^{2k}(D \cup B)$ and we set

$$\varphi_1(z) = h^{2k} \circ \varphi_1 \circ G_1^{-2k}(z) \in h^{2k}(\Delta \cup V_+).$$

We have in this way $h^2 \circ \varphi_1 = \varphi_1 \circ G_1^2|_{P_+}$. Extending φ_1 on P_- by

$$\forall\, y < 0 \quad \varphi_1(x,y) = h \circ \varphi_1 \circ G_1^{-1}(x,y) \in \bigcup_{k \in \mathbb{Z}} h^{2k+1}(\Delta \cup V_+),$$

we obtain a continuous map φ_1 defined on $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 \,|\, y \neq 0\}$ and such that $h \circ \varphi_1 = \varphi_1 \circ G_1|_{\mathcal{O}}$. Using the eighth claim and Lemma 2.18 we get $h^k(\Delta \cup V_+) \cap h^l(\Delta \cup V_+) = \emptyset$ for $k \neq l$ which ensures that φ_1 is one-to-one. Finally, it is easy to construct a homeomorphism $\psi : \mathcal{O} \to \mathcal{O}$ such that $G_1 \circ \psi = \psi \circ G|_{\mathcal{O}}$ and such that

$$\forall x \in \mathbb{R} \quad Cl\Big(\psi\big((\{x\} \times \mathbb{R}) \cap \mathcal{O}\big)\Big) \setminus \psi\big((\{x\} \times \mathbb{R}) \cap \mathcal{O}\big) = \{(0,0), \infty\}.$$

Then $\varphi = \varphi_1 \circ \psi$ is a proper topological embedding such that $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$, with $\varphi(\mathcal{O}) = \bigcup_{k \in \mathbb{Z}} h^k(\Delta \cup V_+)$. The proof of Theorem 2.15 is completed. ■

Chapter 3

Appendix

3.1 Winding numbers

The unit circle is $\mathbb{S}^1 = \{t \in \mathbb{C} = \mathbb{R}^2 \,|\, |t| = 1\}$. Let $p : \mathbb{R} \to \mathbb{S}^1$, $x \mapsto p(x) = e^{2i\pi x}$ be the universal covering map of $\mathbb{S}^1$. If $u : \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous map then there exists a continuous map $\tilde{u} : \mathbb{R} \to \mathbb{R}$ such that $p \circ \tilde{u} = u \circ p$ (one says that $\tilde{u}$ is a lift of u) and an integer $d \in \mathbb{Z}$ such that

$$\forall x \in \mathbb{R} \quad \tilde{u}(x+1) = \tilde{u}(x) + d.$$

This integer d depends only on the homotopy class of u in the space $\mathcal{C}(\mathbb{S}^1)$ of the continuous self-maps of $\mathbb{S}^1$, endowed with the topology of the uniform convergence. It is called the *degree* of u.

If we have now a continuous map $\mathbb{S}^1 \ni t \mapsto v_t \in \mathbb{R}^2 \setminus \{0\}$, we define the *winding number* of $(v_t)_t$ as the degree of the map $t \mapsto u(t) := v_t/||v_t||$ $(t \in \mathbb{S}^1)$. Since two homotopic self-maps of $\mathbb{S}^1$ have the same degree we obtain:

Property 3.1 *Suppose that*

$$\begin{array}{lcl} \mathbb{S}^1 \times [0,1] & \to & \mathbb{R}^2 \setminus \{0\} \\ (t,s) & \mapsto & v_{t,s} \end{array}$$

is a continuous map. Then the winding numbers of $(v_{t,o})_t$ and of $(v_{t,1})_t$ are the same.

3.2 Jordan curves and Jordan domains

A *Jordan curve* is a subset of $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ which is homeomorphic to the circle $\mathbb{S}^1$. We have the classical result (see e.g. [Ku, M, N, T, W]):

Theorem 3.2 (Jordan curve Theorem) *Let $J \subset \mathbb{S}^2$ be a Jordan curve. Then $\mathbb{S}^2 \setminus J$ has exactly two connected components and J is their common frontier.*

An open set $U \subset \mathbb{S}^2$ is said to be a *Jordan domain* if it is a connected component of $\mathbb{S}^2 \setminus J$ for some Jordan curve J.

The case of planar Jordan curves: For a Jordan curve $J \subset \mathbb{R}^2$, one usually consider the two connected components of $\mathbb{R}^2 \setminus J$; the bounded one is called the *interior domain* of J and denoted by $int(J)$ while the unbounded one is called the *exterior domain* of J and denoted by $ext(J)$. We have the following usefull caracterization for these two domains. Let $u : \mathbb{S}^1 \to J = u(S^1)$ be a parametrization of J (i.e. a homeomorphism from $\mathbb{S}^1$ onto J) and let $z \in \mathbb{R}^2 \setminus J$. The winding number of $(u(t) - z)_{t \in \mathbb{S}^1}$ is named the *index of the curve u with respect to z* and is denoted by $ind_z(u)$.

Property 3.3 *The integer $ind_z(u)$ depends only on the connected component of $\mathbb{R}^2 \setminus J$ containing z; more precisely*

$$ind_z(u) = \begin{cases} 0 & si\ z \in ext(J), \\ \pm 1 & si\ z \in int(J). \end{cases}$$

Remark. The first assertion in the above property follows directly from the Property 3.1. Indeed, the plane $\mathbb{R}^2$ being locally path-connected, every connected open subset of $\mathbb{R}^2$ is also path-connected hence if $z, z' \in U$ for a connected component U of $\mathbb{R}^2 \setminus J$ one can find a path from z to z' lying in U, i.e. a continuous map $\alpha : [0,1] \to U$ such that $\alpha(0) = z$ and $\alpha(1) = z'$; the assertion then follows by considering the map $(t,s) \mapsto u(t) - \alpha(s) \in \mathbb{R}^2 \setminus \{0\}$ in Property 3.1. The fact that $ind_z(u) = 0$ for $z \in ext(J)$ is also easy: for example choose Δ to be a vertical line such that J in contained in the half-plane H_r on the right of Δ. Then the half-plane H_l on the left of Δ is contained in $ext(J)$ and, for a given $z \in H_l$, all the vectors $u(t) - z$ ($t \in \mathbb{S}^1$) have positive first coordinate from which we deduce that the winding number of $(u(t) - z)_t$ is zero. In contrast, it is more work to find a point $z \in \mathbb{R}^2 \setminus J$ satisfying $ind_z(u) = \pm 1$ and this is actually the most difficult part in some proofs of Jordan Theorem (e.g. [M]).

Property 3.3 will allow us to define the notion of orientation preserving/reversing homeomorphism in a rather intuitive way and avoiding any homology theory. If $ind_z(u) = 1$ for $z \in int(J)$ one says that u is a *positive* or *counterclockwise* parametrization of J. Otherwise u is a *negative* or *clockwise* parametrization.

We have also the following result, due to Kerékjártó ([Ke2]).

Proposition 3.4 *Let J_1, J_2 be two Jordan curves in the plane $\mathbb{R}^2$ such that $int(J_1) \cap int(J_2) \neq \emptyset$. Then every connected component of $int(J_1) \cap int(J_2)$ is also the interior domain of a Jordan curve $J \subset J_1 \cup J_2$.*

3.3 The Schoenflies Theorem

It can be stated as follows;

Theorem 3.5 *Let $J_1, J_2 \subset \mathbb{S}^2$ be two Jordan curves and, for $i = 1, 2$, let U_i be one of the two Jordan domains with frontier J_i. Any homeomorphism from J_1 onto J_2 can be extended to a homeomorphism from $Cl(U_1) = U_1 \cup J_1$ onto $Cl(U_2) = U_2 \cup J_2$ (and then to a self-homeomorphism of $\mathbb{S}^2$).*

See for example [P] for a proof relying on complex analysis or [C, Ku, N]) for more topological arguments. Moreover one can show that any arc $\alpha \subset \mathbb{S}^2$ is contained in a Jordan curve hence we also have:

Theorem 3.6 *Let $u : [0, 1] \to \alpha \subset \mathbb{S}^2$ be a homeomorphism. Then it can be extended to a homeomorphism of the whole sphere.*

3.4 Orientation preserving vs orientation reversing homeomorphisms

Let $h : U \to V = h(U)$ be a homeomorphism between two connected open subsets of $\mathbb{R}^2$. For a given point $z \in U$, let us choose $r > 0$ such that the disc $B(z, r) = \{m \in \mathbb{R}^2 \,|\, ||m - z|| \leq r\}$ is contained in U and consider a Jordan curve J such that $z \in int(J) \subset B(z, r)$. Since $h(int(J)) = int(h(J))$ we have $h(z) \in int(h(J))$. It follows that if $u : \mathbb{S}^1 \to J$ is a parametrization of J (hence $h \circ u$ a parametrization of $h(J)$) there exists $\epsilon = \epsilon(z, u, r) = \pm 1$ such that

$$ind_{h(z)}(h \circ u) = \epsilon \, ind_z(u).$$

Since two Jordan curves J_i ($i = 1, 2$) satisfying $z \in int(J_i) \subset B(z, r_i) \subset U$ are homotopic in $U \setminus \{z\}$, we deduce that ϵ depends only on the point z. Moreover the map $z \mapsto \epsilon(z)$ is locally constant because of Property 3.3 so it is constant on the connected set U. One says that the homeomorphism h *preserves the orientation* if $\epsilon = 1$ and that h *reverses the orientation* if $\epsilon = -1$.

Since this notion has been defined for a local planar homeomorphism, one can extend it to the framework of homeomorphisms on orientable surfaces.

3.5 Lefschetz index

The aim of this section is to provide an elementary Lefschetz fixed point theory. Roughly, given a continuous self-map h of a"nice" topological space X, such a theory associates to an open set $U \subset X$ an integer $Ind(h, U) \in \mathbb{Z}$ which allows to detect fixed points of h in U. We just explain here how to compute this index in a intuitive way when the open set U is a Jordan domain of $\mathbb{R}^2$ or $\mathbb{S}^2$ and we give the main properties in this framework (see e.g. [D] for a general theory). We begin with a definition in the plane $\mathbb{R}^2$.

Definition 3.7 *Let $X \subset \mathbb{R}^2$ and let $h : X \to \mathbb{R}^2$ be a continuous map. Suppose that U is a Jordan domain such that $Cl(U) \subset X$ and that h has no fixed point on the Jordan curve $J = \partial U$ (i.e. if $U \cap \mathrm{Fix}(h)$ is compact). The Lefschetz index of h on J, denoted by $\mathrm{Ind}(h, J)$, is defined as the winding number of $(h(u(t)) - u(t))_{t\in\mathbb{S}^1}$ where $u : \mathbb{S}^1 \to J$ is a counterclockwise parametrization of J. It does not depend on the choice of u. One also speaks of the Lefschetz of h on U and we write then $\mathrm{Ind}(h, U)$.*

We have the following

Properties 3.8

1. *If $\mathrm{Ind}(h, J) \neq 0$ then $\mathrm{Fix}(h) \cap U \neq \emptyset$,*

2. *Homotopy Invariance; If $(h_s : X \to \mathbb{R}^2)_{0\leq s\leq 1}$ is an homotopy from $h_0 = h$ such that $\mathrm{Fix}(h_s) \cap J = \emptyset$ for every s then $\mathrm{Ind}(h, J) = \mathrm{Ind}(h_1, J)$,*

3. *Topological Invariance; Suppose that $\varphi : X \to Y = \varphi(X)$ is a homeomorphism. Then*
$$\mathrm{Ind}(h, J) = \mathrm{Ind}(\varphi \circ h \circ \varphi^{-1}, \varphi(J)),$$

4. *If $h(z) \in Cl(U)$ for every $z \in J$ then $\mathrm{Ind}(h, J) = 1$.*

Proof. 1) Suppose that $\mathrm{Fix}(h) \cap U = \emptyset$. The set $Cl(U)$ is homeomorphic to the closed unit disc of $\mathbb{R}^2$ by Schoenflies Theorem. So, for some $z_0 \in U$, the Jordan curve J is isotopic inside $Cl(U)$ to another Jordan curve J_1 surrounding z_0 and with diameter arbitrary small. If this diameter is small enough, there is a straightline Δ such that J_1 and $h(J_1)$ are on both sides of Δ from which we deduce $\mathrm{Ind}(h, J_1) = 0$. The assertion then follows by considering the map $(t, s) \mapsto h(u_s(t)) - u_s(t) \neq 0$ in Property 3.1.
2) Use Property 3.1 with the map $(t, s) \mapsto h_s(u(t)) - u(t) \neq 0$.
3) We give an argument valid only for an orientation preserving conjugacy map φ. Consider the restricted map $\varphi|_{Cl(U)}$ and extend it to a homeomorphism ϕ of $\mathbb{R}^2$ by using Schoenflies Theorem. Since φ preserves the orientation so does ϕ and consequently there exists an isotopy from $Id_{\mathbb{R}^2}$ to ϕ, i.e. a family $(\phi_s)_{0\leq s\leq 1}$

of homeomorphisms of $\mathbb{R}^2$ such that $\phi_0 = Id_{\mathbb{R}^2}$, $\phi_1 = \phi$ and $(s, z) \mapsto \phi_s(z)$ is continuous. We get the result by using Property 3.1 with the map $(t, s) \mapsto \phi_s \circ h(u(t)) - \phi_s(u(t)) \neq 0$.
4) Because of Schoenflies theorem and 4) above, there is no loss in supposing that $J = \mathbb{S}^1$. For $0 \leq s \leq 1$ define $h_s(z) = (1-s)h(z)$. We have clearly $\mathrm{Ind}(h_1, J) = 1$ and we conclude with 2) above. ■

The topological invariance above allows one to define the Lesfchetz index for Jordan domains in the sphere: consider a continuous map $h : X \to \mathbb{S}^2$ where $X \subset \mathbb{S}^2$ and a Jordan domain U such that $Cl(U) \subset X$, $\partial U \cap \mathrm{Fix}(h) = \emptyset$. Moreover we restrict our attention to the situation $Cl(U) \cup h(Cl(U)) \neq \mathbb{S}^2$ (it is enough for our purpose). Then we can pick a chart $\psi : \mathbb{S}^2 \setminus \{a\} \to \mathbb{R}^2$ where $a \in \mathbb{S}^2 \setminus \big(Cl(U) \cup h(Cl(U))\big)$ and we simply define

$$\mathrm{Ind}(h, U) := \mathrm{Ind}(\psi \circ h \circ \psi^{-1}, \psi(J))$$

It does not depend on the choice of ψ. Properties 3.8 also hold in this context.

We also demand that the Lesfchetz index is "additive"; Let $h : X \to \mathbb{S}^2$ be as above and suppose that U_i, $i \in I$ is a family of pairwise disjoint Jordan domains such that $\mathrm{Fix}(h)$ is disjoint from each ∂U_i and meets only finitely many U_i's. Then we define

$$\mathrm{Ind}(h, \bigcup_{i \in I} U_i) := \sum_{i \in I} \mathrm{Ind}(h, U_i).$$

We have the useful

Property 3.9 *Let $h : X \to \mathbb{S}^2$ be as above and let $U_1, U_2 \subset \mathbb{S}^2$ be two Jordan domains such that $Cl(U_i) \subset X$ and $\partial U_i \cap \mathrm{Fix}(h) = \emptyset$ $(i = 1, 2)$. We also assume that each connected component of $U_1 \cap U_2$ is a Jordan domain.*

If $\mathrm{Fix}(h) \cap U_1 = \mathrm{Fix}(h) \cap U_1 \cap U_2$ then we have

$$\mathrm{Ind}(h, U_1) = \mathrm{Ind}(h, U_1 \cap U_2) := \sum_{V \in \pi_0(U_1 \cap U_2)} \mathrm{Ind}(h, V).$$

Proof. The set $\mathrm{Fix}(h) \cap U_1 \cap U_2 = \mathrm{Fix}(h) \cap U_1$ is compact hence it is covered by finitely many connected components $V_1, \ldots, V_n$ of $U_1 \cap U_2$. The proof is by induction on n. There is nothing to do if $n = 0$. If $n = 1$, first remark that we can slightly alter ∂V_1 is such a way that $Cl(V_1) \subset U_1$. We get $\mathrm{Ind}(h, U_1) = \mathrm{Ind}(h, V_1)$ by considering an isotopy between the Jordan curves ∂U_1 and ∂V_1 inside the annulus $Cl(U_1) \setminus V_1$. Suppose now the result was proved for $n = 0, 1, \ldots, N$ and let us check that it also holds for $N+1$. Consider an arc α contained in U_1 except its two endpoints a, b in $J = \partial U_1$ and write α_1, α_2 for the two subarcs of J having

a, b as endpoints. Such an arc α separates U_1 into two connected components W_1, W_2 which are Jordan domains bounded by respectively the Jordan curves $C_1 = \alpha_1 \cup \alpha$ and $C_2 = \alpha_2 \cup \alpha$. Moreover α can be choosen disjoint from the $Cl(V_i)$'s and in such a way that each Jordan domain W_k contains at least one of the $Cl(V_i)$'s. We conclude by observing that

$$\mathrm{Ind}(h, U) = \mathrm{Ind}(h, W_1) + \mathrm{Ind}(h, W_2)$$

and by using our induction hypothesis with $W_k \cap U_2$ $(k = 1, 2)$. ■.

We end with a result which is a consequence of the so-called Lefschetz-Hopf Theorem (see e.g. [D]) and which cannot be proved using only our elementary point of view.

Theorem 3.10 *If h is an orientation preserving (resp. reversing) of $\mathbb{S}^2$, then we have $\mathrm{Ind}(h, \mathbb{S}^2) = 2$ (resp. $\mathrm{Ind}(h, \mathbb{S}^2) = 0$). In particular, since $2 \neq 0$, every orientation preserving homeomorphism of $\mathbb{S}^2$ admits a fixed point.*

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