A Brouwer-like theorem for orientation reversing homeomorphisms of the sphere

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ABSTRACT: We provide a topological proof that each orientation reversing homeomorphism of the 2-sphere which has a point of period $k \geq 3$ also has a point of period 2. Moreover if such a k-periodic point can be chosen arbitrarily close to an isolated fixed point o then the same is true for the 2-periodic point. We also strengthen this result proving that if an orientation reversing homeomorphism h of the sphere has no 2-periodic point then the complement of the fixed point set can be covered by invariant open sets where h is conjugate either to the map $(x, y) \mapsto (x + 1, -y)$ or to the map $(x, y) \mapsto \frac{1}{2}(x, -y)$.
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1 Introduction

A classical theorem of Brouwer asserts, in its weaker version, that an orientation preserving homeomorphism of the plane $\mathbb{R}^2$ which possesses a k-periodic point, $k \geq 2$, also has a fixed point (see [2] or [3], [7], [12]). The aim of this paper is to give a counterpart of this result in the framework of orientation reversing homeomorphisms. Considering homeomorphisms of the 2-sphere $\mathbb{S}^2$, we first prove that if such a homeomorphism reverses the orientation and has a k-periodic point, $k \geq 3$, then it also admits a 2-periodic point (Theorem 3.1). Using Nielsen-Thurston theory, such a result was already known for C^1-diffeomorphisms ([10]) and one could probably drop the smoothness assumption working again with this powerful theory. We give a topological proof based on the computation of the Lefschetz index on suitable open subsets of $\mathbb{S}^2$. This point of view emphasizes the analogy with the result of Brouwer mentioned above and leads, we hope, to a fairly intuitive proof. For example these arguments allow one to localize the 2-periodic orbit on both sides of a Jordan curve with some index properties and can be readily adapted to give a local version of Theorem 3.1: If an isolated fixed point is the limit of k-periodic points ($k \geq 3$) then it is also the limit of 2-periodic points (Theorem 4.3).

Section 5 is devoted to a "strong version" of our result. This is motivated by the Brouwer plane translation theorem which, roughly speaking, asserts that if h is a fixed point free orientation preserving planar homeomorphism then every point is contained in a simply connected invariant domain where h is conjugated to a translation (see [9], [12], [16] for modern references). It is then natural to expect a version of our result which would assert that an orientation reversing homeomorphism h of $\mathbb{S}^2$ without a 2-periodic point has "obvious" dynamics on some invariant open sets covering the complement of the fixed point set $\mathrm{Fix}(h)$. This is carried out in Theorem 5.1 where such open sets are shown to exist, where h is conjugate either to the map $(x, y) \mapsto (x + 1, -y)$ or to the map $(x, y) \mapsto \frac{1}{2}(x, -y)$.

2 Background

2.1 Notations and basic definitions

The plane $\mathbb{R}^2$ is endowed with its euclidean norm $||\cdot||$ and we think of the 2-sphere $\mathbb{S}^2$ as the one point compactification of $\mathbb{R}^2$, that is $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$. Thus a planar homeomorphism is identified with a homeomorphism of $\mathbb{S}^2$ fixing the point ∞ and our results are also valid for such a homeomorphism.

For $X \subset Y \subset \mathbb{S}^2$, we write respectively $Int_Y(X)$, $Cl_Y(X)$ and $\partial_Y(X) = Cl_Y(X) \setminus Int_Y(X)$ for the interior, the closure and the frontier of X with respect to Y. For the sake of simplicity we omit the subscript Y when $Y = \mathbb{S}^2$. We also denote $\pi_0(X)$ the set of all the connected components of X.

An *arc* is a subset of $\mathbb{S}^2$ homeomorphic to the interval $[0, 1]$ and an *open arc* is an arc with its two endpoints removed. If γ is an arc with a provided orientation and a,b two points met in this order on γ, then $[a, b]_\gamma$ is the subarc from a to b for the chosen orientation of γ.

A *topological closed disc* is a subset of $\mathbb{S}^2$ homeomorphic to the closed unit disc.

For any map $f : E \to F$, the fixed point set $\{z \in E | f(z) = z\}$ is denoted by $\mathrm{Fix}(f)$. A point $z \in E$ is said to be a *k-periodic point of f* if k is the *smallest* positive integer such that the sequence $z, f(z), \ldots, f^k(z)$ is well-defined and $f^k(z) = z$.

2.2 Jordan curves and Jordan domains

A *Jordan curve* is a subset of $\mathbb{S}^2$ homeomorphic to the unit circle $\mathbb{S}^1$. According to the Jordan Theorem, the complement $\mathbb{S}^2 \setminus J$ of a Jordan curve J has exactly two connected components and J is their common frontier. An open subset of $\mathbb{S}^2$ which is a connected component of the complement of a Jordan curve is said to be *a Jordan domain.* If J is a Jordan curve with a given orientation and if $a \neq b$ are two points of J, then $[a,b]_J$ denotes the arc on J from a to b for this orientation of J. Note that our vocabulary slightly differs from the one usually used in the literature, where a Jordan curve J is often defined as a subset of $\mathbb{R}^2$ and a Jordan domain as the bounded component of $\mathbb{R}^2 \setminus J$.

For later use, we collect now a few propositions about Jordan domains. The first one is a straightforward adaptation of a result of Kerékjártó;

Proposition 2.1 *Let U,U' be two Jordan domains such that $U \cap U' \neq \emptyset$ and $\big(\mathbb{S}^2 \setminus Cl(U)\big) \cap \big(\mathbb{S}^2 \setminus Cl(U')\big) \neq \emptyset$. Then every connected component of $U \cap U'$ is also a Jordan domain, whose frontier is contained in $\partial U \cup \partial U'$.*

Indeed, if we assume that $Cl(U)$ and $Cl(U')$ are contained in the plane $\mathbb{R}^2$ then the hypothesis $\big(\mathbb{S}^2 \setminus Cl(U)\big) \cap \big(\mathbb{S}^2 \setminus Cl(U')\big) \neq \emptyset$ is of course satisfied and Proposition 2.1 is a well-known result of Kerékjártó ([14]). In the general case, let us choose a point $z \in \big(\mathbb{S}^2 \setminus Cl(U)\big) \cap \big(\mathbb{S}^2 \setminus Cl(U')\big)$ and a homeomorphism φ of $\mathbb{S}^2$ such that $\varphi(z) = \infty$. We are reduced to the previous situation considering $Cl(\varphi(U)) = \varphi(Cl(U))$ and $Cl(\varphi(U')) = \varphi(Cl(U'))$.

We have also:

Lemma 2.2 *Let U, V be two Jordan domains such that $V \subset U$, $V \neq U$, and $\partial V \cap \partial U$ contains at least two points.*

(1) For any $\mu \in \pi_0(U \cap \partial V)$ we have:

(i) μ is an open arc lying in U with its two endpoints $x = x(\mu)$, $y = y(\mu)$ in ∂U (it is usually said that μ is a cross-cut of U),

(ii) We have a partition $U \setminus \mu = U'_\mu \cup U''_\mu$ where U'_μ (resp. U''_μ) is the Jordan domain contained in U whose frontier is $\mu \cup [x,y]_{\partial U}$ (resp. $\mu \cup [y,x]_{\partial U}$).

(iii) The Jordan domain V is contained either in U'_μ or in U''_μ.

Notation: *We write $U_{\mu,V}$ for the connected component of $U \setminus \mu$ containing V and μ_* for the arc in ∂U with endpoints x, y such that $\mu \cup \mu_* = \partial U_{\mu,V}$.*

(2) If a is a point in $U \setminus Cl(V)$, there exists a unique $\mu = \mu(a) \in \pi_0(U \cap \partial V)$ such that $a \notin U_{\mu,V}$ (see Fig. 1).

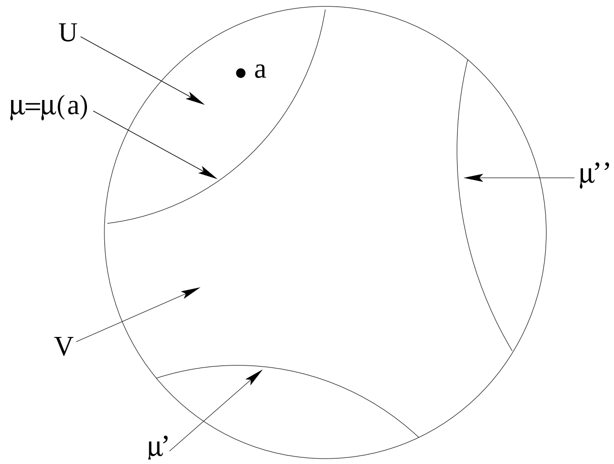

Figure 1: The Jordan domains U, V and the arcs μ

Proof of Lemma 2.2: (1) (i) follows from the fact that $U \cap \partial V = \partial V \setminus \partial U$ is not connected. (ii) is a classical result of plane topology (known as the Θ-curve Lemma). It can be proved with only elementary arguments (see for example [18][Theorem 11.8 page 119]). It can also be seen as a corollary of the Schoenflies Theorem, constructing a homeomorphism of $\mathbb{S}^2$ mapping respectively ∂U and μ onto $\mathbb{S}^1$ and $(-1,1) \times \{0\}$. For (iii), it is enough to remark that V is a connected subset of $U \setminus \mu$.

(2) Choose a' to be any point in V. Since a, a' are separated in $\mathbb{S}^2$ by the closed set ∂V they are also separated in U by a connected component μ of $U \cap \partial V$ (see [18][Theorem 7.1 page 151]) and we have then $a' \in V \subset U_{\mu,V}$, $a \notin U_{\mu,V}$. Now suppose that we can find two connected components $\mu \neq \nu$ of $U \cap \partial V$ such that $a \notin U_{\mu,V} \cup U_{\nu,V}$ and consider the partitions

$$U \setminus \mu = U'_\mu \cup U''_\mu \quad \text{and} \quad U \setminus \nu = U'_\nu \cup U''_\nu$$

given by (1)(ii). Assume for example that $U'_\mu = U_{\mu,V}$ and $U'_\nu = U_{\nu,V}$. Since $\mu \cap \nu = \emptyset$, we have either $\mu \subset U_{\nu,V}$ or $\mu \subset U''_\nu$. This latter case is actually not possible since $\mu \subset Cl(V)$ and $U''_\nu \cap V = \emptyset$. It follows that

$$a \in U''_\nu \subset U \setminus U_{\nu,V} \subset U \setminus \mu \text{ and consequently } U''_\nu \subset U''_\mu.$$

Reversing the roles of ν and μ we obtain $U''_\mu \subset U''_\nu$ so $U''_\mu = U''_\nu$ and finally

$$\mu = U \cap \partial U''_\mu = U \cap \partial U''_\nu = \nu\,,$$

a contradiction. □

2.3 Lefschetz index

Let M be a manifold (more generally, a Euclidean Neighbourhood Retract), U an open subset of M and $\varphi : U \to M$ a continuous map such that $\mathrm{Fix}(\varphi)$ is compact. One can define the *fixed point index*, or *Lefschetz index*, $\mathrm{I}(\varphi) \in \mathbb{Z}$ (see [5]) which possesses the following properties:

Properties 2.3 *(1) $I(\varphi)$ depends only on the set $Fix(\varphi)$; That is, $I(\varphi) = I(\varphi|_{U'})$ for any open set U' such that $Fix(\varphi) \subset U' \subset U$.*

(2) If $Fix(\varphi) = \emptyset$ then $I(\varphi) = 0$.

(3) Additivity; If $U = \bigcup_{i=1}^{n} U_i$ where $U_1, \ldots, U_n$ are open and if the sets $Fix(\varphi|_{U_i})$ are compact and pairwise disjoint ($1 \leq i \leq n$) then

$$I(\varphi) = \sum_{i=1}^{n} I(\varphi|_{U_i}).$$

(4) Homotopy Invariance; If $(\varphi_t : U \to M)_{0\leq t\leq 1}$ is a homotopy from $\varphi_0 = \varphi$ to φ_1 such that $\bigcup_{0\leq t\leq 1} Fix(\varphi_t)$ is compact, then $I(\varphi)=I(\varphi_1)$.

(5) Topological Invariance; If $\psi : M \to M$ is a homeomorphism then the maps φ and $\psi \circ \varphi \circ (\psi^{-1})|_{\psi(U)} : \psi(U) \to M$ have the same Lefschetz index.

Proof of Properties 2.3: The first four ones are stated in [5] so we just give the argument for the fifth one. It is in fact a consequence of the following Commutativity Property of the Lefschetz index (see [5]):
Let U_i be an open set of a manifold M_i ($i \in \{1, 2\}$) and let $k_1 : U_1 \to M_2$, $k_2 : U_2 \to M_1$ be two continuous maps. Then the composite maps

$$\begin{array}{ccc} k_1^{-1}(U_2) & \to & M_1 \\ x & \mapsto & k_2(k_1(x)) \end{array} \quad \text{and} \quad \begin{array}{ccc} k_2^{-1}(U_1) & \to & M_2 \\ x & \mapsto & k_1(k_2(x)) \end{array}$$

have homeomorphic fixed point sets. They also have the same Lefschetz index if their fixed point sets are compact.
We remark that $\mathrm{Fix}(\psi \circ \varphi \circ (\psi^{-1})|_{\psi(U)}) = \psi(\mathrm{Fix}(\varphi))$ is compact, which ensures directly that the Lefschetz index of $\psi \circ \varphi \circ (\psi^{-1})|_{\psi(U)}$ is defined. Now use the Commutativity Property with

$$k_1 = (\psi^{-1})|_{\psi(U)} : \psi(U) \to M \quad \text{and } k_2 = \psi \circ \varphi : U \to M.$$

We obtain

$$\mathrm{I}(\psi \circ \varphi \circ (\psi^{-1})|_{\psi(U)}) = \mathrm{I}(\varphi|_{\varphi^{-1}(U)}).$$

Furthermore we have obviously

$$\mathrm{Fix}(\varphi) \subset \varphi^{-1}(U) \subset U$$

and we conclude with Property 2.3 (1). $\square$

Notations: We will deal in this paper with continuous maps $f : \mathcal{U} \to \mathbb{S}^2$, where $\mathcal{U} \subset \mathbb{S}^2$ is a given open set ($\mathcal{U} = \mathbb{S}^2$ except in Section 4) and we will calculate the Lefschetz index of $\varphi = f|_U$ for various open sets $U \subset \mathcal{U}$. Thus we will speak of the *index of f on U* and we will write $\mathrm{Ind}(f, U)$ instead of $\mathrm{I}(f|_U)$. Moreover, if U contains exactly one fixed point z of f, we recall that $\mathrm{Ind}(f, U)$ is said to be the *Lefschetz index of z (for the map f)* and is also denoted $\mathrm{Ind}(f, z)$.

The following lemma derives from Properties 2.3.

Lemma 2.4 *Let U_1, U_2 be two open subsets of $\mathbb{S}^2$ and let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map. We suppose that $U_i \cap Fix(f)$ is compact ($i \in \{1, 2\}$). Then there are only finitely many connected components V of $U_1 \cap U_2$ such that $Ind(f, V) \neq 0$ and we have*

$$Ind(f, U_1 \cap U_2) = \sum_{V \in \pi_0(U_1 \cap U_2)} Ind(f, V).$$

Proof of Lemma 2.4: We can suppose $U_1 \cap U_2 \neq \emptyset$. The set $U_1 \cap U_2 \cap \mathrm{Fix}(f)$ is compact so there exists a finite open covering

$$U_1 \cap U_2 \cap \mathrm{Fix}(f) \subset V_1 \cup \ldots \cup V_n$$

where $V_1, \ldots, V_n$ are some connected components of $U_1 \cap U_2$. According to Properties 2.3 (1)-(2) we have $\mathrm{Ind}(f, U_1 \cap U_2) = \mathrm{Ind}(f, V_1 \cup \ldots V_n)$ and $\mathrm{Ind}(f, V)$=0 for any $V \in \pi_0(U_1 \cap U_2) \setminus \{V_1, \ldots, V_n\}$. Then we obtain with Property 2.3 (3)

$$\mathrm{Ind}(f, U_1 \cap U_2) = \sum_{i=1}^{n} \mathrm{Ind}(f, V_i) = \sum_{V \in \pi_0(U_1 \cap U_2)} \mathrm{Ind}(f, V).$$

$\square$

Although this is not essential in this paper, let us recall that, for planar maps, there is an intuitive interpretation for the Lefschetz index on Jordan domains:

Proposition 2.5 *Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ and let $f : \mathcal{U} \to \mathbb{R}^2$ be a continuous map. If U is a Jordan domain such that $Cl(U) \subset \mathcal{U}$ and $\partial U \cap Fix(f) = \emptyset$ then $Ind(f, U)$ is the degree of the map*

$$\begin{array}{ccc} \mathbb{S}^1 & \to & \mathbb{S}^1 \\ t & \mapsto & \frac{f(u(t)) - u(t)}{||f(u(t)) - u(t)||} \end{array}$$

where $u : \mathbb{S}^1 \to \partial U = u(\mathbb{S}^1)$ is any homeomorphism which endows ∂U with its counterclockwise orientation.

This result is for example a consequence of [5][exercise 5 page 207]. In other words, if f and U are as in Proposition 2.5 then $\mathrm{Ind}(f, U)$ is the winding number of the vector $f(z) - z$ when z moves along the Jordan curve ∂U in the counterclockwise direction. For this reason $\mathrm{Ind}(f, U)$ is also said to be the index of f on the curve ∂U and is often denoted $\mathrm{Ind}(f, \partial U)$ instead of $\mathrm{Ind}(f, U)$ in the literature.

We end this section with an index zero lemma which will be repeatedly used in this paper;

Lemma 2.6 *Let U, V be two Jordan domains such that $V \subset U$, $V \neq U$, $\partial V \cap \partial U$ contains at least two points and let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map. Assume furthermore that*

(i) f has no fixed point in ∂V,

(ii) $U \cap \partial V \cap f(U) = \emptyset$,

(iii) there exists $\mu \in \pi_0(U \cap \partial V)$ such that, using the notations of Lemma 2.2, $f(\mu_*) \cap U = \emptyset$.

Then we have Ind$(f, V) = 0$.

Proof of Lemma 2.6: Because of (i), $\mathrm{Ind}(f, V)$ is defined. We consider the Jordan domain $U_{\mu,V}$ and the arc μ_* associated to μ, as explained in Lemma 2.2. Since $\partial U_{\mu,V} = \mu \cup \mu_*$ it is easy to construct a homotopy

$$\begin{array}{ccc} Cl(U_{\mu,V}) \times [0,1] & \to & Cl(U_{\mu,V}) \\ (z,t) & \mapsto & r_t(z) \end{array}$$

with the following properties:

1. r_0 is the identity map of $Cl(U_{\mu,V})$,
2. $r_1(Cl(U_{\mu,V})) = \mu_*$,
3. $\forall t \in [0,1]\, \forall z \in \mu_* \quad r_t(z) = z$,
4. if $0 < t \leq 1$ then $r_t(Cl(U_{\mu,V})) \subset U_{\mu,V} \cup \mu_*$.

Essentially, this simply means that $(r_t)_{0 \leq t \leq 1}$ is a strong retracting deformation of $Cl(U_{\mu,V})$ onto μ_*. The additional fourth property ensures that the maps $f \circ r_t$ have no fixed point on ∂V ($0 \leq t \leq 1$). Indeed there is nothing to prove for $f \circ r_0|_{\partial V} = f|_{\partial V}$ and for $0 < t \leq 1$, $z \in \partial V \subset Cl(U_{\mu,V})$, we have:

- If $z \in \mu_*$ then $f \circ r_t(z) = f(z) \neq z$,
- If $z \in U_{\mu,V} \cup \mu$ then with (4)

$$f \circ r_t(z) \in f(U_{\mu,V}) \cup f(\mu_*)$$

and consequently $z \neq f \circ r_t(z)$ since, using (ii) and (iii),

$$\partial V \cap (U_{\mu,V} \cup \mu) \cap f(U_{\mu,V}) \subset \partial V \cap U \cap f(U) = \emptyset$$

and

$$\partial V \cap (U_{\mu,V} \cup \mu) \cap f(\mu_*) \subset U \cap f(\mu_*) = \emptyset.$$

Moreover we have

$$f \circ r_1(V) \cap V \subset f \circ r_1(Cl(U_{\mu,V})) \cap U = f(\mu_*) \cap U = \emptyset$$

which gives $\mathrm{Ind}(f \circ r_1, V) = 0$. The result then follows from Property 2.3 (4) considering the homotopy $(f \circ r_t|_V)_{0 \leq t \leq 1}$. □

2.4 Translation arcs

Definition 2.7 *Let f be a homeomorphism of $\mathbb{S}^2$. An arc γ is said to be* a translation arc for f if

1. one of its two endpoints, say p, is mapped by f onto the other one,
2. we have furthermore $\gamma \cap f(\gamma) = \{p, f(p)\} \cap \{f(p), f^2(p)\}$.

Note that, with the above definition, $\mathrm{Fix}(f)$ is necessarily disjoint from $\bigcup_{k\in\mathbb{Z}} f^k(\gamma)$. For convenience we also make the following convention. If f is a given homeomorphism of $\mathbb{S}^2$ and γ a translation arc for f with endpoints p and $f(p)$ then the arcs $f^k(\gamma)$ are oriented from $f^k(p)$ to $f^{k+1}(p)$ ($k \in \mathbb{Z}$). Of course γ could also be thought as a translation arc for f^{-1} and the arcs $f^k(\gamma)$ would be then oriented from $f^{k+1}(p)$ to $f^k(p)$.

Lemma 2.8 *Let h be a homeomorphism of $\mathbb{S}^2$ such that $h^2 \neq Id_{\mathbb{S}^2}$ and let m be a point in $\mathbb{S}^2 \setminus Fix(h^2)$. Then at least one of the following two assertions holds:*

A1 : There exists a translation arc α for h, with endpoints p and $h(p)$, such that $\alpha \cap h(\alpha) = \{h(p)\}$, $\alpha \cap h^2(\alpha) = \{p\} \cap \{h^3(p)\}$ and $m \in \alpha \setminus \{p, h(p)\}$.

A2 : There exists a translation arc β for h^2, with endpoints q and $h^2(q)$, such that $\beta \cap h(\beta) = \emptyset$ and $m \in \beta \setminus \{q, h^2(q)\}$.

Proof of Lemma 2.8: Let U be the connected component of $\mathbb{S}^2 \setminus Fix(h^2)$ which contains m. We know that $h^2(U) = U$ (see [4]) and, since an open connected subset of $\mathbb{S}^2$ is arcwise connected, there exists an arc γ lying in U with endpoints m and $h^2(m)$. We can slightly enlarge this arc γ and obtain a topological closed disc Δ such that $\gamma \subset Int(\Delta) \subset \Delta \subset U$. For any $R > 0$, let us denote by D_R the closed disc in $\mathbb{R}^2$ with center the origin $o = (0,0)$ and radius R. Up to conjugacy in $\mathbb{S}^2$, we can suppose that $m = o$ and $\Delta = D_1$. Define respectively $R_1 > 0$ and $R_2 > 0$ to be the unique real numbers such that

$$\partial D_{R_1} \cap h(\partial D_{R_1}) = D_{R_1} \cap h(D_{R_1}) \neq \emptyset$$

and

$$\partial D_{R_2} \cap h^2(\partial D_{R_2}) = D_{R_2} \cap h^2(D_{R_2}) \neq \emptyset.$$

Observe that, since $h^2(o) \in Int(D_1) \cap h^2(Int(D_1))$, we have necessarily $R_2 < 1$ so

$$D_{R_2} \subset D_1 \subset \mathbb{S}^2 \setminus \mathrm{Fix}(h^2) \subset \mathbb{S}^2 \setminus \mathrm{Fix}(h).$$

Lemma 2.8 then follows from the comparison of R_1 and R_2:

• If $R_1 \leq R_2$, let us choose a point $p \in \partial D_{R_1}$ such that $h(p) \in \partial D_{R_1}$. Since $D_{R_1} \subset D_{R_2}$, the points $p, h(p), h^2(p)$ are pairwise distinct and any arc α from p to $h(p)$ satisfying $o \in \alpha \setminus \{p, h(p)\} \subset Int(D_{R_1})$ (see Fig. 2) has the properties required in the assertion A1.

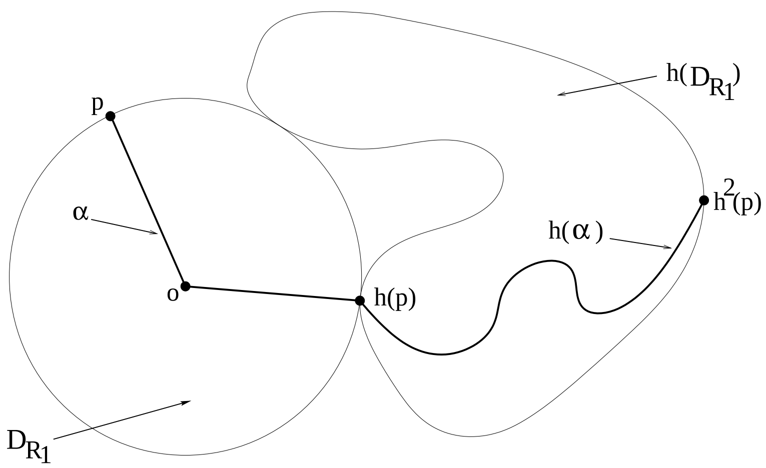

Figure 2: The translation arc α

• If $R_1 > R_2$, let $q \in \partial D_{R_2}$ such that $h^2(q) \in \partial D_{R_2}$. Choose an arc β from q to $h^2(q) \neq q$ such that $o \in \beta \setminus \{q, h^2(q)\} \subset Int(D_{R_2})$. It is clear that β is an arc as described in the assertion A2 (possibly with $q = h^4(q)$). $\square$

2.5 Brouwer's lemma

Let f be an orientation preserving homeomorphism of $\mathbb{S}^2$. Suppose that γ is a translation arc for f, with endpoints $p, f(p)$, such that $\bigcup_{k\in\mathbb{Z}} f^k(\gamma)$ is not a simple curve, i.e. the set $\{k \geq 1 | (\gamma \setminus \{f(p)\}) \cap f^k(\gamma \setminus \{f(p)\}) \neq \emptyset\}$ is nonempty. If n denotes the minimum of this latter set and x the first point on $f^n(\gamma)$ to meet γ, then

$$C = [x, f(p)]_\gamma \bigcup_{i=1}^{n-1} f^i(\gamma) \cup [f^n(p), x]_{f^n(\gamma)}$$

is clearly a Jordan curve and we have:

Proposition 2.9 (Brouwer's lemma) *Let U be a connected component of $\mathbb{S}^2 \setminus C$. Then we have Ind(f, U) = 1. In particular f admits a fixed point in U.*

Usually Brouwer's Lemma ([3], [12][Appendice]) is stated for an orientation preserving homeomorphism f of $\mathbb{R}^2$. The arcs $f^i(\gamma)$ then lie in $\mathbb{R}^2$ and it is shown that $\mathrm{Ind}(f, U)$=1 for the bounded connected component U of $\mathbb{R}^2 \setminus C$. Proposition 2.9 is only a minor adaptation of such a statement. Indeed, let U_1 and U_2 be the two connected components of $\mathbb{S}^2 \setminus C$. According to the Lefschetz-Hopf Theorem (see for example [5]) we have

$$2 = \mathrm{Ind}(f, \mathbb{S}^2) = \mathrm{Ind}(f, U_1) + \mathrm{Ind}(f, U_2).$$

Consequently f admits at least one fixed point $z \in \mathbb{S}^2 \setminus C$, say $z \in U_1$, and it is enough to check $\mathrm{Ind}(f, U_2) = 1$. Choosing a homeomorphism φ of $\mathbb{S}^2$ such that $\varphi(z) = \infty$ it is clear that $\varphi \circ f \circ \varphi^{-1}$ is a planar homeomorphism posseding $\varphi(\gamma)$ as a translation arc, hence the above references give $\mathrm{Ind}(\varphi \circ f \circ \varphi^{-1}, \varphi(U_2))$=1. We derive $\mathrm{Ind}(f, U_2)$=1 from Property 2.3 (5).

3 First result: period $k \geq 3$ implies period 2

We prove in this section the

Theorem 3.1 *Let h be an orientation reversing homeomorphism of the sphere $\mathbb{S}^2$ possessing a point of period at least three. Then h admits also a 2-periodic point. More precisely, there exist a Jordan curve $C \subset \mathbb{S}^2 \setminus Fix(h^2)$ and a point z such that, writing U, U' for the two connected components of $\mathbb{S}^2 \setminus C$, we have*

$$z = h^2(z) \in U, \quad h(z) \in U',$$

$$Ind(h, U) = 0, \quad Ind(h^2, U) = 1.$$

We introduce the following notation in order to avoid unpleasant repetitions.
Notation: f, g being two homeomorphisms of $\mathbb{S}^2$, we write $f \sim g$ if and only if we have

1. $\forall i \in \{1,2\} \quad \mathrm{Fix}(f^i) = \mathrm{Fix}(g^i)$ and $\mathrm{Ind}(f^i, \Omega) = \mathrm{Ind}(g^i, \Omega)$ for any open set $\Omega \subset \mathbb{S}^2$ such that $\Omega \cap \mathrm{Fix}(f^i)$ is compact,

2. $\forall z \in \mathrm{Fix}(f^2) \quad f(z) = g(z)$.

Clearly, $\sim$ defines an equivalence relation and Theorem 3.1 will be proved if its conclusion holds for a homeomorphism $g \sim h$.

Let us explain the main idea to detect the 2-periodic point $z \in U$. We will prove actually the following stronger result (although less meaningful from the dynamical view-point).

An additional index property: There exists a connected component U of $\mathbb{S}^2 \setminus C$ such that, in addition to the above index properties, we have $\mathrm{Ind}(h^2, U \cap h(U)) = 0$ (possibly with $U \cap h(U) = \emptyset$).

In particular this will show $\mathrm{Ind}(h^2, U) \neq \mathrm{Ind}(h^2, U \cap h(U))$ and Property 2.3 (1) then gives $\mathrm{Fix}(h^2) \cap U \neq \mathrm{Fix}(h^2) \cap U \cap h(U)$. In other words there exists a point $z \in U$ such that $h^2(z) = z$ and $h(z) = h^{-1}(z) \in U' = \mathbb{S}^2 \setminus Cl(U)$, as announced in Theorem 3.1.

We remark finally that there is no loss in proving this last index property only for a homeomorphism $g \sim h$:

Claim: Suppose $g \sim h$ and let $\Omega \subset \mathbb{S}^2$ be any open set. Then the indices $\mathrm{Ind}(g^2, \Omega \cap g(\Omega))$ and $\mathrm{Ind}(h^2, \Omega \cap h(\Omega))$ are simultaneously defined or not and, if defined, are equal.

Indeed g and h have exactly the same fixed points and the same 2-periodic orbits so

$$\begin{aligned}\mathrm{Fix}(g^2) \cap \Omega \cap g(\Omega) &= \mathrm{Fix}(g^2) \cap \Omega \cap g(\Omega) \cap h(\Omega) \\ &= \mathrm{Fix}(h^2) \cap \Omega \cap g(\Omega) \cap h(\Omega) = \mathrm{Fix}(h^2) \cap \Omega \cap h(\Omega).\end{aligned}$$

If these sets are compact we let $\Omega' = \Omega \cap g(\Omega) \cap h(\Omega)$. It follows from Property 2.3 (1) and from the definition of $\sim$ that

$$\mathrm{Ind}(g^2, \Omega \cap g(\Omega)) = \mathrm{Ind}(g^2, \Omega') = \mathrm{Ind}(h^2, \Omega') = \mathrm{Ind}(h^2, \Omega \cap h(\Omega)).$$

3.1 A proposition about translation arcs of h

Proposition 3.2 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$. Assume that we can find a translation arc α for h, with endpoints $p, h(p)$, such that:*

- $\alpha \cap h(\alpha) = \{h(p)\}, \quad \alpha \cap h^2(\alpha) = \{p\} \cap \{h^3(p)\}$,
- $\alpha \cap h^k(\alpha) \neq \emptyset$ *for an integer $k \geq 2$, i.e. the set $\bigcup_{k\in\mathbb{Z}} h^k(\alpha)$ is not a simple curve.*

Then there exist a Jordan curve C and a point z as announced in Theorem 3.1.

This proposition is a consequence of the following lemmas. The first one allows us to reduce to the situation where, for a smallest $n \geq 2$, the iterate $h^n(\alpha)$ meets the arc α "in a nice way". This will be convenient to compute some indices on a suitable Jordan domain. We use the same technique as in the proof of [3][Theorem 1], observing that the "perturbations" of h can be constructed without altering not only the fixed point set but also the set of the 2-periodic orbits. For completeness and because similar lemmas will be used farther in this paper, we write a rather detailed proof.

Lemma 3.3 *Let h, α be as in Proposition 3.2. Let us define n to be the minimun of the set $\{k \geq 2 | \alpha \cap h^k(\alpha) \neq \emptyset\}$ and x to be the first point on $h^n(\alpha)$ to meet α. Then there exists an orientation reversing homeomorphism $h_* \sim h$ admitting $\alpha_* = [x, h(p)]_\alpha$ as a translation arc such that $h_*(x) = h(p)$ and*

- $\forall i \in \{1, \ldots, n-1\} \quad h_*^i(\alpha_*) = h^i(\alpha)$,
- $h_*^n(\alpha_*) = [h^n(p), x]_{h^n(\alpha)}$.

Proof of Lemma 3.3: If $n = 2$ then $\{x\} = \{p\} = \{h^3(p)\} = \alpha \cap h^2(\alpha)$ and there is nothing to prove. We suppose from now on $n \geq 3$.

• First step.

If we have already $x = p$, just define $g = h$. Otherwise, since $x \neq h(p)$ by the minimality of n, observe that the arc $[p, x]_\alpha$ has the following two properties:

(i) it is disjoint from its images by h and h^2,

(ii) it is disjoint from $h^i(\alpha)$ for every $i \in \{1, \ldots, n-1\}$.

One can construct a topological closed disc D_1 neighbourhood of $[p, x]_\alpha$, thin enough to satisfy (i) and (ii). Since $\alpha_* = [x, h(p)]_\alpha$ is an arc, one can also construct a homeomorphism φ of $\mathbb{S}^2$ with support in D_1 such that $\varphi(\alpha_*) = \alpha$ (see for example [3][Lemma 2]). Defining $g = h \circ \varphi$, let us check that $g \sim h$ and also that α_* is a translation arc for g with $g(x) = h(p)$ and

$$\forall i \in \{1, \ldots, n\} \quad g^i(\alpha_*) = h^i(\alpha).$$

The Alexander trick gives an isotopy $(\varphi_t)_{0 \le t \le 1}$ with support in D_1 from $\varphi_0 = Id_{\mathbb{S}^2}$ to $\varphi_1 = \varphi$. It is easily seen with $D_1 \cap h(D_1) = \emptyset = D_1 \cap h^2(D_1)$ that, for $i \in \{1, 2\}$ and $t \in [0, 1]$, the homeomorphisms h^i and $(h \circ \varphi_t)^i$ have exactly the same fixed point set. We also observe that

$$z = h^2(z) \Longrightarrow z \notin D_1 \Longrightarrow h \circ \varphi_t(z) = h(z).$$

Furthermore, if $\Omega \subset \mathbb{S}^2$ is an open set such that $\Omega \cap \mathrm{Fix}(h^i)$ is compact, we obtain $\mathrm{Ind}(h^i, \Omega) = \mathrm{Ind}(g^i, \Omega)$ considering the homotopy $((h \circ \varphi_t)^i|_\Omega)_{0 \le t \le 1}$ in Property 2.3 (4). This shows $g \sim h$. By the construction we have $g(\alpha_*) = h \circ \varphi(\alpha_*) = h(\alpha)$ with $g(x) = h \circ \varphi(x) = h(p)$. Since D_1 is disjoint from $\bigcup_{i=1}^{n-1} h^i(\alpha)$, we get $g = h$ on $\bigcup_{i=1}^{n-1} h^i(\alpha)$ so

$$\forall i \in \{1, \ldots, n\} \quad g^i(\alpha_*) = h^i(\alpha) \text{ with } g^i(x) = h^i(p).$$

- Second step.

If $\{x\} = \{g^{n+1}(x)\} = \alpha_* \cap g^n(\alpha_*)$, it suffices to define $h_* = g$. Otherwise, since $x \neq g^n(x) = h^n(p)$, the arc $[x, g^{n+1}(x)]_{g^n(\alpha_*)}$ possesses the following properties:

(iii) it is disjoint from its images by g and g^2,

(iv) it is disjoint from $g^i(\alpha_*)$ for every $i \in \{1, \ldots, n-1\}$.

Choose now a topological closed disc D_2 neighbourhood of $[x, g^{n+1}(x)]_{g^n(\alpha_*)}$ satisfying (iii),(iv) and a homeomorphism ψ of $\mathbb{S}^2$ supported in D_2 such that

$$\psi(g^n(\alpha_*)) = [g^n(x), x]_{g^n(\alpha_*)}.$$

Using the same arguments as in the first step, it is not difficult to check that $h_* = \psi \circ g \sim g$ has the required properties. □

Lemma 3.4 *Let h, α, n be as in Lemma 3.3. We assume furthermore that $\alpha \cap h^n(\alpha) = \{p\} = \{h^{n+1}(p)\}$ and we consider the Jordan curve $C = \bigcup_{i=0}^{n} h^i(\alpha)$. If U is a connected component of $\mathbb{S}^2 \setminus C$, then we have Ind(h, U) = 0 and Ind(h^2, U) = 1.*

Proof of Lemma 3.4: It is easy to construct an orientation reversing homeomorphism g of $\mathbb{S}^2$ possessing the following properties:

1. $g = h$ on $\bigcup_{i=0}^{n-1} h^i(\alpha)$,
2. g maps $h^n(\alpha)$ onto α (hence $g(C) = C$),
3. g interchanges the two connected components of $\mathbb{S}^2 \setminus C$.

Thus $g^{-1} \circ h$ is an orientation preserving homeomorphism of the sphere which coincides with the identity map $Id_{\mathbb{S}^2}$ on the arc $\bigcup_{i=0}^{n-1} h^i(\alpha)$. Using a variation of the Alexander trick (see for example [3]Lemma 1) one can find an isotopy $(\varphi_t)_{0\leq t\leq 1}$ from $\varphi_0 = Id_{\mathbb{S}^2}$ to $\varphi_1 = g^{-1} \circ h$ such that

$$\forall t \in [0,1]\, \forall z \in \bigcup_{i=0}^{n-1} h^i(\alpha) \quad \varphi_t(z) = z.$$

Defining $h_t = g \circ \varphi_t$ $(0 \leq t \leq 1)$, we obtain an isotopy from $h_0 = g$ to $h_1 = h$ such that $h_t = h$ on $\bigcup_{i=0}^{n-1} h^i(\alpha)$ and $(h_t^2)_{0\leq t\leq 1}$ is then an isotopy from g^2 to h^2 such that $h_t^2 = h^2$ on $\bigcup_{i=0}^{n-2} h^i(\alpha)$. Clearly h^2 has no fixed point on α and then also on $\bigcup_{i\in\mathbb{Z}} h^i(\alpha)$. Consequently, for every $t \in [0,1]$, the homeomorphism h_t^2 (and so h_t) has no fixed point on

$$\bigcup_{i=0}^{n-2} h^i(\alpha) \cup h_t\Big(\bigcup_{i=0}^{n-2} h^i(\alpha)\Big) \cup h_t^2\Big(\bigcup_{i=0}^{n-2} h^i(\alpha)\Big) = \bigcup_{i=0}^{n} h^i(\alpha) = C.$$

Hence all the indices $\mathrm{Ind}(h_t, U)$ and $\mathrm{Ind}(h_t^2, U)$ are defined and, according to Property 2.3 (4), we have $\mathrm{Ind}(g, U) = \mathrm{Ind}(h, U)$ and $\mathrm{Ind}(g^2, U) = \mathrm{Ind}(h^2, U)$. We conclude observing that $U \cap g(U) = \emptyset$ gives $\mathrm{Ind}(g, U)$=0 and, as it is well known, $U = g^2(U)$ implies $\mathrm{Ind}(g^2, U)$=1. □

Remark 3.5 *If in Lemma 3.4 we have $n \geq 3$, then $\alpha \cup h(\alpha)$ is a translation arc for the orientation preserving homeomorphism h^2 and Brouwer's lemma gives directly Ind(h^2, U)=1.*

Lemma 3.6 *Let h, α, n and C be as in Lemma 3.4. Then there exists a connected component U of $\mathbb{S}^2 \setminus C$ such that Ind($h^2, U \cap h(U)$) = 0.*

Proof of Lemma 3.6: Let U_1 and $U_2 = \mathbb{S}^2 \setminus Cl(U_1)$ be the two connected components of $\mathbb{S}^2 \setminus C$. We can assume $U_i \cap h(U_i) \neq \emptyset$ for both $i = 1$ and $i = 2$ since otherwise the result is obvious. Let us choose for example $U = U_1$. According to Lemma 2.4, it suffices to prove that $\mathrm{Ind}(h^2, V) = 0$ for any given $V \in \pi_0(U \cap h(U))$. Since h reverses the orientation, every point $z \in C \setminus h^n(\alpha)$ admits a neighbourhood N_z such that $h(N_z \cap U) = h(N_z) \cap U_2$ and $h(N_z \cap U_2) = h(N_z) \cap U$. Consequently we have $(C \setminus \alpha) \cap Cl(U \cap h(U)) = \emptyset$. In particular this shows $h^{\pm 1}(U) \not\subset U$ and we obtain the following properties for every $V \in \pi_0(U \cap h(U))$:

(1) $V \subset U$ with $V \neq U$,

(2) V is a Jordan domain such that $\partial V \subset \alpha \cup h^{n+1}(\alpha)$,

(3) $\partial V \cap C$ contains at least two points.

The first one is clear since $U \not\subset h(U)$. We know from Proposition 2.1 that V is a Jordan domain such that $\partial V \subset C \cup h(C) = \bigcup_{i=0}^{n+1} h^i(\alpha)$ and, since $Cl(V) \subset Cl(U \cap h(U))$ is disjoint from $C \setminus \alpha$, we obtain more precisely $\partial V \subset \alpha \cup h^{n+1}(\alpha)$. The third property follows since otherwise we would have

$$\partial V = Cl(\partial V \setminus C) \subset h^{n+1}(\alpha)$$

which is absurd because an arc cannot contain a Jordan curve.

Thus we can use Lemma 2.2. Every connected component μ of $U \cap \partial V$ is an open arc and the property (2) above also shows that such a μ is a subset of $h^{n+1}(\alpha)$ and has its two endpoints in α. Consequently $C \setminus \alpha$ is contained in the frontier of one of the two connected components of $U \setminus \mu$ and is disjoint from the frontier of the other one (see Lemma 2.2 (1)). Now, since $C \setminus \alpha \subset C \setminus Cl(V)$, there exists a path γ from a point $a \in U$ to a point $b \in C \setminus \alpha$ such that $\gamma \setminus \{b\} \subset U \setminus Cl(V)$. Let $\mu = \mu(a) \in \pi_0(U \cap \partial V)$ be such that $a \notin U_{\mu,V}$ (Lemma 2.2 (2)). Thus b is in the frontier of the connected component of $U \setminus \mu$ which does not contain V so $(C \setminus \alpha) \cap \partial U_{\mu,V} = \emptyset$ and then $\mu_* \subset \alpha$.

We obtain finally $\mathrm{Ind}(h^2, V) = 0$ applying Lemma 2.6 with $f = h^2$ because

$$\begin{aligned}
&U \cap \partial V \cap h^2(U) \subset h^{n+1}(\alpha) \cap h^2(U) = h^2(h^{n-1}(\alpha) \cap U) = \emptyset, \\
&h^2(\mu_*) \cap U \subset h^2(\alpha) \cap U = \emptyset.
\end{aligned}$$

$\square$

Proof of Proposition 3.2: We consider the integer $n \geq 2$ and the point $x \in h^n(\alpha)$ defined in Lemma 3.3. The set

$$C = [x, h(p)]_\alpha \bigcup_{i=1}^{n-1} h^i(\alpha) \cup [h^n(p), x]_{h^n(\alpha)}$$

is then a Jordan curve. If necessary we can replace h,α with h_*,α_* given by Lemma 3.3 so there is no loss in supposing $x = p = h^{n+1}(p)$ and $C = \bigcup_{i=0}^{n} h^i(\alpha)$. We complete the proof using Lemmas 3.4 and 3.6. $\square$

3.2 A proposition about translation arcs of h^2

Proposition 3.7 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$. Assume that we can find a translation arc β for h^2, with endpoints $q, h^2(q)$, such that*

- *$\beta \cap h(\beta) = \emptyset$,*
- *either $q = h^4(q)$ or $h^k(\beta) \cap \beta \neq \emptyset$ for an integer $k \geq 3$, i.e. the sets $\bigcup_{i\in\mathbb{Z}} h^{2i}(\beta)$ and $\bigcup_{j\in\mathbb{Z}} h^{2j+1}(\beta)$ are not two disjoint simple curves.*

Then there exist a Jordan curve C and a point z as announced in Theorem 3.1.

Beginning of the proof of Proposition 3.7: It will be convenient to define an integer $n \geq 2$ and a point $x \in h^n(\beta)$ as follows:

- if $q = h^4(q)$ then $n = 2$ and $x = q = h^4(q)$,
- if $q \neq h^4(q)$ then n is the minimum of the set $\{k \geq 3 | \beta \cap h^k(\beta) \neq \emptyset\}$ and x is the first point on $h^n(\beta)$ to intersect β.

Let us remark that, because of the minimality of n, we have necessarily $x \notin \{h^2(q), h^n(q)\}$. We also note that h^2 (and so h) has no fixed point on $\bigcup_{k\in\mathbb{Z}} h^k(\beta)$. The proof of Proposition 3.7 depends on the parity of n, as explained below.

3.2.1 n is even

We consider the set

$$C = [x, h^2(q)]_\beta \bigcup_{2i=2}^{n-2} h^{2i}(\beta) \cup [h^n(q), x]_{h^n(\beta)}.$$

It is a Jordan curve contained in $\bigcup_{2i=0}^{n} h^{2i}(\beta)$ (we have simply $C = \beta \cup h^2(\beta)$ if $n = 2$). It follows from the minimality of n that

$$\left(\bigcup_{2i=0}^{n} h^{2i}(\beta)\right) \cap \left(\bigcup_{2j+1=1}^{n-1} h^{2j+1}(\beta)\right) = \emptyset.$$

Hence $\bigcup_{2j+1=1}^{n-1} h^{2j+1}(\beta)$ is disjoint from C and, by connectedness, is contained in one of the two connected components U_1, U_2 of $\mathbb{S}^2 \setminus C$, say in U_2. Thus we have also $\bigcup_{2i=2}^{n} h^{2i}(\beta) \subset h(U_2)$. Observe that this implies $h^{\pm 1}(U_1) \not\subset U_1$ and $U_2 \cap h(U_2) \neq \emptyset$. Since β is a translation arc for h^2, Brouwer's lemma gives $\mathrm{Ind}(h^2, U_1)$=1 and $U_1 \cap \mathrm{Fix}(h^2) \neq \emptyset$. We can suppose $U_1 \cap h(U_1) \neq \emptyset$ since otherwise we have $U_1 \cap \mathrm{Fix}(h) = \emptyset$, hence $\mathrm{Ind}(h, U) = 0$, and every fixed point z of h^2 in U_1 satisfies $h(z) \in U_2$. Writing simply $U = U_1$, we know from Proposition 2.1 that every connected component V of $U \cap h(U)$ is a Jordan domain such that $\partial V \subset C \cup h(C)$. Since $Cl(V) \subset Cl(U) \cap Cl(h(U))$ is disjoint from $\bigcup_{k=1}^{n} h^k(\beta)$ we get in fact

$$\partial V \subset [x, h^2(q)]_\beta \cup [h^{n+1}(q), h(x)]_{h^{n+1}(\beta)}.$$

Choosing a point $a \in U$ close enough to $C \cap \left(\bigcup_{2i=2}^{n} h^{2i}(\beta)\right)$ and considering $\mu = \mu(a) \in \pi_0(U \cap \partial V)$ such that $a \notin U_{\mu,V}$ (Lemma 2.2) one can check that necessarily $\mu_* \subset [x, h^2(q)]_\beta$ (see Fig. 3). All this can be done as in Lemma 3.6 and details are left to the reader. Furthermore, since

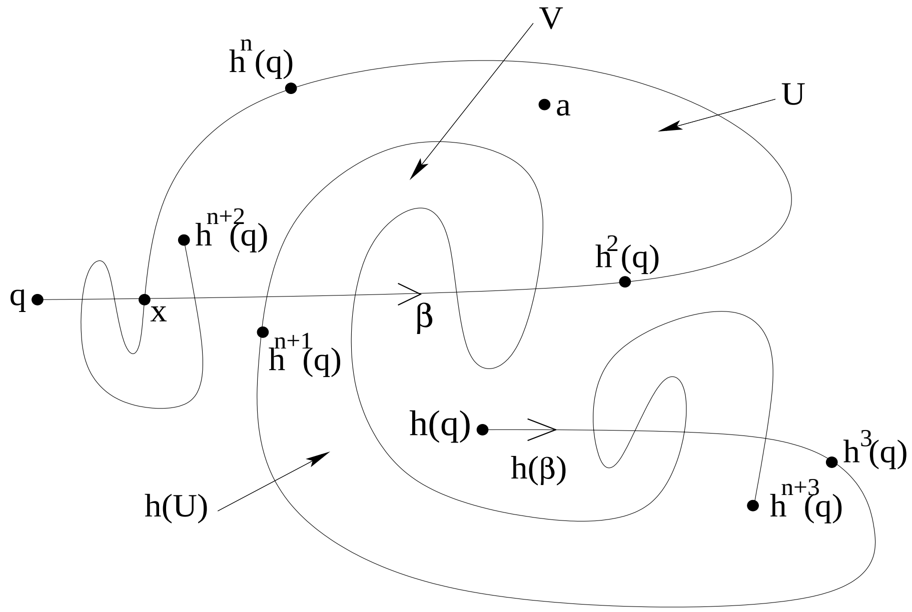

Figure 3: The Jordan domains U and $h(U)$

$$\begin{aligned}
&U \cap \partial V \cap h(U) \subset \partial(U \cap h(U)) \cap U \cap h(U) = \emptyset, \\
&h(\mu_*) \cap U \subset h(\beta) \cap U = \emptyset,
\end{aligned}$$

and

$$\begin{aligned}
&U \cap \partial V \cap h^2(U) \subset h^{n+1}(\beta) \cap h^2(U) = h^2(h^{n-1}(\beta) \cap U) = \emptyset, \\
&h^2(\mu_*) \cap U \subset h^2(\beta) \cap U = \emptyset,
\end{aligned}$$

one can use Lemma 2.6 with successively $f = h$, $f = h^2$ and thus obtain $\mathrm{Ind}(h, V) = 0 = \mathrm{Ind}(h^2, V)$. Since $\mathrm{Fix}(h) \cap U \cap h(U) = \mathrm{Fix}(h) \cap U$ we have with Property 2.3 (1) and Lemma 2.4:

$$0 = \sum_{V \in \pi_0(U \cap h(U))} \mathrm{Ind}(h, V) = \mathrm{Ind}(h, U \cap h(U)) = \mathrm{Ind}(h, U),$$
$$0 = \sum_{V \in \pi_0(U \cap h(U))} \mathrm{Ind}(h^2, V) = \mathrm{Ind}(h^2, U \cap h(U)).$$

This proves Proposition 3.7 when n is even.

3.2.2 n is odd and $h^{n+1}(\beta) \cap \beta = \emptyset$

We begin with a lemma which plays the same role as Lemma 3.3. Note that the assumption $h^{n+1}(\beta) \cap \beta = \emptyset$ is useless in this proof.

Lemma 3.8 (see Fig. 4) *There exists an orientation reversing homeomorphism $h_* \sim h$ such that h_*^2 admits $\beta_* = [x, h^2(q)]_\beta$ as a translation arc with $h_*^2(x) = h^2(q)$ and*

- $h_*(\beta_*) = [h(x), h^3(q)]_{h(\beta)}$,
- $\forall i \in \{2, \dots, n-1\} \quad h_*^i(\beta_*) = h^i(\beta)$,
- $h_*^n(\beta_*) = [h^n(q), x]_{h^n(\beta)}$,
- $h_*^{n+1}(\beta_*) = [h^{n+1}(q), h(x)]_{h^{n+1}(\beta)}$.

Proof of Lemma 3.8 (outline): As in Lemma 3.3 the proof divides into two steps;
- First step.

If $x = q$ we rename $h = g$. Otherwise observe that the arc $[h^2(q), h^2(x)]_{h^2(\beta)}$ has the following properties:

(i) it is disjoint from its images by h and h^2 ,

(ii) it is disjoint from $h^i(\beta)$ for every integer $i \in \{1\} \cup \{3, \dots, n+1\}$.

One can construct a homeomorphism φ of $\mathbb{S}^2$ mapping $[h^2(x), h^4(q)]_{h^2(\beta)}$ onto $h^2(\beta)$ whose support is contained in a topological closed disc D_1 so close to $[h^2(q), h^2(x)]_{h^2(\beta)}$ that it satisfies also (i) and (ii). Defining $g = \varphi \circ h$, we have then $g \sim h$ and $g = h$ on $\beta \cup \bigcup_{i=2}^n h^i(\beta)$, hence $g(\beta_*) = [h(x), h^3(q)]_{h(\beta)}$ with $g(x) = h(x)$ and

$$\forall i \in \{2, \dots, n+1\} \quad g^i(\beta_*) = h^i(\beta) \text{ with } g^i(x) = h^i(p).$$

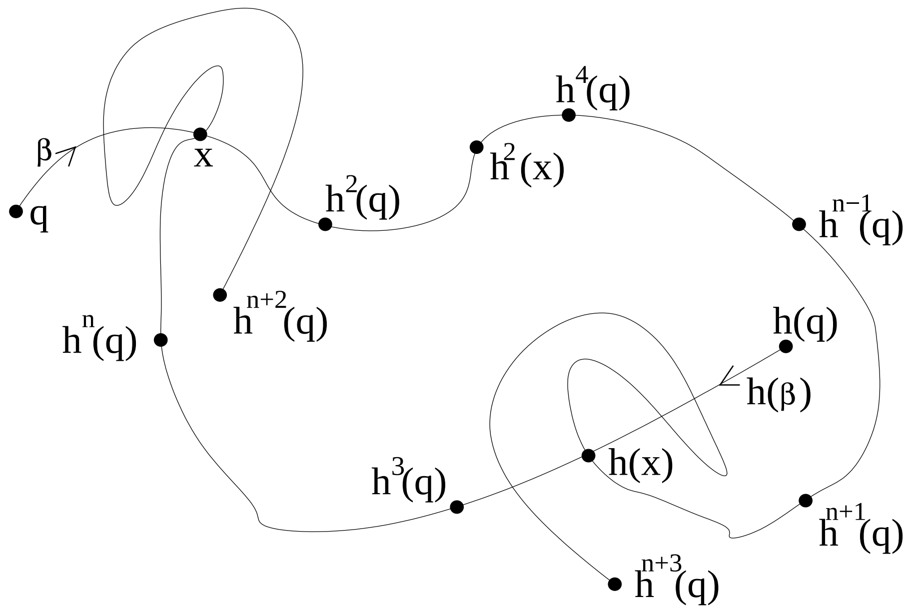

Figure 4: The arcs $h^i(\beta)$, $0 \leq i \leq n+1$

- Second step.

If $\{x\} = \{g^{n+2}(x)\} = \beta_* \cap g^n(\beta_*)$ it is enough to define $h_* = g$. Otherwise we remark that the arc $[x, g^{n+2}(x)]_{g^n(\beta_*)}$ is disjoint from its images by g and g^2 and also from the set $\left(\bigcup_{i=1}^{n-1} g^i(\beta_*)\right) \cup g^{n+1}(\beta_*)$. It is possible to have the same for a topological closed disc D_2 containing the support of a homeomorphism ψ of $\mathbb{S}^2$ such that $\psi(g^n(\beta_*)) = [g^n(x), x]_{g^n(\beta_*)}$. Then $h_* = \psi \circ g \sim g$ possesses the announced properties. $\square$

Continuation of the proof of Proposition 3.7: We consider now the sets

$$\gamma_- = [x, h^2(q)]_\beta \cup \bigcup_{2i=2}^{n-1} h^{2i}(\beta) \cup [h^{n+1}(q), h(x)]_{h^{n+1}(\beta)},$$

$$\gamma_+ = [h(x), h^3(q)]_{h(\beta)} \bigcup_{2j+1=3}^{n-2} h^{2j+1}(\beta) \cup [h^n(q), x]_{h^n(\beta)}$$

and finally $C = \gamma_- \cup \gamma_+$. Keeping in mind that $\beta \cap h^{n+1}(\beta) = \emptyset$, we see that γ_- and γ_+ are two arcs which meet only in their common endpoints $x, h(x)$. Consequently C is a Jordan curve. Replacing h, β with respectively h_*,β_* given by Lemma 3.8, one can suppose that $x = q = h^{n+2}(q)$, that is

$$\gamma_- = \bigcup_{2i=0}^{n+1} h^{2i}(\beta), \quad \gamma_+ = \bigcup_{2j+1=1}^{n} h^{2j+1}(\beta) \text{ and } C = \bigcup_{i=0}^{n+1} h^i(C).$$

Lemma 3.9 *Let U be a connected component of $\mathbb{S}^2 \setminus C$. Then we have Ind(h, U) = 0 and Ind(h^2, U) = 1.*

Proof of Lemma 3.9: It is similar to the one of Lemma 3.4. One can easily construct an orientation reversing homeomorphism g of $\mathbb{S}^2$ such that

1. $g = h$ on the set $\bigcup_{i=0}^{n} h^i(\beta)$,
2. g maps $h^{n+1}(\beta)$ onto β (hence $g(C) = C$),
3. g interchanges the two connected components of $\mathbb{S}^2 \setminus C$.

Thus $g^{-1} \circ h$ is an orientation preserving homeomorphism of the sphere which coincides with $Id_{\mathbb{S}^2}$ on the arc $\bigcup_{i=0}^{n} h^i(\beta)$. Using the same variation of the Alexander trick as in Lemma 3.4, one can find an isotopy $(\varphi_t)_{0\leq t\leq 1}$ from $\varphi_0 = Id_{\mathbb{S}^2}$ to $\varphi_1 = g^{-1} \circ h$ such that

$$\forall t \in [0, 1]\, \forall z \in \bigcup_{i=0}^{n} h^i(\beta) \quad \varphi_t(z) = z.$$

Defining $h_t = g \circ \varphi_t$ $(0 \leq t \leq 1)$, we get an isotopy from g to h such that $h_t = h$ on $\bigcup_{i=0}^{n} h^i(\beta)$ and also an isotopy $(h_t^2)_{0\leq t\leq 1}$ from g^2 to h^2 such that $h_t^2 = h^2$ on $\bigcup_{i=0}^{n-1} h^i(\beta)$. It follows, for every $t \in [0, 1]$, that h_t^2 has no fixed point on

$$\bigcup_{i=0}^{n-1} h^i(\beta) \cup h_t^2\Big(\bigcup_{i=0}^{n-1} h^i(\beta)\Big) = \bigcup_{i=0}^{n+1} h^i(\beta) = C.$$

Using again Property 2.3 (4), we get $\mathrm{Ind}(g, U)$=$\mathrm{Ind}(h, U)$ and $\mathrm{Ind}(g^2, U) = \mathrm{Ind}(h^2, U)$. We obtain finally $\mathrm{Ind}(g, U) = 0$ (resp. $\mathrm{Ind}(g^2, U)$=1) because $U \cap g(U) = \emptyset$ (resp. $U = g^2(U)$). $\square$

Continuation of the proof of Proposition 3.7: Let U_1, U_2 be the two connected components of $\mathbb{S}^2 \setminus C$. According to Lemma 3.9 we have $\mathrm{Ind}(h, U_i)$ = 0 and $\mathrm{Ind}(h^2, U_i) = 1$. In particular we have $U_i \cap \mathrm{Fix}(h^2) \neq \emptyset$ $(i \in \{1, 2\})$. If one can find $i \in \{1, 2\}$ such that $U_i \cap h(U_i) = \emptyset$ then the result is easy. Otherwise we consider for example $U = U_1$. Let V be any connected component of $U \cap h(U)$. Since h reverses the orientation, every point $z \in C \setminus h^{n+1}(\beta)$ possesses a neighbourhood N_z such that $h(N_z \cap U) = h(N_z) \cap U_2$ and $h(N_z \cap U_2) = h(N_z) \cap U$. It follows that $C \setminus \beta$ is disjoint from $Cl(U \cap h(U))$ and in particular from $Cl(V)$. Using one more time Proposition 2.1, we obtain that V is a Jordan domain such that $\partial V \subset \beta \cup h^{n+2}(\beta)$. As in the proof of Lemma 3.6 one can use Lemma 2.2 and find $\mu \in \pi_0(U \cap \partial V)$ such that the corresponding arc μ_* satisfies $\mu_* \subset \beta$. We have then

$$\begin{aligned} &\partial V \cap U \cap h^2(U) \subset h^{n+2}(\beta) \cap h^2(U) = h^2(h^n(\beta) \cap U) = \emptyset, \\ &h^2(\mu_*) \cap U \subset h^2(\beta) \cap U = \emptyset, \end{aligned}$$

and Lemma 2.6 gives $\mathrm{Ind}(h^2, V) = 0$. We deduce from Lemma 2.4 that

$$0 = \sum_{V \in \pi_0(U \cap h(U))} \mathrm{Ind}(h^2, V) = \mathrm{Ind}(h^2, U \cap h(U)).$$

3.2.3 n is odd and $h^{n+1}(\beta) \cap \beta \neq \emptyset$

The following remarks allow us to reduce to the cases studied in Sections 3.2.1 and 3.2.2. We consider the last point y on β to intersect $h^n(\beta) \cup h^{n+1}(\beta)$. Since $\beta \cap h(\beta) = \emptyset$, y does not belong simultaneously to $h^n(\beta)$ and $h^{n+1}(\beta)$ and $y \neq q$. We have also $y \neq h^2(q)$ because of the minimality of n. We can then assert:

Lemma 3.10 *There exists an orientation reversing homeomorphism $\hat{h} \sim h$ such that $\hat{h}^2$ admits $\hat{\beta} = [y, h^2(q)]_\beta$ as a translation arc with $\hat{h}^2(y) = h^2(q)$ and*

- $\hat{h}(\hat{\beta}) = [h(y), h^3(q)]_{h(\beta)}$,
- $\forall i \in \{2, \ldots, n+1\} \quad \hat{h}^i(\hat{\beta}) = h^i(\beta)$.

Proof of Lemma 3.10: It is enough to replace x with y in the construction of the intermediate homeomorphism g in the proof of Lemma 3.8. □

End of the proof of Proposition 3.7: By the definition of y we have:
- if $y \in h^n(\beta) = \hat{h}^n(\hat{\beta})$ then $\hat{h}^{n+1}(\hat{\beta}) \cap \hat{\beta} = h^{n+1}(\beta) \cap \hat{\beta} = \emptyset$ and we reduce to the situation of Section 3.2.2. replacing h, β with $\hat{h}, \hat{\beta}$.
- if $y \in h^{n+1}(\beta) = \hat{h}^{n+1}(\hat{\beta})$ then $\hat{h}^4(y) = h^4(q) \neq y$ and $n+1$ is the smallest integer $k \in \{3, \ldots, n+1\}$ such that $\hat{h}^k(\hat{\beta}) = h^k(\beta)$ intersects $\hat{\beta}$. We reduce to the case treated in Section 3.2.1. replacing h, β and n with $\hat{h}, \hat{\beta}$ and $n+1$. Proposition 3.7 is proved. □

3.3 Proof of Theorem 3.1

Choose m to be a k-periodic point of h ($k \geq 3$) and consider an arc α or β given by Lemma 2.8. Then $m = h^k(m) \in \alpha \cap h^k(\alpha)$ or $m = h^k(m) \in \beta \cap h^k(\beta)$ and either Proposition 3.2 or Proposition 3.7 applies. □

3.4 Some consequences

Corollary 3.11 *Let h be an orientation reversing homeomorphism of the sphere $\mathbb{S}^2$ without a 2-periodic point. If a connected and simply connected compact set $K \subset \mathbb{S}^2$ satisfies $K \cap h(K) = \emptyset = K \cap h^2(K)$ then we have $K \cap h^k(K) = \emptyset$ for every integer $k \neq 0$. Consequently, the only non-wandering points are the fixed points of h.*

Proof of Corollary 3.11: Construct a topological closed disc D containing K and so close to K that it is disjoint from its iterates $h(D)$ and $h^2(D)$. We claim that $D \cap h^k(D) = \emptyset$ for every integer $k \neq 0$; otherwise, there exists a smallest $k \geq 3$ such that $D \cap h^k(D) \neq \emptyset$. Replacing the disc D with a slightly larger one if necessary, we can suppose $Int(D) \cap h^k(Int(D)) \neq \emptyset$. Then we can choose $m \in Int(D) \cap h^{-k}(Int(D))$ and a homeomorphism φ with support in D such that $\varphi(h^k(m)) = m$. Thus $\varphi \circ h$ is an orientation reversing homeomorphism of $\mathbb{S}^2$ posseding m as a k-periodic point. According to Theorem 3.1, $\varphi \circ h$ admits a 2-periodic point. The support of φ is disjoint from its images by h and h^2 so, for $i \in \{1, 2\}$, the homeomorphisms $(\varphi \circ h)^i$ and h^i have the same fixed point set. Thus we get a 2-periodic point for h, a contradiction.

Now, for any point $m \in \mathbb{S}^2 \setminus \mathrm{Fix}(h) = \mathbb{S}^2 \setminus \mathrm{Fix}(h^2)$ we can choose K to be a neighbourhood of m so m is a wandering point. □

Remark 3.12 *As an immediate consequence of Corollary 3.11, any area preserving and orientation reversing homeomorphism of the 2-sphere (or of the closed 2-disc) possesses a 2-periodic point.*

Remark 3.13 *If in Theorem 3.1 we assume furthermore that h^2 has only finitely many fixed points then the 2-periodic point z can be chosen such that Ind(h^2, z) = Ind($h^2, h(z)$) is a positive integer (resp. an odd integer).*

The equality $\mathrm{Ind}(h^2, z) = \mathrm{Ind}(h^2, h(z))$ is a consequence of Property 2.3 (5) and of the obvious relation $h^2 = h \circ h^2 \circ h^{-1}$. Keeping the notations of Theorem 3.1 let us define

$$F = \mathrm{Fix}(h^2) \cap U, \quad F_1 = F \cap h(U), \quad F_2 = F \setminus F_1 = \mathrm{Fix}(h^2) \cap U \cap h^{-1}(U').$$

We have then

$$1 = \mathrm{Ind}(h^2, U) = \sum_{z \in F} \mathrm{Ind}(h^2, z) = \sum_{z \in F_1} \mathrm{Ind}(h^2, z) + \sum_{z \in F_2} \mathrm{Ind}(h^2, z).$$

Recall we have shown $\mathrm{Ind}(h^2, U \cap h(U)) = 0$ hence $\sum_{z \in F_1} \mathrm{Ind}(h^2, z) = 0$ and the assertion follows.

4 A local version of Theorem 3.1

We first remind two recent results:

Theorem 4.1 ([1]) *Let V, W be two connected open subsets of $\mathbb{R}^2$ containing the origin o and let $h : V \to W = h(V)$ be an orientation reversing homeomorphism which possesses o as an isolated fixed point.*

Then Ind$(h, o) \in \{-1, 0, 1\}$.

The iterate homeomorphisms $h^k : V_k \to h_k(V_k)$, $k \geq 1$, are defined inductively on the open sets V_k by $V_1 = V$, $h^1 = h$ and, for $k \geq 2$, $V_k = h^{-1}(V_{k-1}) \subset V_{k-1}, h^k(z) = h^{k-1}(h(z))$.

We have then

Theorem 4.2 ([11]) *Let h be as in Theorem 4.1. Assume that the whole sequence (Ind$(h^k, o))_{k \geq 1}$ is defined, i.e. o is an isolated fixed point of h^k for every integer $k \geq 1$.*

Then (Ind$(h^{2k+1}, o))_{k \geq 0}$ is a constant sequence.

We can now state:

Theorem 4.3 *Let h be as in Theorem 4.1. If there exists an integer $k \geq 3$ such that any neighbourhood of o contains a k-periodic point of h then there is also a 2-periodic point in every neighbourhood of o. In other words, the whole sequence (Ind$(h^k, o))_{k \geq 1}$ is defined if and only if the second term Ind(h^2, o) is defined.*

Proof of Theorem 4.3: Let Ω be an open disc with center o, so small that $\Omega \cap \mathrm{Fix}(h) = \{o\}$ and $\mathrm{Cl}(\Omega) \subset V$. Let us show that Ω necessarily contains a 2-periodic point of h. Using the Schoenflies Theorem, we can extend $h|_{\mathrm{Cl}(\Omega)}$ to a homeomorphism H of the whole sphere. Let Ω' be an open disc with center o such that $\Omega' \subset \Omega \cap H^{-1}(\Omega)$ and let N be the connected component of $\bigcap_{i=0}^{k+1} H^{-i}(\Omega')$ containing o. The set $N \cap H^{-2}(N)$ is a neighbourhood of o so it contains a k-periodic point m of h, and such a point m is also k-periodic for H because $h^j = H^j$ on $\bigcap_{i=0}^{k-1} H^{-i}(\Omega')$ for $j = 1, \ldots, k$. Since $h = H$ and $h^2 = H^2$ on $\Omega \cap H^{-1}(\Omega)$ it is enough to prove that H has a 2-periodic point in Ω'. Suppose this is not true. Then we have

$$\{m, H^2(m)\} \subset N \setminus \{o\} = N \setminus \mathrm{Fix}(H^2).$$

The contradiction is obtained by following carefully the proof of Theorem 3.1 for the homeomorphism H. Since an open connected subset of $\mathbb{R}^2$ is arcwise connected, there exists an arc $\gamma \subset N \setminus \{o\}$ joining m and $H^2(m)$. Going back to the proof of Lemma 2.8, we now slightly enlarge γ to obtain a topological closed disc Δ such that

$$\gamma \subset \mathrm{Int}(\Delta) \subset \Delta \subset N \setminus \{o\}.$$

Then we obtain a suitable translation arc for either H or H^2, which is denoted by respectively α or β. Observe that, by the construction, α and β are contained in Δ and so in $\bigcap_{i=0}^{k+1} H^{-i}(\Omega')$. Remark finally that all the Jordan curves C constructed in the proof of Theorem 3.1 are subset of $\bigcup_{i=0}^{k} H^i(\alpha) \subset \Omega'$ or of $\bigcup_{i=0}^{k+1} H^i(\beta) \subset \Omega'$, hence $\mathbb{S}^2 \setminus C$ has a connected component which is contained in the disc Ω'. Thus we obtain a 2-periodic point of H in Ω', a contradiction. $\square$

5 A strong version of Theorem 3.1

We prove in this section the following result;

Theorem 5.1 *Let h be an orientation reversing homeomorphism of the sphere $\mathbb{S}^2$ without a 2-periodic point. Then for any point $m \in \mathbb{S}^2 \setminus Fix(h)$ there exists a topological embedding (i.e. a continuous one-to-one map) $\varphi : \mathcal{O} \to \mathbb{S}^2 \setminus Fix(h)$ such that*

- *$\mathcal{O}$ is either $\mathbb{R}^2$ or $\{(x, y) \in \mathbb{R}^2 | y \neq 0\}$ or $\mathbb{R}^2 \setminus \{(0, 0)\}$,*
- *$m \in \varphi(\mathcal{O})$,*
- *if $\mathcal{O} = \mathbb{R}^2$ or $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 | y \neq 0\}$ then*
  - *(i) $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$ where $G(x, y) = (x + 1, -y)$,*
  - *(ii) for every $x \in \mathbb{R}$, $\varphi\big((\{x\} \times \mathbb{R}) \cap \mathcal{O}\big)$ is a closed subset of $M \setminus Fix(h)$ (it is said that φ is a proper embedding),*
- *if $\mathcal{O} = \mathbb{R}^2 \setminus \{(0, 0)\}$ then*
  - *(iii) $h \circ \varphi = \varphi \circ H|_{\mathcal{O}}$ where $H(x, y) = \frac{1}{2}(x, -y)$.*

Sections 5.2 and 5.3 below contain preparatory results. Section 5.2 gives some dynamical properties for orientation reversing homeomorphisms of the sphere, derived from results in Section 3. In particular it is shown that recurrence of discs, just as recurrence of points, implies the existence of a 2-periodic point. Section 5.3 recall the notion of brick decomposition of a surface introduced by P. Le Calvez and A. Sauzet ([16], [19]) to give a dynamical proof of the Brouwer plane translation theorem. Theorem 5.1 is proved in Section 5.3.

Note that, since we are looking for conjugacy outside the fixed point set, the map H in the statement of Theorem 5.1 can be replaced with any map $(x, y) \mapsto \lambda(x, -y)$ where $\lambda \in \mathbb{R} \setminus \{0, \pm 1\}$.

5.1 Some recurrence properties

Lemma 5.2 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$ without a 2-periodic point and let V be an open connected subset of $\mathbb{S}^2$ such that $V \cap h(V) = \emptyset = V \cap h^2(V)$. Then we have $V \cap h^k(V) = \emptyset$ for any integer $k \neq 0$.*

Proof of Lemma 5.2: Suppose this is not true. Then we have $V \cap h^k(V) \neq \emptyset$ for an integer $k \geq 3$ and we can choose $z \in V \cap h^k(V)$. Since an open connected subset of $\mathbb{S}^2$ is arcwise connected, there exists an arc $K \subset V$ with endpoints $h^{-k}(z)$ and z. Such an arc is disjoint from its two first iterates $h(K), h^2(K)$ but meets $h^k(K)$. This contradicts Corollary 3.11. $\square$

The next lemmas can be regarded as the counterpart of Frank's Lemma ([8][Proposition 1.3]) in the case of an orientation reversing homeomorphism.

Lemma 5.3 *Let h be an orientation reversing homeomorphism of $\mathbb{S}^2$. Assume that there exists a finite sequence of topological closed discs $D_1, \ldots, D_n$ satisfying*

(i) $\forall i, j \in \{1, \ldots, n\} \quad D_i = D_j$ *or* $Int(D_i) \cap Int(D_j) = \emptyset$,

(ii) $\forall i \in \{1, \ldots, n\} \quad h(D_i) \cap D_i = \emptyset = h^2(D_i) \cap D_i$,

(iii) $\forall i, j \in \{1, \ldots, n\} \quad D_j$ *meets at most one of the two sets* $h^{-1}(D_i)$ *or* $h(D_i)$,
Equivalently: $h(D_i) \cap D_j \neq \emptyset \Longrightarrow h(D_j) \cap D_i = \emptyset$,

(iv) $\forall i \in \{1, \ldots, n-1\}\ \exists k_i \geq 1$ *such that* $h^{k_i}(D_i) \cap Int(D_{i+1}) \neq \emptyset$ *and* $\exists k_n \geq 1$ *such that* $h^{k_n}(D_n) \cap Int(D_1) \neq \emptyset$.

Then h possesses a 2-periodic point.

Proof of Lemma 5.3: Let us choose a sequence $D_1, \dots, D_{n_0}$ satisfying (i)-(iv) and whose length n_0 is minimal among all these sequences. If $n_0 = 1$ then the result is contained in Corollary 3.11 so we can assume $n_0 \geq 2$. Moreover we can suppose that the integers $k_1, \dots, k_{n_0}$ are minimal for the property (iv). In order to simplify the notations we also define $D_{n_0+1} = D_1$. We have clearly

$$h^{k_i}(D_i) \cap Int(D_{i+1}) \neq \emptyset \Longleftrightarrow h^{k_i}(Int(D_i)) \cap Int(D_{i+1}) \neq \emptyset$$

so we can choose for every $i \in \{1, \dots, n_0\}$ a point $x_i \in Int(D_i)$ such that $h^{k_i}(x_i) \in Int(D_{i+1})$. Since the sequence $D_1, \dots, D_{n_0}$ has minimal length we have

$$1 \leq i \neq j \leq n_0 \Longrightarrow Int(D_i) \cap Int(D_j) = \emptyset$$

so there exists an orientation preserving homeomorphism ψ of $\mathbb{S}^2$ with support in $D_1 \cup \dots \cup D_{n_0}$ preserving setwise each disc D_i ($1 \leq i \leq n_0$) and such that

$$\forall i \in \{1, \dots, n_0 - 1\} \quad \psi(h^{k_i}(x_i)) = x_{i+1}, \qquad \psi(h^{k_{n_0}}(x_{n_0})) = x_1.$$

Furthermore we have for every $i, j \in \{1, \dots, n_0\}$

$$1 \leq k \leq k_i - 1 \Longrightarrow h^k(x_i) \notin D_j$$

since otherwise either $D_1, \dots, D_i, D_j, \dots, D_{n_0}$ or $D_j, \dots, D_i$ would define a sequence satisfying (i)-(iv) with length $\leq n_0 - 1$. Thus the homeomorphism $g = \psi \circ h$ reverses the orientation and possesses x_1 as a periodic point with period $k_1 + \dots + k_{n_0} \geq 2$. Theorem 3.1 then gives a 2-periodic point for g and it is enough to check that $\mathrm{Fix}(h) = \mathrm{Fix}(g)$ and $\mathrm{Fix}(h^2) = \mathrm{Fix}(g^2)$.

• The first equality is well known and follows easily from the fact that $D_i \cap h(D_i) = \emptyset$ for every $i \in \{1, \dots, n_0\}$.

• Let us check that $\mathrm{Fix}(g^2) = \mathrm{Fix}(h^2)$.

- First we observe that if $m \in h^{-1}(D_j)$ for an index $j \in \{1, \dots, n_0\}$ then necessarily $m \neq g^2(m)$: For such a point m we have $g(m) = \psi(h(m)) \in \psi(D_j) = D_j$ so $h(g(m)) \in h(D_j)$. If $h(g(m)) \notin \bigcup_{i=1}^{n_0} D_i$ then $g^2(m) = \psi(h(g(m))) = h(g(m))$ and consequently $m \neq g^2(m)$ since $h^{-1}(D_j) \cap h(D_j) = \emptyset$. If one can find $i \in \{1, \dots, n_0\}$ such that $h(g(m)) \in D_i$ then we obtain $h(g(m)) \in D_i \cap h(D_j) \neq \emptyset$ and (iii) implies $D_i \cap h^{-1}(D_j) = \emptyset$. Since $g^2(m) = \psi(h(g(m))) \in \psi(D_i) = D_i$, it follows that $g^2(m) \neq m$.

- Secondly we remark that if $m \notin \bigcup_{i=1}^{n_0} h^{-1}(D_i)$ but $m \in h^{-2}(D_j)$ for a $j \in \{1, \dots, n_0\}$ then we also have $m \neq g^2(m)$. Indeed we have then $g(m) = h(m)$, $g^2(m) = \psi(h^2(m)) \in \psi(D_j) = D_j$ and consequently $m \neq g^2(m)$ since $h^{-2}(D_j) \cap D_j = \emptyset$.

Thus we obtain:

$$m = g^2(m) \Longrightarrow m \notin \left(\bigcup_{i=1}^{n_0} h^{-1}(D_i)\right) \cup \left(\bigcup_{i=1}^{n_0} h^{-2}(D_i)\right) \Longrightarrow g^2(m) = h^2(m).$$

On the other hand, it is easily seen with (ii) that

$$m = h^2(m) \Longrightarrow m \notin \left(\bigcup_{i=1}^{n_0} h^{-1}(D_i)\right) \cup \left(\bigcup_{i=1}^{n_0} h^{-2}(D_i)\right) \Longrightarrow g^2(m) = h^2(m).$$

$\square$

We will use actually the following slightly stronger lemma which relax the hypothesis (iv) of Lemma 5.3. This technical improvement allows to suppress, in the original work of Le Calvez and Sauzet ([16]), a hypothesis of "transversality" for the brick decompositions of a surface (see Section 5.2 below for a definition). These refinements are due to F. Le Roux ([17]) for orientation preserving homeomorphisms. We use the same arguments to write a proof adapted to our situation.

Lemma 5.4 *If in Lemma 5.3 we replace the condition (iv) with the weaker*

(iv') $\forall i \in \{1, \dots, n-1\}\, \exists k_i \geq 1$ *such that* $h^{k_i}(D_i) \cap D_{i+1} \neq \emptyset$ *and*
$\exists k_n \geq 1$ *such that* $h^{k_n}(D_n) \cap D_1 \neq \emptyset$*,*

then the conclusion still holds.

Proof of Lemma 5.4: Let $D_1, \dots, D_{n_0}$ be a sequence of topological closed discs satisfying (i)-(iii),(iv') and whose length n_0 is minimal among all these sequences. We have then

$$1 \leq i \neq j \leq n_0 \Longrightarrow Int(D_i) \cap Int(D_j) = \emptyset.$$

Let us choose for each $i \in \{1, \dots, n_0\}$ a point $x_i \in D_i$ such that $h^{k_i}(x_i) \in D_{i+1}$ (again with the convention $D_{n_0+1} = D_1$).
First we observe that the points $x_1, \dots, x_{n_0}$ are pairwise distinct because

$$(1 \leq i, j \leq n_0 \ \text{ and } \ x_i = x_j) \quad \Longrightarrow \quad h^{k_j}(x_i) = h^{k_j}(x_j) \in h^{k_j}(D_i) \cap D_{j+1}$$

and the fact that $D_1, \dots, D_{n_0}$ has minimal length gives $i = j$.
Now, if we can find $i, j \in \{1, \dots, n_0\}$ and an integer $k \geq 1$ such that $h^k(x_i) = x_j$ then h possesses a 2-periodic point. Indeed, this implies

$$h^{k_j+k}(x_i) = h^{k_j}(x_j) \in h^{k_j+k}(D_i) \cap D_{j+1}.$$

Again because $D_1, \dots, D_{n_0}$ has minimal length, this is possible only for $i = j$ and consequently $h^k(x_i) = x_i$. Because of (ii) we have necessarily $k \geq 3$ and

Theorem 3.1 then gives a 2-periodic point for h. Thus we can assume without loss of generality

$$(**) \quad \forall i, j \in \{1, \ldots, n_0\} \, \forall k \neq 0 \quad h^k(x_i) \neq x_j.$$

For convenience we let $k_0 = k_{n_0}$ and $x_0 = x_{n_0}$. Then we choose for each $i \in \{1, \ldots, n_0\}$ an arc γ_i with endpoints x_i and $h^{k_{i-1}}(x_{i-1}) \neq x_i$ lying entirely in $Int(D_i)$ except possibly its endpoints in ∂D_i. Since $\gamma_i \subset D_i$, these arcs possesse the same properties (ii),(iii) as the discs D_i. Moreover, remembering that the D_i's have disjoint interiors and the x_i's are pairwise distinct ($1 \leq i \leq n_0$), we obtain using $(**)$:

(i') $i \neq j \Longrightarrow \gamma_i \cap \gamma_j = \emptyset$.

By the construction we have also

$$\forall i \in \{1, \ldots, n_0 - 1\} \quad h^{k_i}(\gamma_i) \cap \gamma_{i+1} \neq \emptyset \quad \text{and} \quad h^{k_{n_0}}(\gamma_{n_0}) \cap \gamma_1 \neq \emptyset.$$

One can construct for each $i \in \{1, \ldots, n_0\}$ a topological closed disc D_i' neighbourhood of γ_i and so close to γ_i that (i'),(ii),(iii) are still true with the D_i' 's in place of the γ_i's. Such a sequence $D_1', \ldots, D_{n_0}'$ then satisfies the conditions (i)-(iv) of Lemma 5.3. □

5.2 Brick decompositions

As mentioned above, this notion is due to P. Le Calvez and A. Sauzet ([16], [19]). It is also used with some variants in [13],[15] and [17].

Definition 5.5 *A brick decomposition $\mathcal{D}$ of a nonempty open set $U \subset \mathbb{S}^2$ is a collection $\{B_i\}_{i \in I}$ of topological closed discs where I is a finite or countable set and such that*

1. $\bigcup_{i \in I} B_i = U$,
2. *if $i \neq j$ then $B_i \cap B_j$ is either empty or an arc contained in $\partial B_i \cap \partial B_j$,*
3. *for every point $z \in U$, the set $I(z) = \{i \in I \mid z \in B_i\}$ contains at most three elements and $\bigcup_{i \in I(z)} B_i$ is a neighbourhood of z in U.*

The B_i's are called the *bricks* of the decomposition. Of course the set I is finite only for $U = \mathbb{S}^2$ and we will not be concerned with this situation. For a point $z \in U$, the neighbourhood $\bigcup_{i \in I(z)} B_i$ is necessarily of one of the three kinds pictured in Fig.5(up to a homeomorphism).

We have then the following property which is one of the main motivation for the use of brick decompositions.

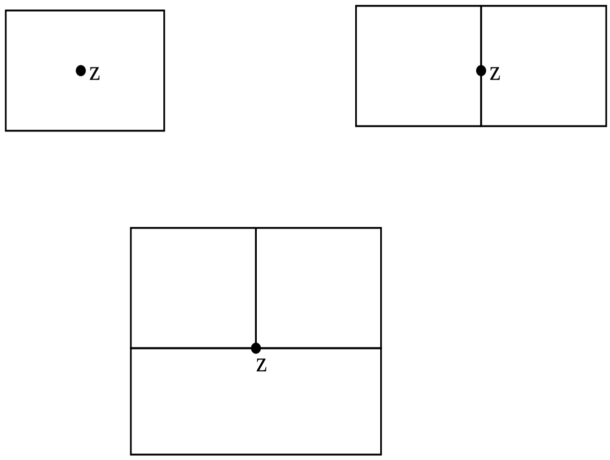

Figure 5: The neighbourhood $\bigcup_{i\in I(z)} B_i$ for a point $z \in U$

Property 5.6 *Let $\mathcal{D} = \{B_i\}_{i\in I}$ be a brick decomposition of an open set $U \subset \mathbb{S}^2$ and let J be a nonempty subset of I. Then $\bigcup_{i\in J} B_i$ is a closed subset of U. Furthermore $\partial_U(\bigcup_{i\in J} B_i)$ is a 1-dimensional submanifold without boundary of U. In particular, its connected components are homeomorphic either to $\mathbb{S}^1$ or to $\mathbb{R}$.*

Proof of Property 5.6: If $z \in Cl_U(\bigcup_{i\in J} B_i)$ it is clear from the definition that $I(z) \cap J \neq \emptyset$ hence $\bigcup_{i\in J} B_i$ is closed in U. Consider now a point $z \in \partial_U(\bigcup_{i\in J} B_i)$. Its neighbourhood $\bigcup_{i\in I(z)} B_i$ contains necessarily two or three bricks, (at least) one of them is in $\{B_i\}_{i\in J}$ and (at least) one of them is not this family. The result is then obvious with Fig.5. $\square$

Let $\mathcal{D} = \{B_i\}_{i\in I}$ be a brick decomposition of an open set $U \subset \mathbb{S}^2$ and let h be a homeomorphism of $\mathbb{S}^2$ such that $h(U) = U$. For a given brick $B_{i_0} \in \mathcal{D}$, we recall the notions of *attractor* and *repeller* associated to B_{i_0} (and h). We define

$$I_0 = \{i_0\}, \quad \mathcal{A}_0 = \mathcal{R}_0 = \bigcup_{i\in I_0} B_i = B_{i_0}$$

and inductively, for $n \in \mathbb{N}$,

$$I_{n+1} = \{i \in I \,|\, h(\mathcal{A}_n) \cap B_i \neq \emptyset\}, \quad \mathcal{A}_{n+1} = \bigcup_{i\in I_{n+1}} B_i,$$

$$I_{-n-1} = \{i \in I \,|\, h^{-1}(\mathcal{R}_{-n}) \cap B_i \neq \emptyset\}, \quad \mathcal{R}_{-n-1} = \bigcup_{i\in I_{-n-1}} B_i.$$

Definition 5.7 *With the above notations, the two sets*

$$\mathcal{A} = \bigcup_{n\geq 1} \mathcal{A}_n \quad \text{and} \quad \mathcal{R} = \bigcup_{n\geq 1} \mathcal{R}_{-n}$$

are said to be respectively the attractor *and the* repeller *associated to the brick* B_{i_0}.

Note that, according to Property 5.6, $\mathcal{A}$ and $\mathcal{R}$ are closed subsets of U. The following easy property is left to the reader.

Property 5.8 *We have* $h(\mathcal{A}\cup B_{i_0}) \subset Int(\mathcal{A})$. *Consequently* $h^k(\partial_U \mathcal{A}) \cap h^l(\partial_U \mathcal{A}) = \emptyset$ *for any two integers* $k \neq l$ *in* $\mathbb{Z}$.

We will use brick decompositions with a homeomorphism of $\mathbb{S}^2$ which reverses the orientation and without a 2-periodic point. The next result describes what are the "good" brick decompositions in this setting and then gives two essential properties for $\mathcal{A}$ and $\mathcal{R}$.

Lemma 5.9 *Let* $\mathcal{D} = \{B_i\}_{i\in I}$ *be a brick decomposition of an open set* $U \subset \mathbb{S}^2$ *and let* h *be an orientation reversing homeomorphism of* $\mathbb{S}^2$ *which has no 2-periodic point and satisfying* $h(U) = U$. *Assume furthermore that* $\mathcal{D}$ *satisfies the two following hypotheses:*

- *H1: Every brick* B_i *satisfies* $B_i \cap h(B_i) = \emptyset = B_i \cap h^2(B_i)$,
- *H2: For any two bricks* B_i, B_j, *at most one of the two sets* $B_i \cap h(B_j)$ *or* $B_i \cap h^{-1}(B_j)$ *is nonempty.*

Then, for any brick $B_{i_0} \in \mathcal{D}$, *the attractor* $\mathcal{A}$ *and the repeller* $\mathcal{R}$ *associated to* B_{i_0} *are such that*

- *(i)* $Int(B_{i_0}) \cap \mathcal{A} = \emptyset$,
- *(ii)* $\mathcal{A} \cap Int(\mathcal{R}) = \emptyset$.

Proof of Lemma 5.9:
(i) Observe that $Int(B_{i_0}) \cap \mathcal{A} \neq \emptyset$ simply means that there exist an integer $n \geq 1$ and a sequence of bricks $B_{i_0}, B_{i_1}, \ldots, B_{i_n} = B_{i_0}$ such that

$$\forall k \in \{0, \ldots, n-1\} \quad h(B_{i_k}) \cap B_{i_{k+1}} \neq \emptyset.$$

This contradicts Lemma 5.4.
(ii) If $\mathcal{A} \cap Int(\mathcal{R}) \neq \emptyset$ then there exist two integers $n, m \geq 1$ and two sequences of bricks, say $B_{i_0}, B_{i_1}, \ldots, B_{i_m}$ and $B_{j_0}, B_{j_1}, \ldots, B_{j_n}$, such that

$$B_{i_0} = B_{j_0} \text{ and } B_{i_m} = B_{j_n},$$
$$\forall k \in \{0, \dots, m-1\} \quad h(B_{i_k}) \cap B_{i_{k+1}} \neq \emptyset,$$
$$\forall l \in \{0, \dots, n-1\} \quad h^{-1}(B_{j_l}) \cap B_{j_{l+1}} \neq \emptyset.$$

As an immediate consequence, each brick in the sequence

$$B_{j_n}, \dots, B_{j_1}, B_{j_0} = B_{i_0}, \dots, B_{i_{m-1}}$$

has its image by h which meets the next brick. This contradicts again Lemma 5.4 since $h(B_{i_{m-1}})$ meets $B_{i_m} = B_{j_n}$. □

5.3 Proof of Theorem 5.1

Let h and m be as in Theorem 5.1. We define $U = \mathbb{S}^2 \setminus \mathrm{Fix}(h) = \mathbb{S}^2 \setminus \mathrm{Fix}(h^2)$. Of course we have $h(U) = U \neq \emptyset$ and, according to the Lefschetz-Hopf Theorem, $U \neq \mathbb{S}^2$. Let us remark that there is a situation where our result is easily seen. According to a theorem of Epstein, a connected component K of $\mathrm{Fix}(h)$ is either a point or an arc or a Jordan curve and, in the last two cases, h interchanges locally the two sides of K: see [6][Theorem 2.5]. If one can choose K to be a Jordan curve then $\mathbb{S}^2 \setminus K$ has exactly two connected components, say U_1 and U_2 with $m \in U_1$, which are interchanged by h (this implies also $K = \mathrm{Fix}(h)$). Since the U_i's are homeomorphic to $\mathbb{R}^2$ we can use the Brouwer plane translation theorem with $h^2|_{U_1}$ to find a proper topological embedding $\varphi : \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \to U_1$ such that $\varphi(0, 1) = m$ and $h^2 \circ \varphi(x, y) = \varphi \circ \tau(x, y)$ for $y > 0$, where $\tau(x, y) = (x + 2, y) = G^2(x, y)$. We obtain a proper topological embedding $\varphi : \mathcal{O} = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\} \to U$ such that $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$ defining

$$\forall\, y < 0 \quad \varphi(x, y) = h \circ \varphi \circ G^{-1}(x, y) \in U_2.$$

Thus we can suppose that $\mathbb{S}^2 \setminus K$ is connected for every connected component K of $\mathrm{Fix}(h)$ and this implies that $U = \mathbb{S}^2 \setminus \mathrm{Fix}(h)$ is connected (see for example [18][Theorem 14.3 page 123]). According to Lemma 2.8, Propositions 3.2 and 3.7, at least one of the two following properties is true:

- P1: There exists a translation arc α for h containing the point m and such that $\bigcup_{k\in\mathbb{Z}} h^k(\alpha)$ is a simple curve contained in U.
- P2: There exists a translation arc β for h^2 containing the point m and such that $\bigcup_{k\in\mathbb{Z}} h^{2k}(\beta)$ and $\bigcup_{k\in\mathbb{Z}} h^{2k+1}(\beta)$ are two disjoint simple curves contained in U.

5.3.1 Proof when P1 is true

Up to conjugacy in $\mathbb{S}^2$, we can suppose that

$h^{-1}(\alpha) = [-1, 0] \times \{0\}$,

$h(x, y) = (x + 1, -y)$ for every $(x, y) \in h^{-1}(\alpha) \cup \alpha = [-1, 1] \times \{0\}$,

$m = (\frac{3}{4}, 0)$.

For $\epsilon > 0$ we consider the three rectangles (see Fig. 7)

$$D_{-1} = \{(x, y) \in \mathbb{R}^2 \,|\, -\frac{1}{4} \leq x \leq \frac{1}{4} \text{ and } -\epsilon \leq y \leq \epsilon\},$$

$$D_0 = \{(x, y) \in \mathbb{R}^2 \,|\, \frac{1}{4} \leq x \leq \frac{3}{4} \text{ and } -\epsilon \leq y \leq \epsilon\},$$

$$D_1 = \{(x, y) \in \mathbb{R}^2 \,|\, \frac{3}{4} \leq x \leq \frac{5}{4} \text{ and } -\epsilon \leq y \leq \epsilon\}.$$

Lemma 5.10 (adapted from [19]) *There exist $\epsilon > 0$ and a brick decomposition $\mathcal{D} = \{B_i\}_{i \in \mathbb{N}}$ of U such that:*

1. *D_{-1}, D_0 and D_1 are bricks of $\mathcal{D}$,*
2. *$\mathcal{D}$ satisfies the hypotheses H1 and H2 in Lemma 5.9.*

Proof of Lemma 5.10: Clearly one can choose $\epsilon > 0$ so small that, for k, l in $\{0, \pm 1\}$, we have

- $D_k \cap h(D_k) = \emptyset = D_k \cap h^2(D_k)$,
- $D_k \cap h(D_l) = \emptyset$ or $D_k \cap h^{-1}(D_l) = \emptyset$.

Let us denote $V = U \setminus Int(D_{-1} \cup D_0 \cup D_1)$. This is a non compact bordered surface with only one boundary component, namely $\partial(D_{-1} \cup D_0 \cup D_1)$, so there exists a countable triangulation $\mathcal{T}$ of V. Let $\mathcal{T}'$ be (for example) the barycentric subdivision of $\mathcal{T}$. If the middle p of some edge E of $\mathcal{T}$ is a vertex of a rectangle D_k ($k = 0, \pm 1$), we slightly alter $\mathcal{T}'$ remplacing p with a close point $q \in E$. Remark that this requires only finitely many modifications since there are only finitely triangles of $\mathcal{T}$ meeting $\partial(D_{-1} \cup D_0 \cup D_1)$. We continue to write $\mathcal{T}'$ for this "perturbation" of $\mathcal{T}'$. Then we define

$$\mathcal{D}_1 = \{D_{-1}, D_0, D_1\} \cup \{Star(v, \mathcal{T}') \,|\, v \text{ vertex of } \mathcal{T}\},$$

where $Star(v, \mathcal{T}')$ is the union of the triangles of $\mathcal{T}'$ containing v. It is left to the reader that $\mathcal{D}_1$ is a brick decomposition of U and in particular that, for any vertex v of $\mathcal{T}$, the set $Star(v, \mathcal{T}') \cap D_k$ is either empty or an arc ($k = 0, \pm 1$). If the condition H1 does not hold with $\mathcal{D}_1$, we can achieve it as follows; the bricks $B \in \mathcal{D}_1$ such that $B \cap (h(B) \cup h^2(B)) \neq \emptyset$ can be numbered $B_1, B_2, \ldots$ and $D_0, D_{\pm 1}$ *are not* in this (finite or countable) sequence because of the choice of ϵ. Up to a homeomorphism, we can suppose that B_1 is a rectangle. It can be subdivided into finitely many sub-rectangles $B_{1,1}, B_{1,2}, \ldots, B_{1,n_1}$ disjoint from their images by h and h^2; it suffices that all these $B_{1,j}$ have diameter less than the infimum of $\mathrm{d}(z, h(z))$ and $\mathrm{d}(z, h^2(z))$, z lying in B_1, where $\mathrm{d}(\cdot,\cdot)$ is any distance defining the topology of $\mathbb{S}^2$. Furthermore, since B_1 meets only finitely many bricks of $\mathcal{D}_1$, this can be done in such a way that we have still a brick decomposition of U replacing B_1 with $B_{1,1}, B_{1,2}, \ldots, B_{1,n_1}$ in $\mathcal{D}_1$ (see Fig. 6). We write $\mathcal{D}_{1,1}$ for this new brick decomposition of U. Now construct similarly a brick decomposition of U

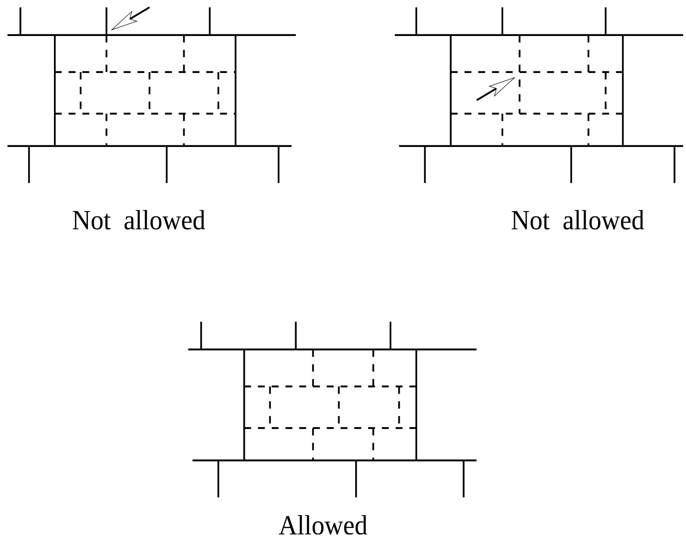

Figure 6: Subdivision of a brick

$$\mathcal{D}_{1,2} = (\mathcal{D}_{1,1} \setminus \{B_2\}) \cup \{B_{2,1}, B_{2,2}, \ldots, B_{2,n_2}\}$$

where $B_{2,1}, B_{2,2}, \ldots, B_{2,n_2}$ come from a suitable subdivision of B_2, and so on. Then $\mathcal{D}_2 = (\mathcal{D}_1 \setminus \{B_1, B_2, \ldots\}) \bigcup_{i,j} \{B_{i,j}\}$ is a brick decomposition satisfying H1 and possessing D_0, $D_{\pm 1}$ as bricks.

Suppose now that H2 is not satisfied with $\mathcal{D}_2$, i.e. the set

$$\{B \in \mathcal{D}_2 \mid \exists\, B' \in \mathcal{D}_2 \quad \text{such that} \quad B \cap h^{-1}(B') \neq \emptyset \quad \text{and} \quad B \cap h(B') \neq \emptyset\}$$

is nonempty. The bricks in this latter set and *other than* $D_0, D_{\pm 1}$ are numbered $B_1, B_2, \ldots$. Since $h(B_1) \cup h^{-1}(B_1)$ is compact, there are only finitely many corresponding B', say $B'_1, \ldots, B'_n$. Each B'_k is disjoint from its image by h^2 so there exists $\delta > 0$ such that

$$\forall k \in \{1, \ldots, n\} \quad \mathrm{dist}(h^{-1}(B'_k), h(B'_k)) > \delta,$$

where $\mathrm{dist}(X, Y)$ is the distance between two subsets X, Y of $\mathbb{S}^2$. As above we can subdivide B_1 into "sub-bricks" with diameter less than δ and so carefully that we have still a brick decomposition of U, say $\mathcal{D}_{2,1}$, when B_1 is replaced with its sub-bricks in $\mathcal{D}_2$. Afterwards, define $\mathcal{D}_{2,2}$ replacing B_2 with suitable sub-bricks in $\mathcal{D}_{2,1}$, etc. It is easily seen that, replacing in $\mathcal{D}_2$ the bricks $B_1, B_2, \ldots$ with their sub-bricks, we obtain a brick decomposition of U which satisfies the hypotheses H1,H2 and which possesses D_0, $D_{\pm 1}$ as bricks. $\square$

Let us consider the attractor $\mathcal{A}$ and the repeller $\mathcal{R}$ associated to $B_{i_0} = D_0$. We remark that

$$D_1 \subset \mathcal{A}_1 \subset \mathcal{A} \quad \text{and} \quad D_{-1} \subset \mathcal{R}_{-1} \subset \mathcal{R}$$

since respectively $(\frac{5}{4}, 0) \in h(D_0) \cap D_1$ and $(-\frac{1}{4}, 0) \in h^{-1}(D_0) \cap D_{-1}$.

According to Lemma 5.9 (i), the vertical segment $\{\frac{3}{4}\} \times [-\epsilon, \epsilon]$ is contained in a connected component Δ of $\partial_U \mathcal{A}$. Furthermore we know from Property 5.6 that Δ is either a Jordan curve or homeomorphic to $\mathbb{R}$. Before to deal with these two situations, we give the following notations and an elementary but important lemma.

Notations 5.11

$$\gamma_- = \{(x, 0) \mid -\frac{1}{4} < x < \frac{3}{4}\},$$
$$\gamma_+ = \{(x, 0) \mid \frac{3}{4} < x < \frac{7}{4}\} = h(\gamma_-),$$
$$\gamma = \{(x, 0) \mid -\frac{1}{4} < x < \frac{7}{4}\} = \gamma_- \cup \{(\frac{3}{4}, 0)\} \cup \gamma_+.$$

Lemma 5.12 *The set $h^{-1}(\Delta) \cup \gamma_-$ (resp. $\gamma_+ \cup h(\Delta)$) is connected and contained in $U \setminus \mathcal{A}$ (resp. in $Int(\mathcal{A})$).*

Proof of Lemma 5.12: These sets are obviously contained in U. For the connectedness, it is enough to remark that

$$(-\frac{1}{4}, 0) \in h^{-1}(\Delta) \cap Cl(\gamma_-) \quad \text{and} \quad (\frac{7}{4}, 0) \in Cl(\gamma_+) \cap h(\Delta).$$

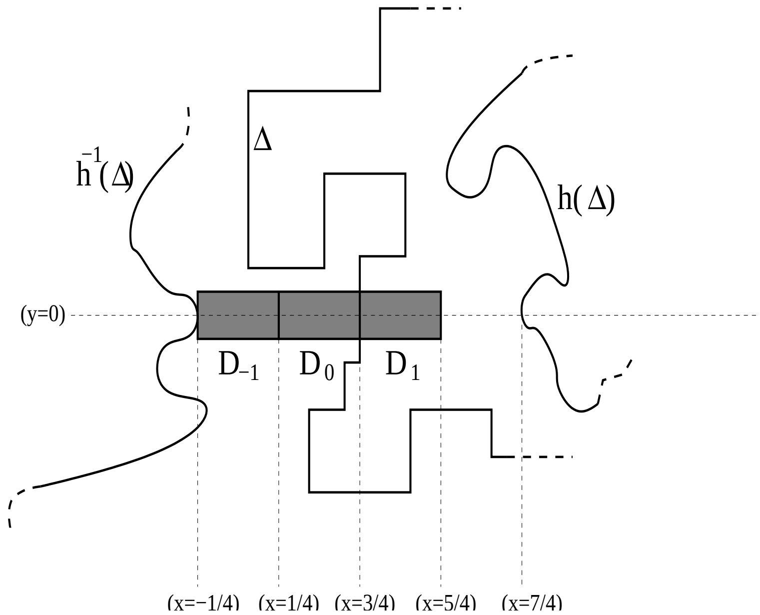

Figure 7: The bricks D_0, $D_{\pm 1}$ and Δ, $h^{\pm 1}(\Delta)$ close to these bricks

Property 5.8 gives $h(\Delta) \subset h(\mathcal{A}) \subset Int(\mathcal{A})$ and also

$$h^{-1}(\Delta) \cap \mathcal{A} = h^{-1}(\Delta \cap h(\mathcal{A})) \subset h^{-1}(\partial_U \mathcal{A} \cap Int(\mathcal{A})) = \emptyset.$$

According to Lemma 5.9 we have

$$Int(D_{-1}) \cap \mathcal{A} \subset Int(\mathcal{R}) \cap \mathcal{A} = \emptyset \quad \text{and} \quad Int(D_0) \cap \mathcal{A} = \emptyset$$

hence $\gamma_- \cap \mathcal{A} = \emptyset$.

It remains to be checked that $\gamma_+ \subset Int(\mathcal{A})$. This follows from

$$\{(x,0) \,|\, \frac{3}{4} < x < \frac{5}{4}\} \subset Int(D_1) \subset Int(\mathcal{A})$$

and, with Property 5.8, from

$$\{(x,0) \,|\, \frac{5}{4} \leq x < \frac{7}{4}\} \subset h(D_0) \subset Int(\mathcal{A}).$$

$\square$

First case: The set Δ is a Jordan curve.
Claim 1: The set Δ separates $h^{-1}(\Delta)$ and $h(\Delta)$ in $\mathbb{S}^2$.
Proof: If this was not true, Lemma 5.12 would show that the segment γ intersects Δ transversely and meets only one connected component of $\mathbb{S}^2 \setminus \Delta$, which is absurd. $\square$

Let us write V_+ for the connected component of $\mathbb{S}^2 \setminus \Delta$ containing $h(\Delta)$. We have $\partial h(V_+) = h(\Delta) \subset V_+$ so $h(V_+) \cap V_+ \neq \emptyset$ and actually $h(Cl(V_+)) \subset V_+$ since, according to the above claim,

$$h(V_+) \cap \partial V_+ = h(V_+) \cap \Delta = h(V_+ \cap h^{-1}(\Delta)) = \emptyset.$$

It is now a routine to construct a topological embedding φ defined on $\mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\}$ and conjugating h and H. We just sketch such a construction; defining $\Omega = V_+ \setminus h(Cl(V_+))$, we have clearly $Cl(\Omega) = \Delta \cup \Omega \cup h(\Delta) \subset U$. Let $\varphi : \mathbb{S}^1 \to \Delta$ be a homeomorphism. It can be extended to a homeomorphism

$$\varphi : \mathbb{S}^1 \cup H(\mathbb{S}^1) \to \Delta \cup h(\Delta)$$

defining $\varphi|_{H(\mathbb{S}^1)} = h \circ \varphi \circ H^{-1}|_{H(\mathbb{S}^1)}$. Using suitably the Schoenflies Theorem, one can extend again φ to a homeomorphism from the compact annulus $A = \{z \in \mathbb{C} \,|\, \frac{1}{2} \leq |z| \leq 1\}$ onto $Cl(\Omega)$. Finally, for any point $z \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exists a unique $k \in \mathbb{Z}$ such that $z \in H^k(A \setminus \partial^- A)$, where $\partial^- A = \{z \in \mathbb{C} \,|\, |z| = \frac{1}{2}\}$, and we define

$$\varphi(z) = h^k \circ \varphi \circ H^{-k}(z) \in h^k(Cl(\Omega)).$$

One can easily check that $\varphi : \mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\} \to U$ is a well-defined one-to-one continuous map such that $h \circ \varphi = \varphi \circ H|_{\mathcal{O}}$ and $\varphi(\mathcal{O}) = \bigcup_{k\in\mathbb{Z}} h^k(Cl(\Omega))$.

Second case: The set Δ is homeomorphic to $\mathbb{R}$.
Since Δ is a closed subset of U we have

$$\emptyset \neq Cl(\Delta) \setminus \Delta \subset \mathrm{Fix}(h).$$

Moreover, $Cl(\Delta) \setminus \Delta$ has at most two connected components, say L_1 and L_2 with possibly $L_1 = L_2$, and each L_i is contained in a connected component K_i of $\mathrm{Fix}(h)$. It will be convenient to compactify $\mathbb{S}^2 \setminus (K_1 \cup K_2)$ as follows; let us choose a_1 and a_2 in $\mathbb{S}^2$ with the convention that $a_1 = a_2$ if and only if $K_1 = K_2$. Since U has been assumed to be connected, we have the same for $\mathbb{S}^2 \setminus (K_1 \cup K_2)$ and it is then very classical that this latter set is homeomorphic to $\mathbb{S}^2 \setminus \{a_1, a_2\}$ (see for example [18]Chapter VI). Now, if ψ is any homeomorphism from $\mathbb{S}^2 \setminus (K_1 \cup K_2)$ onto $\mathbb{S}^2 \setminus \{a_1, a_2\}$, we define $\hat{h} : \mathbb{S}^2 \to \mathbb{S}^2$ by

$$\hat{h}(z) = \begin{cases} z & \text{if } z \in \{a_1, a_2\}, \\ \psi \circ h \circ \psi^{-1}(z) & \text{if } z \notin \{a_1, a_2\}. \end{cases}$$

One can check that $\hat{h}$ is a homeomorphism and that $Cl(\psi(\Delta)) \setminus \psi(\Delta) = \{a_1, a_2\}$. Furthermore, since we are looking for a (proper) topological embedding φ taking its values in $U \subset \mathbb{S}^2 \setminus (K_1 \cup K_2)$, it is enough to prove our theorem for $\hat{h}$ instead of h. In other words, there is no loss in supposing that K_i (and so L_i) is reduced to one point ($i \in \{1, 2\}$). This will be assumed from now on.
Claim 2: We have necessarily $K_1 = K_2$.
Proof: Suppose this is not true and define

$$C = Cl(\Delta \cup h(\Delta)) = \Delta \cup h(\Delta) \cup K_1 \cup K_2.$$

Thus C is a Jordan curve. Let us remark that the sets $h^{-1}(\Delta) \cup \gamma_-$ and γ_+ are both connected and contained in $U \setminus (\Delta \cup h(\Delta)) \subset \mathbb{S}^2 \setminus C$; for $h^{-1}(\Delta) \cup \gamma_-$, this is contained in Lemma 5.12 since we know from Property 5.8 that $\Delta \cup h(\Delta) \subset \mathcal{A}$. Lemma 5.12 gives also

$$\gamma_+ \cap \Delta \subset Int(\mathcal{A}) \cap \partial_U \mathcal{A} = \emptyset$$

and

$$\gamma_+ \cap h(\Delta) = h(\gamma_- \cap \Delta) \subset h(\gamma_- \cap \mathcal{A}) = \emptyset.$$

Now, since the segment γ intersects $\Delta \subset C$ transversely, we deduce that the connected components V_-, V_+, of $\mathbb{S}^2 \setminus C$ containing respectively $h^{-1}(\Delta) \cup \gamma_-$ and γ_+ are different. It follows that

$$\partial h^{-1}(V_+) \cap V_+ = h^{-1}(C) \cap V_+ = h^{-1}(\Delta) \cap V_+ = \emptyset$$

so we have either $V_+ \subset h^{-1}(V_+)$ or $V_+ \cap h^{-1}(V_+) = \emptyset$. We remark now that none of these two situations is possible. The first one would imply

$$\gamma_+ \cup \gamma_- = \gamma_+ \cup h^{-1}(\gamma_+) \subset h^{-1}(V_+)$$

which is absurd because the segment γ intersects $\Delta \subset h^{-1}(C)$ transversely. Suppose now that $h^{-1}(V_+) \cap V_+ = \emptyset$. We first remark that we cannot have $h^{-1}(Cl(V_+)) \cup Cl(V_+) = \mathbb{S}^2$ since this would imply $h^{-1}(\Delta) = h(\Delta)$ which contradicts Property 5.8. So the set $h^{-1}(Cl(V_+)) \cup Cl(V_+)$ is contained in the domain of a single chart of $\mathbb{S}^2$ and can be represented as in Fig 8. Keeping in mind

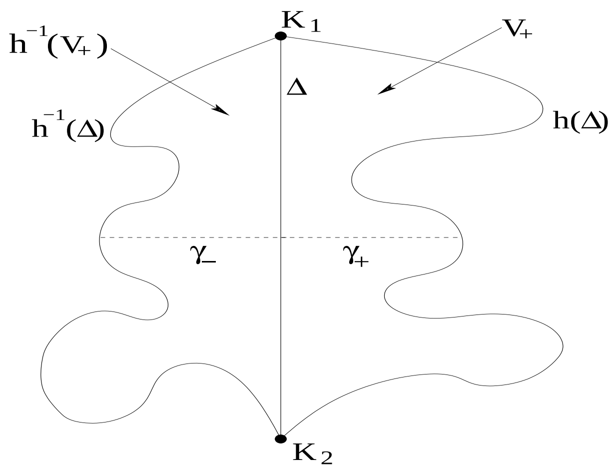

Figure 8: $V_+ \cap h^{-1}(V_+) = \emptyset$ is not possible

that K_1, K_2 are fixed points of h, this contradicts the fact that h reverses the orientation. $\square$

Thus $Cl(\Delta) = \Delta \cup K_1$ is a Jordan curve. Again, γ intersects $\Delta \subset Cl(\Delta)$ transversely so we can write with Lemma 5.12:

Claim 3: The set $Cl(\Delta)$ separates $h^{-1}(\Delta)$ and $h(\Delta)$ in $\mathbb{S}^2$.

Now, let V_+ be the connected component of $\mathbb{S}^2 \setminus Cl(\Delta)$ containing $h(\Delta)$. Since $h(\Delta) \subset \partial h(V_+) \cap V_+$ we have $h(V_+) \cap V_+ \neq \emptyset$ and in fact $h(V_+ \cup \Delta) \subset V_+$ because the third claim implies

$$h(V_+) \cap \partial V_+ = h(V_+) \cap Cl(\Delta) = h(V_+) \cap \Delta = h(V_+ \cap h^{-1}(\Delta)) = \emptyset.$$

We conclude as follows. Let us define $\Omega = V_+ \setminus h(Cl(V_+))$. We have obviously $Cl(\Omega) \setminus K_1 = \Delta \cup \Omega \cup h(\Delta) \subset U$. Using the Schoenflies Theorem, one can construct a homeomorphism

$$\varphi : \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1\} \cup \{\infty\} \to Cl(\Omega)$$

such that $\varphi(\infty) = K_1$, $\varphi(\{0\} \times \mathbb{R}) = \Delta$ and

$$\forall y \in \mathbb{R} \quad \varphi(1, y) = h \circ \varphi \circ G^{-1}(1, y) \in h(\Delta).$$

Now, if $k \leq x < k + 1$ ($k \in \mathbb{Z}$) we let

$$\varphi(x, y) = h^k \circ \varphi \circ G^{-k}(x, y) \in h^k(\Delta \cup \Omega).$$

It is easily seen that $\varphi : \mathcal{O} = \mathbb{R}^2 \to U$ defined in this way is a proper topological embedding, with image $\varphi(\mathcal{O}) = \bigcup_{k \in \mathbb{Z}} h^k(\Delta \cup \Omega)$, such that $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$. This completes the proof of Theorem 5.1 when Property P1 is true.

5.3.2 Proof when P2 is true

Up to conjugacy in $\mathbb{S}^2$, we can suppose that

$$h^{-2}(\beta) = [-2,0] \times \{-1\},$$

$$h(x,y) = (x+1,-y) \text{ if } (x,y) \in \bigcup_{k=-2}^{1} h^k(\beta) = [-2,2] \times \{-1\} \cup [-1,3] \times \{1\},$$

$$m = (\frac{3}{2},-1).$$

For $\epsilon > 0$, let us consider the five rectangles (see Fig. 9)

$$D_i = \{(x,y) \,|\, \frac{i+1}{2} \leq x \leq \frac{i+3}{2} \text{ and } -1-\epsilon \leq y \leq -1+\epsilon\} \quad \text{for } i \in \{0,\pm 2\},$$

$$D_i = \{(x,y) \,|\, \frac{i+1}{2} \leq x \leq \frac{i+3}{2} \text{ and } 1-\epsilon \leq y \leq 1+\epsilon\} \quad \text{for } i = \pm 1.$$

The proof of the next lemma is similar to the one of Lemma 5.10 and will be omitted.

Lemma 5.13 *There exist $\epsilon > 0$ and a brick decomposition $\mathcal{D} = \{B_i\}_{i\in\mathbb{N}}$ of U such that:*

1. *$D_0, D_{\pm 1}$ and $D_{\pm 2}$ are bricks of $\mathcal{D}$,*
2. *$\mathcal{D}$ satisfies the hypotheses H1 and H2 of Lemma 5.9.*

We consider the attractor $\mathcal{A}$ and the repeller $\mathcal{R}$ associated to the brick $B_{i_0} = D_0$. First we remark that

$$D_1 \cup D_2 \subset \mathcal{A} \quad \text{and} \quad D_{-1} \cup D_{-2} \subset \mathcal{R}$$

since, on one hand, $(2,1) \in h(D_0) \cap D_1$ and $(2,-1) \in h(D_1) \cap D_2$, and on the other hand, $(0,1) \in h^{-1}(D_0) \cap D_{-1}$ and $(0,-1) \in h^{-1}(D_{-1}) \cap D_{-2}$. Using Lemma 5.9, we see that the vertical segment $\{\frac{3}{2}\} \times [-1-\epsilon, -1+\epsilon]$ is contained in a connected component Δ of $\partial_U \mathcal{A}$. As in Section 5.3.1 we give some convenient notations and a basic lemma before to study the situation where Δ is homeomorphic to $\mathbb{S}^1$ (resp. to $\mathbb{R}$).

Notations 5.14

$$\gamma_- = \{(x,-1) \,|\, -\frac{1}{2} < x < \frac{3}{2}\},$$

$$\gamma_+ = \{(x,-1) \,|\, \frac{3}{2} < x < \frac{7}{2}\} = h^2(\gamma_-),$$

$$\gamma = \{(x,-1) \,|\, -\frac{1}{2} < x < \frac{7}{2}\} = \gamma_- \cup \{(\frac{3}{2},-1)\} \cup \gamma_+.$$

Lemma 5.15 *The set $h^{-2}(\Delta)\cup\gamma_-$ (resp. $\gamma_+\cup h^2(\Delta)$) is connected and contained in $U\setminus\mathcal{A}$ (resp. in $Int(\mathcal{A})$).*

The proof is similar to the one of Lemma 5.12 and is left to the reader.

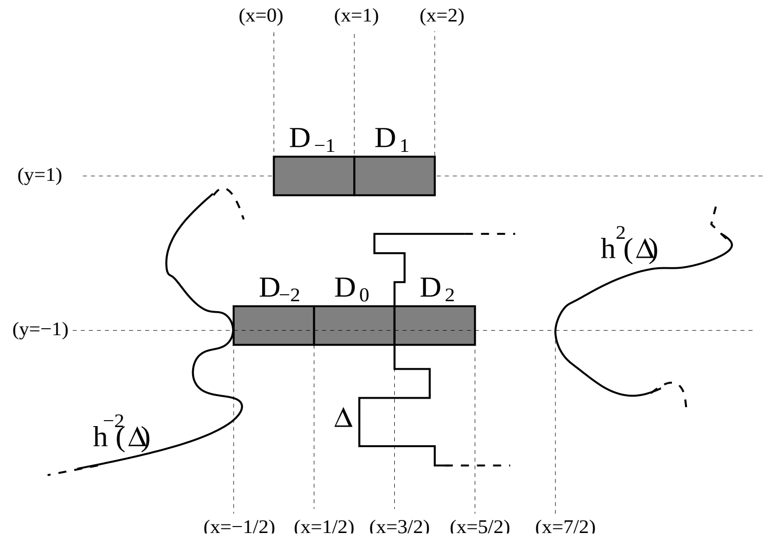

Figure 9: The bricks $D_0, D_{\pm1}, D_{\pm2}$ and $\Delta, h^{\pm2}(\Delta)$ close to these bricks

First case: The set Δ is a Jordan curve.
Claim 4: The set Δ separates $h^{-1}(\Delta)$ and $h(\Delta)$ in $\mathbb{S}^2$.
Proof: First we remark that Δ separates $h^{-2}(\Delta)$ and $h^2(\Delta)$ in $\mathbb{S}^2$: this follows from Lemma 5.15 and from the fact that γ intersects Δ transversely. Let us denote V_-, V_+ the connected components of $\mathbb{S}^2\setminus\Delta$ containing respectively $h^{-2}(\Delta)$ and $h^2(\Delta)$. As in Section 5.3.1 (with h^2 in the place of h), one can check that $h^2(Cl(V_+))\subset V_+$ or equivalently $Cl(V_-)\subset h^2(V_-)$. According to the Brouwer fixed point Theorem, h^2 possesses two fixed points $z_-\in V_-$ and $z_+\in V_+$ and these points are also fixed points of h since h has no 2-periodic point. In particular we have

$$V_+\cap h(V_+)\neq\emptyset\neq V_-\cap h^{-1}(V_-).$$

We deduce now from $h(\Delta)\cap\Delta=\emptyset$ that $h(\Delta)\subset V_+$: otherwise we would have $h(\Delta)\subset V_-$ and consequently

$$V_+\cap\partial h(V_+)=V_+\cap h(\Delta)=\emptyset$$

so $V_+\subset h(V_+)\subset h^2(V_+)$ which contradicts $h^2(Cl(V_+))\subset V_+$. We get similarly $h^{-1}(\Delta)\subset V_-$ replacing h,V_+ with h^{-1},V_-. □

Defining $\Omega = V_+ \setminus h(Cl(V_+))$, we proceed now exactly as in Section 5.3.1 to construct a topological embedding

$$\varphi : \mathcal{O} = \mathbb{R}^2 \setminus \{(0,0)\} \to U$$

with image $\varphi(\mathcal{O}) = \bigcup_{k\in\mathbb{Z}} h^k(Cl(\Omega))$ such that $h \circ \varphi = \varphi \circ H|_{\mathcal{O}}$.

Second case: The set Δ is homeomorphic to $\mathbb{R}$.
We denote again L_1, L_2 the connected components of the nonempty set $Cl(\Delta) \setminus \Delta \subset \mathrm{Fix}(h)$, with possibly $L_1 = L_2$. Each L_i is contained in a connected component K_i of $\mathrm{Fix}(h)$ and, as explained in Section 5.3.1, there is no loss in supposing that K_i (and so L_i) is reduced to one point.

For convenience we will use the following notations for the two half-planes on both sides of the x-axis:

$$P_+ = \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} \quad \text{and} \quad P_- = \{(x,y) \in \mathbb{R}^2 \,|\, y < 0\}.$$

We first suppose $K_1 = K_2$.

Then $Cl(\Delta) = \Delta \cup K_1$ is a Jordan curve. Using again Lemma 5.15 and since $\gamma \cap \Delta$ is a transverse intersection, one can write:
Claim 5: The set $Cl(\Delta)$ separates $h^{-2}(\Delta)$ and $h^2(\Delta)$ in $\mathbb{S}^2$.
We consider now the two connected components V_-, V_+ of $\mathbb{S}^2 \setminus Cl(\Delta)$, with $h^2(\Delta) \subset V_+$ and $h^{-2}(\Delta) \subset V_-$. One can easily derive from the claim above that $h^2(V_+ \cup \Delta) \subset V_+$, i.e. $V_- \cup \Delta \subset h^2(V_-)$.
Claim 6: There are three possible situations:

S1: $h(V_+ \cup \Delta) \subset V_+$,

S2: $h(V_+ \cup \Delta) \subset V_-$,

S3: $h(V_- \cup \Delta) \subset V_+$.

Proof: Suppose that we are neither in the situation S1 nor in the situation S2. Then $h(V_+ \cup \Delta)$ meets $\partial V_+ = \partial V_- = Cl(\Delta)$. Since $h(\Delta) \cap \Delta = \emptyset$, this implies $h(V_+) \cap \Delta \neq \emptyset$ and then $\Delta \subset h(V_+)$. Consequently $h(V_- \cup \Delta)$ is a connected subset of $\mathbb{S}^2 \setminus Cl(\Delta)$ and we get either $h(V_- \cup \Delta) \subset V_+$ or $h(V_- \cup \Delta) \subset V_-$. The latter is actually not possible because of $V_- \cup \Delta \subset h^2(V_-)$. □

We construct now a proper topological embedding $\varphi : \mathcal{O} \to U$ conjugating h and G which will be defined on $\mathcal{O} = \mathbb{R}^2$ in the first situation and on $\mathcal{O} = \{(x,y) \in$

$\mathbb{R}^2 \mid y \neq 0\}$ in the last two ones.
• In the situation S1 we proceed exactly as in Section 5.3.1.
• Remark now that

$$h(V_- \cup \Delta) \subset V_+ \Longleftrightarrow V_- \cup \Delta \subset h(V_+) \Longleftrightarrow h^{-1}(V_- \cup \Delta) \subset V_+$$

which shows that the situation S3 can be reduced to the situation S2 replacing h with h^{-1}. Since it is equivalent to prove Theorem 5.1 for h or for h^{-1}, it suffices to consider S2. In this case, let us denote $\Omega = V_+ \setminus h^2(Cl(V_+))$. We have then $Cl(\Omega) \setminus K_1 = \Delta \cup \Omega \cup h^2(\Delta) \subset U$. We construct the required embedding φ as follows. We consider for example the set $D = \{(x, \frac{1}{x}) \mid x > 0\}$ and we write B for the domain between D and $G^2(D)$ in the upper half-plane P_+. Using the Schoenflies Theorem, one can construct a homeomorphism

$$\varphi : Cl(B) = Cl_{\mathbb{R}^2}(B) \cup \{\infty\} \to Cl(\Omega)$$

such that $\varphi(\infty) = K_1$, $\varphi(D) = \Delta$ and $\varphi \circ G^2|_D = h^2 \circ \varphi|_D$. Then we define the map φ on the half-plane P_+ observing that for every point $z \in P_+$ there exists a unique even integer $2k \in \mathbb{Z}$ such that $z \in G^{2k}(D \cup B)$ and then defining

$$\varphi(z) = h^{2k} \circ \varphi \circ G^{-2k}(z) \in h^{2k}(\Delta \cup \Omega).$$

In particular we have at this stage

$$h^2 \circ \varphi = \varphi \circ G^2|_{P_+}.$$

Afterwards we extend φ on P_- by

$$\forall y < 0 \quad \varphi(x, y) = h \circ \varphi \circ G^{-1}(x, y) \in \bigcup_{k \in \mathbb{Z}} h^{2k+1}(\Delta \cup \Omega).$$

It is easily seen that we have obtained in this way a continuous map

$$\varphi : \mathcal{O} = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\} \to U$$

satisfying $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$ and such that, for every $x \in \mathbb{R}$, $\varphi((\{x\} \times \mathbb{R}) \cap \mathcal{O})$ is a closed subset of U . It is not totally obvious that this map φ is one-to-one (in contrast to the previously constructed embeddings). To check this property, it is enough to see that the sets $h^k(\Delta \cup \Omega)$, $k \in \mathbb{Z}$, are pairwise disjoint. This turns out to be true because $h^k(\Delta) \cap h^l(\Delta) = \emptyset$ for $k \neq l$ (Property 5.8) and because

$$\Omega \cap h(\Omega) \subset V_+ \cap h(V_+) = \emptyset, \qquad \Omega \cap h^2(\Omega) \subset \Omega \cap h^2(V_+) = \emptyset$$

which implies, according to Lemma 5.2, $h^k(\Omega) \cap h^l(\Omega) = \emptyset$ for $k \neq l$.

We suppose now $K_1 \neq K_2$.

Let us define $C = Cl(\Delta \cup h^2(\Delta)) = \Delta \cup h^2(\Delta) \cup K_1 \cup K_2$. Thus C is a Jordan curve.

Claim 7: The set C separates $h^{-2}(\Delta)$ and γ_+ in $\mathbb{S}^2$.

Proof: Property 5.8 gives $\Delta \cup h^2(\Delta) \subset \mathcal{A}$ so, with Lemma 5.15, $h^{-2}(\Delta) \cup \gamma_-$ is contained in a connected component V_- of $\mathbb{S}^2 \setminus C$. This lemma also gives $\gamma_+ \cap \Delta \subset \gamma_+ \cap \partial_U \mathcal{A} = \emptyset$ and $\gamma_+ \cap h^2(\Delta) \subset h^2(\gamma_- \cap \mathcal{A}) = \emptyset$ hence γ_+ is also contained in a connected component V_+ of $\mathbb{S}^2 \setminus C$. We have necessarily $V_- \neq V_+$ since the segment γ intersects $\Delta \subset C$ transversely. $\square$

We keep the notations V_-, V_+ of the proof above, that is V_- (resp. V_+) is the connected component of $\mathbb{S}^2 \setminus C$ containing $h^{-2}(\Delta)$ (resp. γ_+). In particular we have $\partial V_- = \partial V_+ = C$.

Claim 8: We have $h^2(V_+) \cap V_+ = \emptyset = h(V_+) \cap V_+$.

Proof: According to the previous claim we have

$$\partial h^{-2}(V_+) \cap V_+ = (h^{-2}(\Delta) \cup \Delta) \cap V_+ = \emptyset$$

so we have either $h^{-2}(V_+) \cap V_+ = \emptyset$ or $V_+ \subset h^{-2}(V_+)$. The latter would imply that γ is contained in $h^{-2}(V_+)$ except for the point $(\frac{3}{2}, -1)$ which is absurd since this segment intersects $\Delta \subset h^{-2}(C)$ transversely. This proves $h^2(V_+) \cap V_+ = \emptyset$. For the other equality, we first observe that the situations $h^{\pm 1}(V_+) \subset V_+$ are not possible since they contradict $h^2(V_+) \cap V_+ = \emptyset$. Suppose now $V_+ \cap h(V_+) \neq \emptyset$. Then we have

$$h(V_+) \cap C \neq \emptyset \quad \text{and} \quad V_+ \cap h(C) \neq \emptyset,$$

that is

$$h(V_+) \cap (\Delta \cup h^2(\Delta)) \neq \emptyset \quad \text{and} \quad V_+ \cap (h(\Delta) \cup h^3(\Delta)) \neq \emptyset.$$

For convenience we define four sets $E_1, \ldots, E_4$ by

$$E_1 = h(V_+) \cap \Delta,\ E_2 = h(V_+) \cap h^2(\Delta),\ E_3 = V_+ \cap h(\Delta),\ E_4 = V_+ \cap h^3(\Delta).$$

Since $h^k(\Delta) \cap h^l(\Delta) = \emptyset$ for $k \neq l$ we see that E_i is either empty or equal, for respectively $i = 1, 2, 3, 4$, to the whole set Δ, $h^2(\Delta), h(\Delta), h^3(\Delta)$.

It turns out that necessarily $E_1 = \emptyset$, hence $E_2 = h^2(\Delta)$. Otherwise we would have $\Delta \subset h(V_+)$, i.e. $h^{-1}(\Delta) \subset V_+$, and $h^{-1}(Cl(\Delta))$ would be a connected set joining K_1 and K_2 in $Cl(V_+)$. Moreover, $Cl(\gamma_+)$ is an arc contained in V_+ except one endpoint on Δ and the other one on $h^2(\Delta)$ so it separates K_1 and K_2 in

$Cl(V_+)$. This implies $h^{-1}(\Delta) \cap \gamma_+ \neq \emptyset$. On the other hand, since $\gamma_+ \subset \mathcal{A}$, we get with Property 5.8

$$h^{-1}(\Delta) \cap \gamma_+ = h^{-1}(\Delta \cap h(\gamma_+)) \subset h^{-1}(\partial_U \mathcal{A} \cap Int(\mathcal{A})) = \emptyset,$$

a contradiction.

We observe also that the two sets E_2 and E_4 cannot be simultaneously nonempty since this would give $h^3(\Delta) \subset h^2(V_+) \cap V_+ = \emptyset$. It remains to be studied the situation $h(\Delta) \subset V_+$, i.e. $h^2(\Delta) \subset h(V_+)$. We first observe that we cannot have $Cl(V_+) \cup h(Cl(V_+)) = \mathbb{S}^2$ because this would imply $\Delta \subset h(V_+)$ and then $h(\Delta) \subset h^2(V_+) \cap V_+ = \emptyset$. Thus the whole set $Cl(V_+) \cup h(Cl(V_+)$ is contained in the domain of a single chart of $\mathbb{S}^2$. In such a chart, the situation is as in Fig. 10 and, K_1 and K_2 being fixed points, we obtain a contradiction with the fact that h reverses the orientation. The claim is proved. $\square$

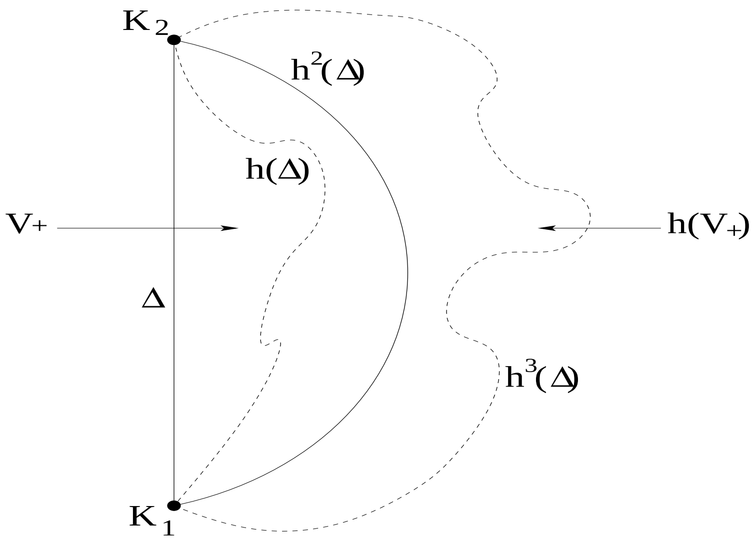

Figure 10: The situation $h(\Delta) \subset V_+$ is not possible

We consider now a new "model" homeomorphism G_1 defined by

$$\forall\, (x, y) \in \mathbb{R}^2 \quad G_1(x, y) = (x + |y|, -y).$$

Let $D = \{(0, y) \in \mathbb{R}^2 \,|\, y > 0\}$ and let B be the domain between D and $G_1^2(D)$ in the half-plane P_+. Using again the Schoenflies Theorem, one can construct a homeomorphism $\varphi_1 : Cl(B) \to Cl(V_+)$ such that $\varphi_1(0, 0) = K_1$, $\varphi_1(\infty) = K_2$, $\varphi_1(D) = \Delta$ and $\varphi_1 \circ G_1^2|_D = h^2 \circ \varphi_1|_D$. For every point $z \in P_+$ there exists a unique even integer $2k \in \mathbb{Z}$ such that $z \in G_1^{2k}(D \cup B)$ and we set

$$\varphi_1(z) = h^{2k} \circ \varphi_1 \circ G_1^{-2k}(z) \in h^{2k}(\Delta \cup V_+).$$

We have in this way $h^2 \circ \varphi_1 = \varphi_1 \circ G_1^2|_{P_+}$. Extending φ_1 on P_- by

$$\forall\, y < 0 \quad \varphi_1(x,y) = h \circ \varphi_1 \circ G_1^{-1}(x,y) \in \bigcup_{k\in\mathbb{Z}} h^{2k+1}(\Delta \cup V_+),$$

we obtain a continuous map φ_1 defined on $\mathcal{O} = \{(x,y) \in \mathbb{R}^2 \,|\, y \neq 0\}$ and such that $h \circ \varphi_1 = \varphi_1 \circ G_1|_{\mathcal{O}}$. Using the eighth claim and Lemma 5.2 we get $h^k(\Delta \cup V_+) \cap h^l(\Delta \cup V_+) = \emptyset$ for $k \neq l$ which ensures that φ_1 is one-to-one. Finally, it is easy to construct a homeomorphism $\psi : \mathcal{O} \to \mathcal{O}$ such that $G_1 \circ \psi = \psi \circ G|_{\mathcal{O}}$ and such that

$$\forall x \in \mathbb{R} \quad Cl\Big(\psi\big((\{x\} \times \mathbb{R}) \cap \mathcal{O}\big)\Big) \setminus \psi\big((\{x\} \times \mathbb{R}) \cap \mathcal{O}\big) = \{(0,0), \infty\}.$$

Then $\varphi = \varphi_1 \circ \psi$ is a proper topological embedding such that $h \circ \varphi = \varphi \circ G|_{\mathcal{O}}$, with $\varphi(\mathcal{O}) = \bigcup_{k\in\mathbb{Z}} h^k(\Delta \cup V_+)$. The proof of Theorem 5.1 is completed. $\square$

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