# Nielsen theory and linked periodic orbits of homeomorphisms of $\mathbb{S}^2$

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## 1 Linked periodic orbits

It is well known that every orientation preserving homeomorphism of the plane $\mathbb{R}^2$ with a $k$-periodic orbit $\mathcal{O}$ ($k \geq 2$) necessarily has a fixed point (see [4] or [5], [7], [11]). It is then interesting to know if $\mathcal{O}$ is linked with one of these fixed points. The difficulty to answer this question depends a lot on the precise sense we give to the word "linked". Before to discuss briefly this point, let us observe this general problem has a very natural counterpart in the framework of orientation reversing homeomorphisms of the sphere $\mathbb{S}^2$. Indeed, every such homeomorphism possessing a $k$-periodic orbit $\mathcal{O}$ ($k \geq 3$) also has a 2-periodic orbit ([2]) and one can ask if $\mathcal{O}$ is linked with one of these 2-periodic orbits. The aim of this paper is to answer this question in some sense.

For completeness, let us recall what seems to be the deeper form of the "linking problem", due to J. Franks: if an orientation preserving homeomorphism $h$ of $\mathbb{R}^2$ has a $k$-periodic orbit $\mathcal{O}$ ($k \geq 2$), is it possible to find a fixed point $z$ such that the rotation number $\rho(\mathcal{O}, h|_{A_z})$ of $\mathcal{O}$ in the open annulus $A_z = \mathbb{R}^2 \setminus \{z\}$ is non-zero modulo $\mathbb{Z}$ ? The only partial results in this direction are [1] and [10]. Similarly one can ask if an orientation reversing homeomorphism $h$ of $\mathbb{S}^2$ with a periodic orbit $\mathcal{O}$ of period at least three also has a 2-periodic orbit $\mathcal{O}'$ such that $\rho(\mathcal{O}, h|_{A_{\mathcal{O}'}}) \neq 0$ in $\mathbb{Q}/\mathbb{Z}$, where $A_{\mathcal{O}'}$ is the annulus $\mathbb{S}^2 \setminus \mathcal{O}'$. Observe that this makes sense because $h|_{A_{\mathcal{O}'}}$ interchanges the two ends and reverses the orientation of $A_{\mathcal{O}'}$ so its lifts to the universal cover $\widetilde{A_{\mathcal{O}'}} = \mathbb{R}^2$ commutes with the deck translations, which allows to define the rotation number $\rho(\mathcal{O}, h|_{A_{\mathcal{O}'}})$ as usual. We rather deal in this paper with another notion of linking, due to J.M. Gambaudo. In what follows, a *disc* (resp. a *Jordan curve*) in a surface $S$ is a subset of $S$ homeomorphic to the closed unit disc of $\mathbb{R}^2$ (resp. to $\mathbb{S}^1$). The frontier and the interior of a disc $D \subset S$ are denoted respectively $\partial D$ and $Int(D)$.

**Definition 1.1** ([8]) *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two periodic orbits of a homeomorphism $h$ of a surface $S$. We say that $\mathcal{O}_1$ and $\mathcal{O}_2$ are* unlinked *if there exist* *two discs $D_i \subset S$ ($i = 1, 2$) with the following properties:*

- $\forall i = 1, 2 \quad \mathcal{O}_i \subset Int(D_i)$,
- $D_1 \cap D_2 = \emptyset$,
- $\forall i = 1, 2 \quad h(\partial D_i)$ *is freely isotopic to* $\partial D_i$ *in* $S \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$.

*Otherwise the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are said to be* linked.

B. Kolev proved in [14] that every $k$-periodic orbit $\mathcal{O}$ ($k \geq 2$) of an orientation preserving $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}^2$ is linked with a fixed point in this sense. His result actually holds for a homeomorphism by using a local smoothing near the orbit $\mathcal{O}$ similarly as in Section 3. The same is also true for an orientation preserving $\mathcal{C}^1$-embedding of the 2-disc ([8]). We also recall the following definition of P. Boyland which appear to be related with our result.

**Definition 1.2** ([[3], p.265]) *Let $h$ be a homeomorphism of an (open or compact) annulus $A$. A periodic orbit $\mathcal{O}$ of $h$ is* trivially embedded in $A$ *if there exists a disc $D \subset A$ such that*

- $\mathcal{O} \subset Int(D)$,
- *the (non essential) Jordan curves $\partial D$ and $h(\partial D)$ are freely isotopic in $A \setminus \mathcal{O}$.*

We can now state the result which is proved in this paper.

**Theorem 1.1** *Let $h$ be an orientation reversing homeomorphism of the sphere $\mathbb{S}^2$ with a periodic orbit $\mathcal{O}$ of period $k \geq 3$. Then there exists a 2-periodic orbit $\mathcal{O}'$ such that one cannot find any Jordan curve $C \subset \mathbb{S}^2$ separating $\mathcal{O}$ and $\mathcal{O}'$ which is freely isotopic to $h(C)$ in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$. In particular*

- *$\mathcal{O}$ and $\mathcal{O}'$ are linked in the sense of [8],*
- *$\mathcal{O}$ is not trivially embedded in the open annulus $A_{\mathcal{O}'} = \mathbb{S}^2 \setminus \mathcal{O}'$ in the sense of [3].*

Although the existence of a 2-periodic orbit is a purely topological fact (see [2]), we will need some part of Nielsen theory based on hyperbolic geometry to obtain the linking property. Of course a great deal of progress about surface homeomorphisms has been made since Nielsen, especially from Thurston's work. One could probably give an alternate proof of Theorem 1.1 using Thurston's machinery. Maybe an advantage in using Theorem 2.1 below is to avoid any result involving persistence of periodic orbits under isotopy. The reader interested in the relationship between Nielsen and Thurston theories is referred to [9] and [12].

## 2 Nielsen classes and their index

If $M$ is a compact connected surface with universal covering $\pi : \tilde{M} \to M$, one can define a *Nielsen class* of a continuous map $\psi : M \to M$ as the set $\pi(\text{Fix}(\tilde{\psi}))$ where $\tilde{\psi} : \tilde{M} \to \tilde{M}$ is a lift of $\psi$ and $\text{Fix}(\tilde{\psi})$ its fixed point set (this definition allows a Nielsen class to be empty). The nonempty Nielsen classes of $\psi$ define a partition of $\text{Fix}(\psi)$. Moreover each Nielsen class $X$ is an isolated subset of $\text{Fix}(\psi)$ hence one can define its Lefschetz index $\text{Ind}(\psi, X) \in \mathbb{Z}$. There are only finitely many Nielsen classes. See [6] or [13] for more details.

We suppose from now on that $M$ is orientable, has negative Euler characteristic $\chi(M)$ and that $\psi$ is a homeomorphism. Theorem 2.1 below, due to Nielsen (and maybe a little forgotten), is the main ingredient for proving our result. It is proved for closed surfaces in [15] (see [17] for an english translation). The adaptation to the case of surfaces with boundary can be found in [16]. The paper [9] is also useful to understand Nielsen's work. We just recall here the minimal background to make this theorem usable.

One can endow $M$ with a complete hyperbolic structure, with geodesic boundary if any. Then $\tilde{M}$ can be identified with a convex subset of the hyperbolic Poincaré disc $\mathbb{H}^2$ and its fundamental group $F$ with a group of orientation preserving isometries of $\mathbb{H}^2$. We have simply $\tilde{M} = \mathbb{H}^2$ if $M$ has no boundary and a precise description of $\tilde{M}$ will be recalled below in the case of a bordered surface. For simplicity, we use the same letter for an isometry $l$ of $\mathbb{H}^2$ and for the linear fractional transformation extending it to the whole plane $\mathbb{C}$. The fixed point set of every $l \in F \setminus \{Id_{\tilde{M}}\}$ consists of exactly two points on the boundary circle $\mathbb{S}^1$ of $\mathbb{H}^2$. Considering the set

$$G_F = \bigcup_{l \in F \setminus \{Id_{\tilde{M}}\}} \text{Fix}(l) \subset \mathbb{S}^1$$

and its closure $E_\infty = \overline{G}_F$, it turns out that $E_\infty$ is the whole circle $\mathbb{S}^1$ (resp. a nowhere dense perfect subset of $\mathbb{S}^1$) if $M$ is a closed (resp. bordered) surface. If $M$ has boundary $\partial M$, then $\tilde{M}$ is obtained by considering all the intervals $\{I_n\}_{n \in \mathbb{N}}$ connected component of $\mathbb{S}^1 \setminus E_\infty$ and by removing from $\mathbb{H}^2$ all the hyperbolic half-planes bounded by such an $I_n$ and the geodesic joining its endpoints. Then the frontier of $\tilde{M}$ w.r.t. $\tilde{M} \cup E_\infty$ is $\pi^{-1}(\partial M) \cup E_\infty$ and is topologically a circle. A key result, valid for both closed and bordered surfaces, is that every lift $\tilde{\psi} : \tilde{M} \to \tilde{M}$ of $\psi$ extends to a homeomorphism (also denoted $\tilde{\psi}$) of $\tilde{M} \cup E_\infty$. Although Nielsen only deals with orientation preserving homeomorphisms, it is well known that the extended homeomorphism $\tilde{\psi}$ preserves (resp. reverses) the orientation of the disc $\tilde{M} \cup E_\infty$ if $\psi$ preserves (resp. reverses) the orientation of $M$. We assume now that $\psi$ preserves the orientation. For every lift $\tilde{\psi}$ of $\psi$ we consider $N(\tilde{\psi}) = \{l \in F \mid l \circ \tilde{\psi} = \tilde{\psi} \circ l\}$. Nielsen proved that $N(\tilde{\psi})$ is a finitely generated sugbgroup of $F$ whose minimal number of generators is denoted by $\nu(\tilde{\psi})$, with $\nu(\tilde{\psi}) = 0$ if $\mathrm{N}(\tilde{\psi}) = \{Id_{\tilde{M}}\}$. Defining

$$M(\tilde{\psi}) = \mathrm{Fix}(\tilde{\psi}|_{E_\infty}) \quad \text{and} \quad G_{N(\tilde{\psi})} = \bigcup_{l \in N(\tilde{\psi}) \setminus \{Id_{\tilde{M}}\}} \mathrm{Fix}(l),$$

it follows from the definition of $\tilde{\psi}$ on $E_\infty$ that $\overline{G}_{N(\tilde{\psi})} \subset M(\tilde{\psi})$. Nielsen defines an *isolated point* of $M(\tilde{\psi})$ as being the common endpoint of two intervals connected components of $\mathbb{S}^1 \setminus M(\tilde{\psi})$, both of them containing points of $E_\infty$. We follow [9] and we rather use the word N-isolated ("N" for Nielsen). In a connected component of $\mathbb{S}^1 \setminus M(\tilde{\psi})$ adjacent to a N-isolated point of $M(\tilde{\psi})$, the points of $E_\infty$ are all moved in the same direction by $\tilde{\psi}$. Thus we get in the obvious way a notion of *attractive* or *repulsive* or *neutral* N-isolated point of $M(\tilde{\psi})$. An important remark for our purpose is the following: if $M(\tilde{\psi})$ contains a neutral N-isolated point then $N(\tilde{\psi})$ is an abelian group generated by one element $l$ such that $M(\tilde{\psi}) = \mathrm{Fix}(l)$. We recall finally that the points of $M(\tilde{\psi})$ in a given connected component of $\mathbb{S}^1 \setminus \overline{G}_{N(\tilde{\psi})}$ do not accumulate. Consequently the points of $M(\tilde{\psi})$ in such an interval are N-isolated and alternately attractive and repulsive. The number of $N(\tilde{\psi})$-orbits of attractive N-isolated points of $M(\tilde{\psi})$ is finite and denoted by $\mu(\tilde{\psi}) \in \mathbb{N}$. If $F$ is generated by only one element $l$, this definition of $\mu(\tilde{\psi})$ must be modified by considering only the attractive N-isolated points different from the fixed points of $l$. We can now state:

**Theorem 2.1** (Nielsen, [15], [16]) Let $\psi : M \to M$ be an orientation preserving homeomorphism and let $X = \pi(\mathrm{Fix}(\tilde{\psi}))$ be a Nielsen class of $\psi$. Then we have

$$\mathrm{Ind}(\psi, X) = 1 - \mu(\tilde{\psi}) - \nu(\tilde{\psi})$$

with the only following exception: if $M$ has no boundary and $\psi$ is isotopic to $Id_M$, then $\mathrm{Ind}(\psi, X) \in \{2 - 2g, 0\}$, where $g$ is the genus of $M$.

# 3 Proof of Theorem 1.1

We first remark that there is no loss in supposing that $h$ is a $\mathcal{C}^1$-diffeomorphism at each point of $\mathcal{O}$. Indeed, for any given $\epsilon > 0$ one can construct an isotopy $(h_t)_{0 \le t \le 1}$ from $h_0 = h$ such that every $h_t$ coincides with $h$ on $\mathcal{O}$ and outside a $\epsilon$-neighbourhood of $\mathcal{O}$ and $h_1$ is a $\mathcal{C}^1$-diffeomorphism on $\mathcal{O}$. For a small enough $\epsilon$, all the $h_t$ have exactly the same 2-periodic orbits. Moreover, for any 2-periodic orbit $\mathcal{O}'$ of $h$ and any Jordan curve $C \subset \mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$, we have a free isotopy $(h_t(C))_{0 \le t \le 1}$ from $h(C)$ to $h_1(C)$ in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$ which shows that Theorem 1.1 is true for $h$ iff it is true for $h_1$.

Thus we can consider the homeomorphism $\varphi : M \to M$ obtained by blowing-up $h|_{\mathbb{S}^2 \setminus \mathcal{O}}$ (see e.g. [[3], p.234]). The surface $M$ is a compact $k$-holed sphere (hence $\chi(M) = 2 - k \le -1$) having $\mathbb{S}^2 \setminus \mathcal{O}$ as interior. The homeomorphisms $\varphi$ and $h$ coincide on $\mathbb{S}^2\backslash\mathcal{O}$ and the $k$ boundary components of $M$ are setwise permuted by $\varphi$ with the period $k$ so $\varphi$ and $h$ have exactly the same 2-periodic orbits. We use in the following the notations from Section 2.

**Lemma 3.1** *The image $\varphi(X)$ of a Nielsen class $X$ of $\varphi^2$ is also a Nielsen class of $\varphi^2$.*

*Proof.* Let $\tilde{\psi}$ be a lift of $\varphi^2$ such that $\pi(\mathrm{Fix}(\tilde{\psi})) = X$ and let $\tilde{\varphi}$ be any lift of $\varphi$. We have

$$\mathrm{Fix}(\tilde{\varphi} \circ \tilde{\psi} \circ \tilde{\varphi}^{-1}) = \tilde{\varphi}(\mathrm{Fix}(\tilde{\psi})).$$

Since $\tilde{\varphi} \circ \tilde{\psi} \circ \tilde{\varphi}^{-1}$ is also a lift of $\varphi^2$ this proves that $\pi(\tilde{\varphi}(\mathrm{Fix}(\tilde{\psi}))) = \varphi(X)$ is also a Nielsen class of $\varphi^2$. $\square$

The next lemma was inspired by [14].

**Lemma 3.2** *Let $\mathcal{O}'$ be a 2-periodic orbit of $h$ (i.e. of $\varphi$). Suppose that the conclusion of Theorem 1.1 does not hold with $\mathcal{O}'$. Then the two points of $\mathcal{O}'$ are in the same Nielsen class $X$ of $\varphi^2$. In particular this Nielsen class is such that $\varphi(X) = X$.*

*Proof.* Under this assumption, there exist a Jordan curve $C \subset \mathbb{S}^2$ separating $\mathcal{O}$ and $\mathcal{O}'$ and an isotopy $(\alpha_t : \mathbb{S}^1 \to \mathbb{S}^2\backslash(\mathcal{O}\cup\mathcal{O}') \subset M)_{0\leq t\leq 1}$ from $\alpha_0(\mathbb{S}^1) = C$ to $\alpha_1(\mathbb{S}^1) = h(C) = \varphi(C)$. According to the Jordan Theorem, $\mathbb{S}^2 \setminus C$ has exactly two connected components whose closures are two discs $D, \Delta$ such that $\partial D = \partial\Delta = C$, say with $\mathcal{O} \subset Int(D)$ and $\mathcal{O}' \subset Int(\Delta)$. In particular $C$ bounds the disc $\Delta \subset M$ hence $(\alpha_t)_{0\leq t\leq 1}$ can be lifted to an isotopy $(\tilde{\alpha}_t : \mathbb{S}^1 \to \tilde{M})_{0\leq t\leq 1}$. For every $t \in [0, 1]$ we write $C_t = \alpha_t(\mathbb{S}^1)$ and $\tilde{C}_t = \tilde{\alpha}_t(\mathbb{S}^1)$. The Jordan curve $\tilde{C}_t$ bounds only one disc $\tilde{\Delta}_t$ in the surface $\tilde{M}$. The map $\pi$ induces a homeomorphism from every connected component of $\pi^{-1}(\Delta)$ onto the disc $\Delta \subset M$. Since $\tilde{C}_0$ is the frontier of such a connected component, one deduces that $\tilde{\Delta}_0$ is a connected component of $\pi^{-1}(\Delta)$ and that $\tilde{\Delta}_0 \cap \pi^{-1}(\mathcal{O}')$ consists of exactly two points $\tilde{a}_0, \tilde{b}_0$. The set $\varphi(\Delta) \subset M$ is also a disc containing $\mathcal{O}'$ hence the same arguments show that $\tilde{\Delta}_1$ is a connected component of $\pi^{-1}(\varphi(\Delta))$ and $\tilde{\Delta}_1 \cap \pi^{-1}(\mathcal{O}') = \{\tilde{a}_1, \tilde{b}_1\}$ with $\tilde{a}_1 \neq \tilde{b}_1$. There exists a lift $\tilde{\varphi} : \tilde{M} \to \tilde{M}$ of $\varphi$ such that $\tilde{\varphi}(\tilde{\Delta}_0) = \tilde{\Delta}_1$, say with $\tilde{\varphi}(\tilde{a}_0) = \tilde{b}_1$ and $\tilde{\varphi}(\tilde{b}_0) = \tilde{a}_1$. Now observe that we have necessarily $\tilde{a}_0 = \tilde{a}_1$ and $\tilde{b}_0 = \tilde{b}_1$ since otherwise one could find $t \in [0, 1]$ such that $\{\tilde{a}_0, \tilde{b}_0\} \cap \tilde{C}_t \neq \emptyset$ so $\mathcal{O}' \cap C_t \neq \emptyset$, a contradiction. This proves that $\mathcal{O}'$ is contained in the Nielsen class $X = \pi(\mathrm{Fix}(\tilde{\varphi}^2))$ of $\varphi^2$. Lemma 3.1 and $\varphi(\mathcal{O}') = \mathcal{O}'$ then imply $\varphi(X) = X$. $\square$

**Remark 1** *Let $\mathcal{O}'$ be as in Lemma 3.2. We keep the notations $C, D, \Delta$ and $(\alpha_t)_{0\leq t\leq 1}$ of the proof above. Considering now the universal covering $p : \mathbb{R}^2 \to A_{\mathcal{O}'}$ of the open annulus $A_{\mathcal{O}'} = \mathbb{S}^2 \setminus \mathcal{O}'$, observe that the isotopy*

*$(\alpha_t : \mathbb{S}^1 \to \mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}') \subset A_{\mathcal{O}'})_{0\leq t\leq 1}$ can be lifted to an isotopy $(\breve{\alpha}_t : \mathbb{S}^1 \to \mathbb{R}^2)_{0\leq t\leq 1}$ because $C$ bounds the disc $D \subset A_{\mathcal{O}'}$. Similarly as in the proof of Lemma 3.2, the Jordan curves $\breve{\alpha}_0(\mathbb{S}^1)$, $\breve{\alpha}_1(\mathbb{S}^1)$ bound two discs $\breve{D}_0$, $\breve{D}_1$ in $\mathbb{R}^2$ which are connected components of respectively $p^{-1}(D)$, $p^{-1}(h(D))$ and there exists a lift $\breve{h} : \mathbb{R}^2 \to \mathbb{R}^2$ of $h|_{A_{\mathcal{O}'}}$ such that $\breve{h}(\breve{D}_0) = \breve{D}_1$. Because $\mathcal{O} \cap \alpha_t(\mathbb{S}^1) = \emptyset$ for every $t \in [0,1]$, one deduces that $p^{-1}(\mathcal{O}) \cap \breve{D}_0 = p^{-1}(\mathcal{O}) \cap \breve{D}_1$, hence this latter set is a k-periodic orbit of $\breve{h}$ and consequently $\rho(\mathcal{O}, A_{\mathcal{O}'}) = 0$. This proves that if $\mathcal{O}$ was linked with a 2-periodic orbit $\mathcal{O}'$ in the "strong sense" adapted from Franks (see Section 1) then $\mathcal{O}'$ would also satisfy the conclusion of Theorem 1.1.*

**Lemma 3.3** *If a Nielsen class $X$ of $\varphi^2$ satisfies $\varphi(X) = X$ then we have Ind$(\varphi^2, X) \leq 0$.*

*Proof.* Let $\tilde{\psi}$ be a lift of $\varphi^2$ such that $\pi(\text{Fix}(\tilde{\psi})) = X$. We can suppose $\text{Fix}(\tilde{\psi}) \neq \emptyset$ since otherwise $X = \emptyset$ has index zero. According to Theorem 2.1, it is enough to prove $\mu(\tilde{\psi}) \geq 1$ under the assumption $\nu(\tilde{\psi}) = 0$, i.e. $N(\tilde{\psi}) = \{Id_{\tilde{M}}\}$. We have then $G_{N(\tilde{\psi})} = \emptyset$ hence $M(\tilde{\psi}) = M(\tilde{\psi}) \setminus \overline{G}_{N(\tilde{\psi})}$ and this set has finite cardinality because it does not accumulate in $\mathbb{S}^1 \setminus \overline{G}_{N(\tilde{\psi})} = \mathbb{S}^1$. It cannot contain any neutral point because this would imply $\nu(\tilde{\psi}) = 1$. Since there are as many attractive points as repulsive points in $M(\tilde{\psi}) = M(\tilde{\psi}) \setminus \overline{G}_{N(\tilde{\psi})}$, we just have to check that $M(\tilde{\psi})$ is nonempty. Recall that the frontier of $\tilde{M}$ w.r.t. $\tilde{M} \cup E_\infty$, namely $\pi^{-1}(\partial M) \cup E_\infty$, is homeomorphic to $\mathbb{S}^1$ and that an orientation reversing homeomorphism of a circle has (exactly) two fixed points. Since $\tilde{\psi}$ has no fixed point on $\pi^{-1}(\partial M)$, it is enough to find a lift $\tilde{\varphi}$ of $\varphi$ such that $\tilde{\psi} = \tilde{\varphi}^2$. Pick $\tilde{x} \in \text{Fix}(\tilde{\psi})$. Since $\varphi(X) = X$ there exists a lift $\tilde{\varphi}_1$ of $\varphi$ such that $\tilde{\varphi}_1(\tilde{x}) \in \text{Fix}(\tilde{\psi})$. For the same reason there exists a lift $\tilde{\varphi}_2$ of $\varphi$ such that $\tilde{\varphi}_2(\tilde{\varphi}_1(\tilde{x})) \in \text{Fix}(\tilde{\psi})$. Since $\tilde{\varphi}_2 \circ \tilde{\varphi}_1$ is a lift of $\varphi^2$ there exists $l \in F$ such that $l \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_1 = \tilde{\psi}$ and we have $l(\tilde{\varphi}_2(\tilde{\varphi}_1(\tilde{x}))) = \tilde{\psi}(\tilde{x}) = \tilde{x}$. One can easily deduce from $N(\tilde{\psi}) = \{Id_{\tilde{M}}\}$ that $\pi$ induces a one-to-one map from $\text{Fix}(\tilde{\psi})$ onto $X$ hence we get necessarily $l = Id_{\tilde{M}}$. Thus we obtain $\tilde{\psi} = \tilde{\varphi}_2 \circ \tilde{\varphi}_1$ and also $\tilde{\psi} = \tilde{\varphi}_1 \circ \tilde{\varphi}_2$ because $\tilde{\psi} = \tilde{\varphi}_2 \circ \tilde{\varphi}_1$ and $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ are two lifts of $\varphi^2$ which agree at the point $\tilde{\varphi}_1(\tilde{x})$. Writing $\tilde{\varphi}_2 = l \circ \tilde{\varphi}_1$ for an $l \in F$, it follows that $\tilde{\varphi}_1 \circ l = l \circ \tilde{\varphi}_1$ and then $l \in N(\tilde{\psi}) = \{Id_{\tilde{M}}\}$. This gives $\tilde{\varphi}_1 = \tilde{\varphi}_2$ and $\tilde{\psi} = \tilde{\varphi}_1^2$. □

We can now prove Theoreme 1.1. The Lefschetz trace formula ([6], [13]) gives

$$\sum_X \text{Ind}(\varphi^2, X) = \sum_{i=0}^{2} (-1)^i \text{Tr}((\varphi^2)_{\star,i}) = 2$$

where the summation is over the Nielsen classes of $\varphi^2$ and $\text{Tr}((\varphi^2)_{\star,i})$ is the trace of the endomorphism $(\varphi^2)_{\star,i}$ induced by $\varphi^2$ on the homology group

$H_i(M, \mathbb{Q})$. Hence there is a nonempty Nielsen class $X_0$ of $\varphi^2$ with positive index. According to Lemmas 3.1-3.3, $X_0$ is disjoint from $\mathrm{Fix}(\varphi)$, i.e. contains only 2-periodic points of $\varphi$, and the orbit of any point $z \in X_0$ satisfies the conclusion of Theorem 1.1. □

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