A topological version of the Poincaré-Birkhoff theorem with two fixed points

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Abstract. The main result of this paper gives a topological property satisfied by any homeomorphism of the annulus $\mathbb{A} = \mathbb{S}^1 \times [-1, 1]$ isotopic to the identity and with at most one fixed point. This generalizes the classical Poincaré-Birkhoff theorem because this property certainly does not hold for an area preserving homeomorphism h of $\mathbb{A}$ with the usual boundary twist condition. We also have two corollaries of this result. The first one shows in particular that the boundary twist assumption may be weakened by demanding that the homeomorphism h has a lift H to the strip $\widetilde{\mathbb{A}} = \mathbb{R} \times [-1, 1]$ possessing both a forward orbit unbounded on the right and a forward orbit unbounded on the left. As a second corollary we get a new proof of a version of the Conley-Zehnder theorem in $\mathbb{A}$: if a homeomorphism of $\mathbb{A}$ isotopic to the identity preserves the area and has mean rotation zero, then it possesses two fixed points.

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1 Introduction

In its more classical version, the Poincaré-Birkhoff theorem is the following.

Theorem 1.1 (Birkhoff, [2, 3]) *Let h be a homeomorphism of the annulus $\mathbb{A} = \mathbb{S}^1 \times [-1, 1]$ isotopic to the identity. If h preserves the area and satisfies the boundary twist condition then it has at least two fixed points.*

The boundary twist condition requires that h has a lift H to the universal cover $\widetilde{\mathbb{A}} = \mathbb{R} \times [-1, 1]$ such that, writing $H(\theta, \pm 1) = (\varphi_\pm(\theta), \pm 1)$, one has $(\varphi_-(\theta) - \theta)(\varphi_+(\theta) - \theta) < 0$ for every $\theta \in \mathbb{R}$. Let us recall that this result was conjectured by Poincaré in [24] and proved by Birkhoff in [2, 3] (see also [5]). In fact the proof in [2] actually ensures the existence of only one fixed point while the full Theorem 1.1 turns out to be a particular case of a more general statement in [3] where the annulus is not necessarily setwise invariant under the homeomorphism. This is an interesting direction for generalizing Theorem 1.1 (in particular for applications to ODE's, see e.g. [7]) but we focus in this paper on self-homeomorphisms of $\mathbb{A}$. Our interest is in the search of more topological versions and on the number of fixed points that one can expect from such variants. These questions seem to originate from [24]; indeed it is well known that Poincaré overlooked the possibility for a homeomorphism of $\mathbb{A}$ to have a single fixed point and so suggested the following strategy for proving his theorem ([24][p.376-377]): assume that T is a homeomorphism of $\mathbb{A}$ satisfying the boundary twist condition and without fixed point; then one could show that T does not preserve the area by constructing an essential Jordan curve $C \subset \mathbb{A}$ disjoint from its image $C' = T(C)$. Precisely this program was achieved by Kerékjártó in [17] and the right generalization for obtaining the second fixed point was obtained by P. H. Carter in [6] [1]. Other references for Kerékjártó's and Carter's results are [11], [14] and [19]. One can loosely say after Carter's work that the area preserving hypothesis in Theorem 1.1 may be replaced with the weaker assumption that there is no essential subannulus $\mathbb{B} \subset \mathbb{A}$ containing its image as a proper subset. Afterwards some authors generalized both the twist and the conservative hypotheses; one can quote C. Bonatti and L. Guillou ([13][Théorème 5.1]), J. Franks ([9, 10]) and H.E. Winkelnkemper ([26]), the results in [10], [13] and [26] giving only one fixed point.

The first result of the present paper is a purely topological version of the Poincaré-Birkhoff theorem allowing to detect two fixed points (Theorem 3.1). More precisely this result shows that a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ isotopic to the identity and possessing at most one fixed point should be either "non-conservative" or "untwisted" in some topological sense. It appears as a natural extension of Bonatti-Guillou's theorem concerning fixed point free homeomorphism of $\mathbb{A}$. Section 5 then relates Theorem 3.1 (and its refinement Theorem 3.2) with other works on the subject. As a first application, we obtain Theorem 5.2 showing that a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ can be thought of as twisted if there is a lift $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ possessing a forward orbit unbounded on the left and another one unbounded on the right. This extends a theorem by Franks where the twist assumption is interpreted by means of rotation numbers ([9][Theorem 3.3]). We get finally a short "geometric" proof of M. Flucher's version of the Conley-Zehnder theorem in the annulus ([8][Theorem 2]) when dealing with homeomorphism preserving the Lebesgue measure (Theorem 5.7).

This article can also be regarded as the continuation of [4] where results close to the Poincaré-Birkhoff theorem were obtained in the isotopy class of the symmetry $S_{\mathbb{A}}$ interchanging the boundary components of $\mathbb{A}$. Theorem 3.2 below and Theorem 1.2 of [4] have indeed strong similarities, as well as their proofs. Nevertheless these two results are also logically independent and the present paper is largely self-contained.

2 Preliminaries

2.1 Definitions and notation

The notation and vocabulary used throughout this paper are the same as in [4]. In particular the strip $\widetilde{\mathbb{A}} = \mathbb{R} \times [-1, 1]$ is regarded as the universal cover of the annulus $\mathbb{A} = \mathbb{S}^1 \times [-1, 1]$ with covering map $\Pi(\theta, r) = (e^{2i\pi\theta}, r)$. The deck transformations are then the iterates $\tau^n : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of the translation $\tau(\theta, r) = (\theta + 1, r)$. We write $\overline{X}, \mathrm{Int}(X), \partial X$ for respectively the closure, interior and frontier relative to $\widetilde{\mathbb{A}}$ of a subset $X \subset \widetilde{\mathbb{A}}$. When we need to consider these notions with respect to another topological space Y, we use the explicit notation $\mathrm{Cl}_Y(X), \mathrm{Int}_Y(X)$ and $\partial_Y X$ for any $X \subset Y$. The reader is refered to the short Section 2.1 of [4] for other details. Recall also that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous map and J a Jordan curve in $\mathbb{R}^2 \setminus \mathrm{Fix}(f)$, then the *index* of J w.r.t. f is the winding number of the vector $f(p) - p$ when p describes J counterclokwised oriented. Given any map $f : E \to E$, a family $\mathcal{E}$ of subsets of E is said to be *f-free* if $f(X) \cap X = \emptyset$ for every $X \in \mathcal{E}$. Finally a subset $\mathbb{B} \subset \mathbb{A}$ is named a *subannulus* of $\mathbb{A}$ if it is homeomorphic to $\mathbb{A}$.

2.2 Nielsen classes

If $h : \mathbb{A} \to \mathbb{A}$ is a continuous map, its Nielsen classes may be defined as the various sets $\Pi(\mathrm{Fix}(H))$ where $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ varies over all the lifts of h (see e.g. [15]). The nonempty Nielsen classes of h realize a finite partition of $\mathrm{Fix}(h)$. We consider throughout this paper only maps $h : \mathbb{A} \to \mathbb{A}$ which are homotopic to the identity; the Nielsen classes $\Pi(\mathrm{Fix}(H))$ and $\Pi(\mathrm{Fix}(H'))$ are then disjoint whenever $H, H' : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ are two distinct lifts of h: this follows easily from the fact that one has in this setting $H \circ \tau = \tau \circ H$ and since $H' = \tau^n \circ H$ for some $n \in \mathbb{Z} \setminus \{0\}$. We will say that two

[1] As Birkhoff in [3], Carter deals with the situation where the annulus is not setwise invariant.

Nielsen classes N, N' are *consecutive* if they are both nonempty and if, for some $k \in \{\pm 1\}$ and for some lift $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of h, one has $N = \Pi(\mathrm{Fix}(H))$ and $N' = \Pi(\mathrm{Fix}(\tau^k \circ H))$.

2.3 Le Calvez-Sauzet's brick decompositions

This notion was introduced in [18, 25] as a convenient tool for proving Brouwer's plane translation theorem. A *brick decomposition* of a surface S consists essentially in a locally finite tiling of S with topological closed discs (the *bricks* of the decomposition) in such a way that any subset $X \subset S$ obtained as the union of some bricks is a subsurface of S. We refer to Section 2.4 of [4] for a definition and for basic properties. Because of their importance, we recall now the related notions of attractor and repeller. Suppose that $f : S \to S$ is a homeomorphism of a surface S endowed with a brick decomposition $\mathcal{D} = \{B_i\}_{i \in I}$. The *attractor* associated to a brick B_{i_0} (and to f) is the union of all the bricks B such that, for some integer $n \geq 1$, there exists a sequence of bricks $B_{i_0}, B_{i_1}, \cdots, B_{i_n} = B$ satisfying

$$\forall j \in \{0, \cdots, n-1\}\, f(B_{i_j}) \cap B_{i_{j+1}} \neq \emptyset.$$

One defines similarly the *repeller* associated to B_{i_0} by replacing f with f^{-1}. If $\mathcal{A}, \mathcal{R}$ denote respectively the attractor and the repeller associated to some brick, one clearly has $f(\mathcal{A}) \subset \mathcal{A}$ and $f^{-1}(\mathcal{R}) \subset \mathcal{R}$. Even better, since the union of the bricks containing a given point $z \in S$ is a neighbourhood of z in S, one has $f(\mathcal{A}) \subset \mathrm{Int}_S(\mathcal{A})$ and $f^{-1}(\mathcal{R}) \subset \mathrm{Int}_S(\mathcal{R})$. We also have the following essential properties.

Proposition 2.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism leaving setwise invariant a subsurface $S \subset \mathbb{R}^2 \setminus \mathrm{Fix}(f)$ and such that no Jordan curve $J \subset \mathbb{R}^2 \setminus \mathrm{Fix}(f)$ has index 1 w.r.t. f. Let B_{i_0} be any brick of a f-free brick decomposition $\mathcal{D}$ of S and write $\mathcal{A}, \mathcal{R}$ for respectively the attractor and the repeller associated to B_{i_0} and f. Then*

(i) B_{i_0} is contained neither in $\mathcal{A}$ nor in $\mathcal{R}$, i.e. $B_{i_0} \cap \mathrm{Int}_S(\mathcal{A}) = \emptyset = B_{i_0} \cap \mathrm{Int}_S(\mathcal{R})$;

(ii) there is no brick contained in $\mathcal{A} \cap \mathcal{R}$, i.e. $\mathrm{Int}_S(\mathcal{A}) \cap \mathcal{R} = \emptyset = \mathrm{Int}_S(\mathcal{R}) \cap \mathcal{A}$.

This follows from Franks' lemma about periodic disc chains ([9][Proposition 1.3]) which is itself a consequence of Brouwer's lemma on translation arcs. Nevertheless one should observe that Franks' statement deals with open (topological) discs whereas we use it with closed discs, namely the bricks of $\mathcal{D}$. The needed refinements to work with closed discs are due to Guillou and Le Roux (see [21][p.38-39]).

3 Statement of the main results

Our first result is the following.

Theorem 3.1 *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the compact annulus $\mathbb{A} = \mathbb{S}^1 \times [-1, 1]$ isotopic to the identity (i.e. h preserves the orientation and the two boundary components of $\mathbb{A}$). If h has no more than one fixed point then at least one of the two following properties holds:*

1. *There exists an essential Jordan curve $J \subset \mathbb{A}$ such that $J \cap h(J) = J \cap \mathrm{Fix}(h)$.*
2. *There exists an arc α crossing $\mathbb{A}$ such that*
    - *$\alpha \cap h(\alpha) = \alpha \cap \mathrm{Fix}(h)$,*
    - *$h(\alpha)$ does not meet the two local sides of α.*

As mentioned in the introduction, Theorem 3.1 is a natural extension of Bonatti-Guillou's version of the Poincaré-Birkhoff theorem since it reduces exactly to [13][Théorème 5.1] under the strongest assumption that h is fixed point free. According to the Lefschetz-Hopf theorem, if $z \in \mathbb{A}$ is the unique fixed point of a continuous map $h : \mathbb{A} \to \mathbb{A}$ homotopic to the identity then its Lefschetz index equals the Euler characteristic $\chi(\mathbb{A}) = 0$ hence Theorem 3.1 is contained in Theorem 3.2 below. This result points out, we hope, what are the truly important assumptions for our constructions and will be useful for Theorem 5.2 and Theorem 5.7.

Theorem 3.2 *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the compact annulus $\mathbb{A} = \mathbb{S}^1 \times [-1, 1]$ isotopic to the identity. If h satisfies the following three assumptions*

(i) There is at least one boundary component of $\mathbb{A}$ where h has no fixed point;

(ii) There are no two consecutive Nielsen classes;

(iii) h only has finitely many fixed points and these fixed points (if any) have Lefschetz index 0;

then, for some Nielsen class N_0 of h (maybe $N_0 = \emptyset$), at least one of the following two properties holds:

1'. There exists an essential subannulus $\mathbb{B} \subset \mathbb{A}$ containing either $h(\mathbb{B})$ or $h^{-1}(\mathbb{B})$ as a proper subset. Moreover the boundary of $\mathbb{B}$ consists of one of the two boundary component of $\mathbb{A}$, call it $\mathrm{Bd}^\sigma(\mathbb{A})$, together with an essential Jordan curve $J \subset \mathbb{A} \setminus \mathrm{Bd}^\sigma(\mathbb{A})$ such that $J \cap h(J) = J \cap N_0$;

2'. There exists an arc α crossing $\mathbb{A}$ such that

- *$\alpha \cap h(\alpha) = \alpha \cap N_0$,*
- *$h(\alpha)$ does not meet the two local sides of α.*

4 Proof of Theorem 3.2

Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity and satisfying the assumptions (i)-(iii) in Theorem 3.2. We can suppose without loss that h is fixed point free on $\mathrm{Bd}^-(\mathbb{A})$ so there exists a lift H_0 of h to $\widetilde{\mathbb{A}}$ such that

$$\forall (\theta, -1) \in \mathrm{Bd}^-(\widetilde{\mathbb{A}}) \quad (\theta, -1) < H_0(\theta, -1) < (\theta + 1, -1)$$

and then also

$$\forall (\theta, -1) \in \mathrm{Bd}^-(\widetilde{\mathbb{A}}) \quad (\theta, -1) < \tau \circ H_0^{-1}(\theta, -1) < (\theta + 1, -1).$$

According to (ii) we can pick $G \in \{H_0, \tau \circ H_0^{-1}\}$ which is fixed point free and we rename H the remaining homeomorphism. Thus H is a lift of h while G is a lift of h^{-1} or conversely, and anyway $G \circ H = H \circ G = \tau$. We let $F = \mathrm{Fix}(H)$ and $S = \widetilde{\mathbb{A}} \setminus F$. Thus S is a subsurface of $\widetilde{\mathbb{A}}$ such that $\mathrm{Bd}^-(\widetilde{\mathbb{A}}) \subset S = \tau(S) = H(S)$ and it is an open subset of $\widetilde{\mathbb{A}}$. We also define N_0 to be the Nielsen class of h determined by H, that means $N_0 = \Pi(F)$. We have the following easy fact which will allows us to use Proposition 2.1.

Lemma 4.1 *The homeomorphism $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ can be extended to an orientation preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that any Jordan curve $J \subset \mathbb{R}^2 \setminus \mathrm{Fix}(f)$ have index 0 w.r.t. f.*

Proof of Lemma 4.1. Just let, for $(x, y) \in \mathbb{R}^2$ and $|y| \geq 1$,

$$f(x, y) = H(x, \frac{y}{|y|}) + (0, y - \frac{y}{|y|})$$

and remark that any connected component of $\mathrm{Fix}(f)$ is either (vertically) unbounded or consists in a fixed point of H in $\widetilde{\mathbb{A}} \setminus \mathrm{Bd}(\widetilde{\mathbb{A}})$. Due to (iii), all the fixed points of H are isolated with index 0 and the result follows since the index of a Jordan curve $J \subset \mathbb{R}^2 \setminus \mathrm{Fix}(f)$ surrounding finitely many fixed points $p_1, \cdots, p_n$ is equal to the sum of the indices of the p_i's. ■

We construct now a brick decomposition of S adapted to our purpose. In particular the fourth item in Lemma 4.2 ensures conveniently that the "dynamics on the bricks" adjacent to $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ looks like the behaviour of H on $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$. Similar ideas where used for Lemma 3.1 of [4].

Lemma 4.2 *There exist an $\epsilon \in (0,1)$ and a brick decomposition $\widetilde{\mathcal{D}}_H = \{B_i\}_{i\in I}$ of the surface S satisfying the following properties:*

1. *$\widetilde{\mathcal{D}}_H$ is τ-equivariant, which means $\{\tau(B_i)\}_{i\in I} = \widetilde{\mathcal{D}}_H$.*

2. *$\widetilde{\mathcal{D}}_H$ is H-free.*

3. *The bricks meeting $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ are rectangles $B_n^- = [a_n, a_{n+1}] \times [-1, -1+\epsilon]$ where $(a_n)_{n\in\mathbb{Z}}$ is a strictly increasing sequences of reals numbers such that $\lim_{\pm\infty} a_n = \pm\infty$.*

4. *For every $n \in \mathbb{Z}$ one has $H(B_n^-) \cap B_{n+1}^- \neq \emptyset$, hence $B_{n+1}^- \subset \mathcal{A}(B_n^-)$ and $B_{n-1}^- \subset \mathcal{R}(B_n^-)$.*

Proof of Lemma 4.2. A family $\mathcal{I}$ of compact intervals of $\mathrm{Bd}^-(\mathbb{A})$ is named an *interval decomposition* of $\mathrm{Bd}^-(\mathbb{A})$ if $\bigcup_{\alpha\in\mathcal{I}} \alpha = \mathrm{Bd}^-(\mathbb{A})$ and if any two distinct α, α' of $\mathcal{I}$ can meet only in a common endpoint. We define similarly an interval decomposition of $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$. Because h has no fixed point on $\mathrm{Bd}^-(\mathbb{A})$ there exists a finite and h-free interval decomposition $\mathcal{I}_0$ of $\mathrm{Bd}^-(\mathbb{A})$ (with at least three intervals). By considering all the connected components of all the sets $\Pi^{-1}(\alpha)$, $\alpha \in \mathcal{I}_0$, one gets a H-free and τ-equivariant interval decomposition $\widetilde{\mathcal{I}_0}$ of $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$. Given two interval decompositions $\widetilde{\mathcal{I}}, \widetilde{\mathcal{I}}'$ of $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$, let us write $\widetilde{\mathcal{I}}' \succeq \widetilde{\mathcal{I}}$ if every interval of $\widetilde{\mathcal{I}}$ is contained in an interval of $\widetilde{\mathcal{I}}'$. This defines a partial ordering on the set of all the interval decompositions of $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ and there exists $\widetilde{\mathcal{I}} \succeq \widetilde{\mathcal{I}_0}$ which is maximal among the interval decompositions of $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ which are both H-free and τ-equivariant. Observe that if $\tilde{\alpha} = [a, a'] \times \{-1\}$ and $\tilde{\alpha}' = [a', a''] \times \{-1\}$ are two consecutive intervals of $\widetilde{\mathcal{I}}$ ($a < a' < a''$) then necessarily $H(\tilde{\alpha}) \cap \tilde{\alpha}' \neq \emptyset$. Indeed one has $\tau(\tilde{\alpha}) \neq \tilde{\alpha}'$ because $H(\tilde{\alpha}) \cap \tilde{\alpha} = \emptyset$ and $(a, -1) < H(a, -1) < (a+1, -1)$ on $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ so one gets another τ-equivariant interval decomposition $\widetilde{\mathcal{I}}' \succeq \widetilde{\mathcal{I}}$ by removing from $\widetilde{\mathcal{I}}$ all the intervals $\tau^n(\tilde{\alpha}), \tau^n(\tilde{\alpha}')$ and replacing them with the $\tau^n(\tilde{\alpha} \cup \tilde{\alpha}')$'s ($n \in \mathbb{Z}$). Due to the maximality of $\widetilde{\mathcal{I}}$, this $\widetilde{\mathcal{I}}'$ cannot be H-free so $H(\tilde{\alpha} \cup \tilde{\alpha}') \cap (\tilde{\alpha} \cup \tilde{\alpha}') \neq \emptyset$. Since H moves the points of $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ towards the right one gets $H(\tilde{\alpha}) \cap \tilde{\alpha}' \neq \emptyset$, as announced. The family $\{\Pi(\tilde{\alpha})\}_{\tilde{\alpha}\in\widetilde{\mathcal{I}}}$ being finite, there exists $\epsilon \in (0,1)$ so small that, for every $\tilde{\alpha} = [a, a'] \times \{-1\} \in \widetilde{\mathcal{I}}$, the rectangle $\tilde{R}_{\tilde{\alpha}} = [a, a'] \times [-1, -1+\epsilon]$ is disjoint from its H-image. This provides the bricks B_i^- adjacent to $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ as described in (3)-(4) and it remains to complete this collection of rectangles to get the required brick decomposition of S. The family $\left\{\Pi(\tilde{R}_{\tilde{\alpha}})\right\}_{\tilde{\alpha}\in\widetilde{\mathcal{I}}}$ consists in finitely many topological closed discs in $\mathbb{A} \setminus N_0$ and one easily constructs a brick decomposition $\mathcal{D}$ of $\mathbb{A} \setminus N_0$ containing the $\Pi(\tilde{R}_{\tilde{\alpha}})$'s as bricks. Subdividing if necessary some bricks of $\mathcal{D}$ other than the $\Pi(\tilde{R}_{\tilde{\alpha}})$'s, one can assume that all the connected components of the sets $\Pi^{-1}(B)$, $B \in \mathcal{D}$, are disjoint from their H-image; hence they provide a brick decomposition of S satisfying (1)-(4). ∎

We fix from now on a brick decomposition $\widetilde{\mathcal{D}}_H$ of S given by Lemma 4.2 and a brick B_i^- adjacent to $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$, say $B_0^- = [a_0, a_1] \times [-1, -1+\epsilon]$. We write respectively $\mathcal{A}, \mathcal{R}$ for the attractor and the repellor associated to B_0^- and H. Property (4) in Lemma 4.2 ensures that $[a_1, +\infty) \times \{-1\} \subset \bigcup_{i\geq 1} B_i^- \subset \mathcal{A}$ so $\mathcal{A}$ is unbounded on the right and moreover, using $B_0^- \cap B_1^- \neq \emptyset$, that $\mathcal{A}$ is connected. Similarly $\mathcal{R}$ is connected and unbounded on the left, with $(-\infty, a_0] \times \{-1\} \subset \bigcup_{i\leq -1} B_i^- \subset \mathcal{R}$. Theorem 3.2 now follows from Propositions 4.3-4.4 below.

Proposition 4.3 *The following implications holds:*

- *$\mathcal{A}$ is unbounded on the left $\Rightarrow \mathcal{R} \cap \mathrm{Bd}^+(\widetilde{\mathbb{A}}) = \emptyset \Rightarrow$ the alternative (1') of Theorem 3.2 occurs.*

- *$\mathcal{R}$ is unbounded on the right $\Rightarrow \mathcal{A} \cap \mathrm{Bd}^+(\widetilde{\mathbb{A}}) = \emptyset \Rightarrow$ the alternative (1') of Theorem 3.2 occurs.*

Proposition 4.4 *If $\mathcal{A}$ is bounded on the left and meets $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$ then the conclusion of Theorem 3.2 holds.*

Proof of Proposition 4.3. We just prove the first point, the second one being similar by reversing the roles of $\mathcal{A}$ and $\mathcal{R}$. First suppose that $\mathcal{R} \cap \mathrm{Bd}^+(\widetilde{\mathbb{A}}) \neq \emptyset$. The set $\mathrm{Int}(\mathcal{R})$ is arcwise connected (as the interior of a connected union of bricks) so there exists an arc $\tilde{\gamma}$ crossing $\widetilde{\mathbb{A}}$ and entirely contained in $\mathrm{Int}(\mathcal{R})$. Proposition 2.1 then gives $\mathcal{A} \cap \tilde{\gamma} \subset \mathcal{A} \cap \mathrm{Int}(\mathcal{R}) = \emptyset$. Since furthermore $\mathcal{A}$ is unbounded on the right and connected, it should be contained in the domain on the right of $\tilde{\gamma}$ and so it is bounded on the left.

Assume now that $\mathcal{R} \cap \mathrm{Bd}^+(\widetilde{\mathbb{A}}) = \emptyset$ and define $X_0 = \bigcup_{n\in\mathbb{Z}} \tau^n(\mathcal{R}) \subset \widetilde{\mathbb{A}} \setminus \mathrm{Bd}^+(\widetilde{\mathbb{A}})$. The brick decomposition $\widetilde{\mathcal{D}}_H$ being τ-equivariant, each $\tau^n(\mathcal{R})$ is a union of bricks of $\widetilde{\mathcal{D}}_H$ and so is X_0. One clearly has $\bigcup_{i\in\mathbb{Z}} B_i^- \subset X_0$. Moreover X_0 is connected because so are the $\tau^n(\mathcal{R})$'s and because, given any two $n, m \in \mathbb{Z}$, the brick B_{-i}^- is contained in $\tau^m(\mathcal{R}) \cap \tau^n(\mathcal{R})$ for a large enough $i \in \mathbb{N}$. We need now Lemma 2.4 of [4] apart from the fact that $F = \mathrm{Fix}(H)$ can meet $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$ in the present paper. We bypass this minor difficulty as follows. Consider the strip $\widehat{\mathbb{A}} = \mathbb{R} \times [-1, 2]$ and a brick decomposition $\widehat{\mathcal{D}}$ of $\widehat{\mathbb{A}} \setminus F$ extending $\widetilde{\mathcal{D}}_H$, that means such that any brick of $\widetilde{\mathcal{D}}_H$ is also a brick of $\widehat{\mathcal{D}}$. Recall that any set $X \subset S$ which is a union of bricks of $\widetilde{\mathcal{D}}_H$ is closed in S hence $\overline{X} \subset X \cup F$; moreover F only has isolated points so $\partial_{\widehat{\mathbb{A}}}\mathrm{Cl}_{\widehat{\mathbb{A}}}(X) = \partial\overline{X}$ when X is disjoint from $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$. We now let X be the union of X_0 with all the bounded connected components of $\widehat{\mathbb{A}} \setminus (F \cup X_0)$. Equivalently, X is the union of X_0 with the connected components of $S \setminus X_0$ which are bounded and disjoint from $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$. Thus X is a connected union of bricks of $\widetilde{\mathcal{D}}_H \subset \widehat{\mathcal{D}}$ and [4][Lemma 2.4] tell us that $\partial_{\widehat{\mathbb{A}}}\mathrm{Cl}_{\widehat{\mathbb{A}}}(X) = \partial\overline{X}$ is a 1-dimensional submanifold of $\widehat{\mathbb{A}}$. Moreover $\mathrm{Bd}^-(\widetilde{\mathbb{A}}) \subset \mathrm{Int}_{\widehat{\mathbb{A}}}(X)$ and $(\mathbb{R} \times \{2\}) \cap \mathrm{Cl}_{\widehat{\mathbb{A}}}(X) = \emptyset$ so there is a connected component $\tilde{J}$ of $\partial\overline{X}$ which is a line properly embedded in $\widetilde{\mathbb{A}}$ and separating the boundary components of $\widehat{\mathbb{A}}$. We have $\tau(X) = X$ hence $\tau(\partial\overline{X}) = \partial\overline{X}$ and then $\tau(\tilde{J}) = \tilde{J}$ since otherwise $\tilde{J}$ and $\tau(\tilde{J}) \subset \partial\overline{X}$ would be two disjoint properly embedded lines in $\widetilde{\mathbb{A}}$ joining the two ends of $\widetilde{\mathbb{A}}$, which is not possible since $\overline{X}$ is connected and contains $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ in its interior. It follows that $J = \Pi(\tilde{J})$ is an essential Jordan curve in $\mathbb{A} \setminus \mathrm{Bd}^-(\mathbb{A})$ and we consider the subannulus $\mathbb{B} \subset \mathbb{A}$ bounded by $\mathrm{Bd}^-(\mathbb{A}) \cup J$. In order to show that $\mathbb{B}$ satisfies the required properties, it is enough to check that $H^{-1}(\tilde{J} \cap S)$ is included in the connected component of $\widetilde{\mathbb{A}} \setminus \tilde{J}$ containing $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$. Since X is closed in S and S is an open subset of $\widetilde{\mathbb{A}}$ one gets $\partial\overline{X} \subset \partial_S X \cup F$. Moreover $\partial_S X \subset \partial_S X_0$ and the closedness of X_0 in S implies $\partial_S X_0 \subset \bigcup_{n\in\mathbb{Z}} \partial_S(\tau^n(\mathcal{R}))$. Hence any point $\tilde{z} \in \tilde{J} \cap S$ belongs to $\partial_S(\tau^n(\mathcal{R})) = \tau^n(\partial_S\mathcal{R})$ for some $n \in \mathbb{Z}$ and consequently

$$H^{-1}(\tilde{z}) \in H^{-1}(\tau^n(\partial_S\mathcal{R})) = \tau^n(H^{-1}(\partial_S\mathcal{R})) \subset \tau^n(\mathrm{Int}(\mathcal{R})) \subset \mathrm{Int}(X).$$

We are done since $\mathrm{Int}(X)$ is connected, disjoint from $\tilde{J}$ and contains $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$. ■

Proposition 4.4 is a consequence of Lemmas 4.5-4.6 below.

Lemma 4.5 *If $\mathcal{A}$ is bounded on the left and meets $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$ then at least one of the two following assertions is true.*

1. *There exists an essential Jordan curve $J \subset \mathbb{A} \setminus \mathrm{Bd}(\mathbb{A})$ such that $J \cap h(J) = \emptyset$. In particular the alternative (1') of Theorem 3.2 occurs.*

2. *There exists an arc $\tilde{\beta}$ crossing $\widetilde{\mathbb{A}}$ such that, writing W_r for the domain on the right of $\tilde{\beta}$, we have*
   - *$\tau(W_r) \subset H(W_r) \subset W_r$; equivalently $G(W_r) \subset W_r \subset H^{-1}(W_r)$ since $G = \tau \circ H^{-1}$.*
   - *$H(\tilde{\beta}) \cap \tilde{\beta} = \tilde{\beta} \cap F$.*

Proof of Lemma 4.5. Suppose that $\mathcal{A}$ is bounded on the left and meets $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$. As for Proposition 4.3, we consider the strip $\widehat{\mathbb{A}} = \mathbb{R} \times [-1, 2]$ and we endow the surface $\widehat{\mathbb{A}} \setminus F$ with a brick decomposition $\widehat{\mathcal{D}}$ extending $\widetilde{\mathcal{D}}_H$. Let X be the union of $\mathcal{A}$ with the bounded connected components of $\widehat{\mathbb{A}} \setminus (F \cup \mathcal{A})$. Thus X is a connected union of bricks of $\widetilde{\mathcal{D}}_H \subset \widehat{\mathcal{D}}$, it is bounded on the left and meets the two boundary components of $\widetilde{\mathbb{A}}$. According to [4][Lemma 2.4], the set $\partial_{\widehat{\mathbb{A}}}\mathrm{Cl}_{\widehat{\mathbb{A}}}(X)$ is a 1-dimensional submanifold of $\widehat{\mathbb{A}}$ and one of its connected component Δ is a half-line properly embedded in $\widetilde{\mathbb{A}}$, originating on $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ and meeting $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$. We define $\tilde{\alpha}$ to be the arc obtained by following Δ from its origin to the first point where it intersects $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$. Note that $\tilde{\alpha}$ is an arc crossing $\widetilde{\mathbb{A}}$ and contained in $\partial\overline{X}$. Let U_r be the domain on the right of $\tilde{\alpha}$ and observe that $\mathrm{Int}(X) \subset U_r$ because $\mathrm{Int}(X)$ is connected, unbounded on the right and disjoint from $\partial\overline{X}$. Let us show that $H(U_r) \subset U_r$. Remember that $\partial U_r = \tilde{\alpha} \subset \partial\overline{X} \subset \partial_S X \cup F \subset \partial_S \mathcal{A} \cup F$ so

$$\partial H(U_r) = H(\tilde{\alpha}) = H(\tilde{\alpha} \cap S) \cup (\tilde{\alpha} \cap F)$$

and

$$H(\tilde{\alpha} \cap S) \subset H(\mathcal{A}) \subset \mathrm{Int}(\mathcal{A}) \subset \mathrm{Int}(X) \subset U_r.$$

This gives $\partial H(U_r) \subset \overline{U_r}$ and consequently $H(U_r) \subset U_r$ because H fixes the two ends of $\widetilde{\mathbb{A}}$.

Let us define $V = \bigcup_{n\in\mathbb{N}} G^n(U_r)$. Recall that G is a lift of h or h^{-1} without fixed point so Winkelnkemper's version of the Poincaré-Birkhoff theorem ([26]) tell us that either G is conjugate to τ or there exists an essential Jordan curve $J \subset \mathbb{A}$ such that $J \cap h^{\pm 1}(J) = \emptyset$. In the latter case, J is certainly disjoint from $\mathrm{Bd}(\mathbb{A})$ and we are done. We suppose now that G is conjugate to τ. It follows that $(\theta, \pm 1) < G(\theta, \pm 1)$ on $\mathrm{Bd}^{\pm}(\widetilde{\mathbb{A}})$ for every point $(\theta, \pm 1) \in \mathrm{Bd}(\widetilde{\mathbb{A}})$ since this is already known to be true for $(\theta, -1) \in \mathrm{Bd}^-(\widetilde{\mathbb{A}})$ hence we have $V = \bigcup_{0\le n\le m} G^n(U_r)$ for some $m \in \mathbb{N}$. This implies $\overline{V} = \bigcup_{0\le n\le m} G^n(\overline{U_r})$ so $\widetilde{\mathbb{A}}\setminus\overline{V}$ has a single unbounded (on the left) connected component which we call W_l. It is a classical result of Kerékjártó that any connected component W of the intersection of two Jordan domains $U_1, U_2 \subset \mathbb{R}^2$ is also a Jordan domain, with frontier $\partial_{\mathbb{R}^2} W \subset \partial_{\mathbb{R}^2} U_1 \cup \partial_{\mathbb{R}^2} U_2$ (see [16] or [20][Section 1]). One deduces that ∂W_l is an arc crossing $\widetilde{\mathbb{A}}$ and contained in $\bigcup_{0\le n\le m} G^n(\tilde{\alpha})$. We let $\tilde{\beta} = \partial W_l$ and we write W_r for the domain on the right of $\tilde{\beta}$. It follows from $G(V) \subset V$ that $W_l \subset G(W_l)$, i.e. $H(W_l) \subset \tau(W_l)$, and then $\tau(W_r) \subset H(W_r)$. Furthermore $H \circ G = G \circ H$ and $H(U_r) \subset U_r$ give $H(V) \subset V$ so $W_l \subset H(W_l)$ and then $H(W_r) \subset W_r$. It remains to check that $\tilde{\beta}$ and $H(\tilde{\beta})$ meet only in F. Just observe that $G(S) = S$ hence

$$\tilde{\beta} \cap S \subset \Big(\bigcup_{n\in\mathbb{N}} G^n(\tilde{\alpha})\Big) \cap S \subset \bigcup_{n\in\mathbb{N}} G^n(\partial_S \mathcal{A})$$

and consequently

$$H(\tilde{\beta} \cap S) \subset \bigcup_{n\in\mathbb{N}} G^n(H(\partial_S \mathcal{A})) \subset \bigcup_{n\in\mathbb{N}} G^n(\mathrm{Int}(\mathcal{A})) \subset \bigcup_{n\in\mathbb{N}} G^n(U_r) = V \subset W_r.$$

■

All the arguments for proving the next lemma are already present in [4]. Indeed one gets (1) below as [4][Lemma 3.6] provided the map H^2 appearing in [4] is replaced with the map H of the present paper. Item (2) is essentially obtained in the same way from [4][Lemma 3.7] and close ideas were previously used in [13]. For convenience, we write a proof with some details omitted and refer to [4] for more complete arguments.

Lemma 4.6 *Let $\tilde{\beta}$ and W_r be as in Lemma 4.5.*

1. *If $\tau(\overline{W_r}) \not\subset W_r$, i.e. if $\tau(\tilde{\beta}) \cap \tilde{\beta} \neq \emptyset$, then the alternative (1') of Theorem 3.2 occurs.*
2. *If $\tau(\overline{W_r}) \subset W_r$, i.e. if $\tau(\tilde{\beta}) \cap \tilde{\beta} = \emptyset$, then the alternative (2') of Theorem 3.2 occurs.*

Proof of Lemma 4.6. (1) Suppose that $\tau(\tilde{\beta}) \cap \tilde{\beta} \neq \emptyset$. In the following, any arc crossing $\widetilde{\mathbb{A}}$ is oriented from its endpoint on $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ to its endpoint on $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$. We write $\tilde{z}_0, \tilde{z}_1$ for respectively the endpoint of $\tilde{\beta}$ on $\mathrm{Bd}^-(\widetilde{\mathbb{A}})$ and on $\mathrm{Bd}^+(\widetilde{\mathbb{A}})$; we also let $\tilde{z}$ be the first point of $\tilde{\beta}$ such that $\tau(\tilde{z}) \in \tilde{\beta}$. We first assume that $\tilde{z}$ and $\tau(\tilde{z})$ are met in this order on $\tilde{\beta}$. Since $\tau(\tilde{\beta})$ approaches $\tilde{\beta}$ from only one side, any two points in $\tilde{\beta} \cap \tau(\tilde{\beta})$ are met in the same order on $\tilde{\beta}$ and on $\tau(\tilde{\beta})$. Consequently we have $\{\tau(\tilde{z})\} = [\tilde{z}_0, \tau(\tilde{z})]_{\tilde{\beta}} \cap \tau(\tilde{\beta})$ and the set $\tilde{\gamma} = [\tilde{z}_0, \tau(\tilde{z})]_{\tilde{\beta}} \cup \tau([\tilde{z}_0, \tilde{z}]_{\tilde{\beta}})$ is the frontier $\partial\Omega$ of some connected component Ω of $W_r \cap \tau(W_l)$, where W_l denotes again the domain on the left of $\tilde{\beta}$. Let us show that $\mathbb{B} = \Pi(\overline{\Omega})$ is a subannulus of $\mathbb{A}$ as required. We let $\tilde{J} = [\tilde{z}, \tau(\tilde{z})]_{\tilde{\beta}} \subset \widetilde{\mathbb{A}} \setminus \mathrm{Bd}(\widetilde{\mathbb{A}})$. Clearly $\tilde{J}$ projects onto an essential loop $J = \Pi(\tilde{J}) \subset \mathbb{A} \setminus \mathrm{Bd}(\mathbb{A})$ hence, in order prove that $\mathbb{B}$ is an essential subannulus of $\mathbb{A}$, one just need to show that the covering map Π is one-to-one when restricted to $\overline{\Omega} \setminus [\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}$. Equivalently, one has to show

$$\forall n \in \mathbb{Z} \setminus \{0\} \quad \tau^n(\overline{\Omega} \setminus [\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}) \cap (\overline{\Omega} \setminus [\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}) = \emptyset.$$

This is true for $n = \pm 1$ because $\Omega \subset W_r \setminus \tau(\overline{W_r})$ so

$$\tau(\overline{\Omega}) \cap \overline{\Omega} \subset \tau(\overline{W_r}) \cap \partial\Omega = \tau([\tilde{z}_0, \tilde{z}]_{\tilde{\beta}}).$$

Moreover $\overline{\Omega}$ is a topological closed disc and it is then classical that the inclusion $\tau(\overline{\Omega}) \cap \overline{\Omega} \subset \partial\Omega$ implies $\tau^n(\overline{\Omega}) \cap \overline{\Omega} = \emptyset$ for every integer $n \notin \{0, \pm 1\}$ (see for example the footnote in [4][p.1916] for details).

It remains to be checked that $\mathbb{B}$ contains its image by h or h^{-1} and that $h(J) \cap J = J \cap N_0$. It is enough to show that $H(\tilde{J}) \subset \overline{\Omega}$ and that $H(\tilde{J}) \cap \tilde{J} = \tilde{J} \cap F$. For this last equality just recall that $H(\tilde{J}) \cap \tilde{J} \subset H(\tilde{\beta}) \cap \tilde{\beta} = \tilde{\beta} \cap F$. Observe now that $\tilde{\beta} \cap \tau(\tilde{\beta}) \subset \tilde{\beta} \cap H(\tilde{\beta}) \subset F$ so $\tau^k(\tilde{z}) \in F$ for any $k \in \mathbb{Z}$. We have $H(\tilde{J}) \cap \Omega \neq \emptyset$ since the H-image of a point of $\tilde{J} \cap S$ close to $\tilde{z}$ is a point of $W_r \cap \tau(W_l)$ close to $H(\tilde{z}) = \tilde{z}$ and since Ω is the only connected component of $W_r \cap \tau(W_l)$ having $\tilde{z}$ in its frontier. Finally $H(\tilde{J})$ lies entirely in $\overline{\Omega}$ since otherwise $H(\tilde{J} \setminus \{\tilde{z}, \tau(\tilde{z})\}) \subset \overline{W_r} \cap \tau(\overline{W_l})$ would contain the point $\tau(\tilde{z}) = H(\tau(\tilde{z}))$, a contradiction. This proves the first item when $\tilde{z}$ precedes $\tau(\tilde{z})$ on $\tilde{\beta}$. In the other case, observe that we also have $\tau^{-1}(W_l) \subset H^{-1}(W_l) \subset W_l$ and replace in the above reasoning $\tilde{z}, \tau(\tilde{z}), \tau, H, W_l, W_r$ with respectively $\tau(\tilde{z}), \tilde{z}, \tau^{-1}, H^{-1}, W_r, W_l$.

(2) Let $X = \bigcap_{i \in \mathbb{N}} G^i(\tilde{\beta})$ and $X_n = \bigcap_{i=0}^n G^i(\tilde{\beta})$ $(n \geq 1)$. Remark that $X = \emptyset$, i.e. $X_N = \emptyset$ for a least integer $N \geq 1$. Otherwise G^{-1} induces a map from X to $G^{-1}(X) \subset X$ and this map preserves the order on X naturally provided by an orientation of $\tilde{\beta}$ because $G(\tilde{\beta})$ approaches $\tilde{\beta}$ from only one side. One would deduce that G has a fixed point in X, a contradiction.

If $N = 1$, i.e. if $\tilde{\beta} \cap G(\tilde{\beta}) = \emptyset$, then $H(\tilde{\beta}) \cap \tau(\tilde{\beta}) = \emptyset$ and we simply define $\alpha = \Pi(\tilde{\beta})$. If $N \geq 2$ a suitable modification of $\tilde{\beta}$ allows to bring down the integer N and then to reduce inductively to the easy case $N = 1$. Let us give a few details. One has

(i) $\tau(X_{N-1}) \subset H(W_r) \subset W_r$, i.e. $G(X_{N-1}) \subset W_r \subset H^{-1}(W_r)$,

(ii) $H(X_{N-1}) \subset W_r$,

(iii) $X_{N-1} \cap \mathrm{Bd}(\widetilde{\mathbb{A}}) = \emptyset$.

Property (i) follows from $G(X_{N-1}) \cap \tilde{\beta} = X_N = \emptyset$ and (ii) from $H(X_{N-1}) \cap \tilde{\beta} \subset H(G(\tilde{\beta})) \cap \tilde{\beta} = \tau(\tilde{\beta}) \cap \tilde{\beta} = \emptyset$. For (iii), just recall again that G is fixed point free. Choose an open neighbourhood $U \subset \widetilde{\mathbb{A}} \setminus \mathrm{Bd}(\widetilde{\mathbb{A}})$ of the compact set X_{N-1} in $\widetilde{\mathbb{A}}$ which is so small that (i)-(iii) remain true when one replaces X_{N-1} with U. Consider a finite covering $X_{N-1} \subset \bigcup_{i=1}^n \tilde{\alpha}_i$ where the $\tilde{\alpha}_i$'s are some connected components of $U \cap \tilde{\beta}$. For each $i = 1, \cdots, n$, pick an arc $\tilde{\gamma}_i$ satisfying the following properties:

- $\tilde{\gamma}_i$ lies entirely in $W_l \cap U$ except for its two endpoints points $\tilde{a}_i, \tilde{b}_i$ in $\tilde{\alpha}_i$;
- the arc $\tilde{\beta}_i \subset \tilde{\alpha}_i$ with the same endpoints $\tilde{a}_i, \tilde{b}_i$ as $\tilde{\gamma}_i$ is long enough to have $X_{N-1} \cap \tilde{\alpha}_i = X_{N-1} \cap (\tilde{\beta}_i \setminus \{\tilde{a}_i, \tilde{b}_i\})$.

Moreover these $\tilde{\gamma}_i$'s can be chosen pairwise disjoint hence we get a new arc $\tilde{\gamma}$ crossing $\widetilde{\mathbb{A}}$ by removing from $\tilde{\beta}$ all the $\tilde{\beta}_i$'s and replacing them with the $\tilde{\gamma}_i$'s. Observe that $\tilde{\gamma} \cap X_{N-1} = \emptyset$ by construction and, letting W_r' be the domain on the right of $\tilde{\gamma}$, that $W_r \subset W_r'$. Using $\tilde{\gamma} \setminus \tilde{\beta} \subset W_l \cap U$, it is not difficult to prove that

(a) $\tau(W'_r) \subset H(W'_r) \subset W'_r$,

(b) $\tau(\overline{W'_r}) \subset W'_r$,

(c) $H(\tilde{\gamma}) \cap \tilde{\gamma} = \tilde{\gamma} \cap F$.

Moreover one checks that $\tilde{\gamma} \cap G(\tilde{\gamma}) \subset \tilde{\beta} \cap G(\tilde{\beta})$ hence $\bigcap_{i=0}^{n} G^i(\tilde{\gamma}) \subset X_n$ for every $n \geq 1$. Taking $n = N-1$ we obtain

(d) $\bigcap_{i=0}^{N-1} G^i(\tilde{\gamma}) \subset \tilde{\gamma} \cap X_{N-1} = \emptyset$.

■

5 Relationship with previous works

Let us write p_1 for the projection $\widetilde{\mathbb{A}} \to \mathbb{R}$, $(\theta, r) \mapsto \theta$. If h is a homeomorphism of $\mathbb{A}$ isotopic to the identity and H a lift of h to $\widetilde{\mathbb{A}}$, the *horizontal displacement function* $D_H : \mathbb{A} \to \mathbb{R}$ is the continuous map defined by $D_H(z) = p_1(H(\tilde{z})) - p_1(\tilde{z})$ where $z \in \mathbb{A}$ and $\tilde{z}$ is any point in $\Pi^{-1}(\{z\})$.

5.1 Link with Franks' twist assumption

The next result is Theorem 3.3 of [9] (see also [12]).

Theorem 5.1 *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity and with every point non wandering, and let $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h. If there exist points $\tilde{x}, \tilde{y} \in \widetilde{\mathbb{A}}$ such that*

$$\liminf_{n \to +\infty} \frac{p_1(H^n(\tilde{y})) - p_1(\tilde{y})}{n} < 0 < \limsup_{n \to +\infty} \frac{p_1(H^n(\tilde{x})) - p_1(\tilde{x})}{n}$$

then h has at least two fixed points.

This roughly means that, for a conservative homeomorphism h of $\mathbb{A}$, the presence of points turning clockwise and counterclockwise with linear speed under forward iteration guarantees the existence of two fixed points. The following consequence of Theorems 3.1-3.2 drops the assumption on the speed of the points turning in $\mathbb{A}$ (see also Question 5.4 below). It also relax the generalized conservative assumption in Theorem 5.1 since if $\mathbb{B} \subset \mathbb{A}$ is a subannulus such that $h(\mathbb{B}) \subsetneq \mathbb{B}$ then $\mathrm{Int}_{\mathbb{A}}(\mathbb{B}) \setminus h(\mathbb{B})$ is a nonempty wandering open set for h (similarly if $h^{-1}(\mathbb{B}) \subsetneq \mathbb{B}$).

Theorem 5.2 *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity such that there is no essential subannulus $\mathbb{B} \subset \mathbb{A}$ containing $h(\mathbb{B})$ or $h^{-1}(\mathbb{B})$ as a proper subset, and let $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h. If there exist points $\tilde{x}, \tilde{y} \in \widetilde{\mathbb{A}}$ such that*

$$\liminf_{n \to +\infty} p_1(H^n(\tilde{y})) = -\infty \quad \textit{and} \quad \limsup_{n \to +\infty} p_1(H^n(\tilde{x})) = +\infty$$

then h has at least two fixed points; more precisely the Nielsen class $\Pi(\mathrm{Fix}(H))$ contains at least two points.

Proof of Theorem 5.2. Let us check separately the first assertion. Suppose that h has at most one fixed point. Consider an arc α given by Theorem 3.1 and a connected component $\tilde{\alpha}$ of $\Pi^{-1}(\alpha)$. Thus $\tilde{\alpha}$ is an arc crossing $\widetilde{\mathbb{A}}$ and it has no point of transverse intersection with $H(\tilde{\alpha})$ because the same is true for α and $h(\alpha)$. It follows that one of the two connected components of $\widetilde{\mathbb{A}} \setminus \tilde{\alpha}$, call it W, contains its image by H. Consequently either every $\tilde{z} \in W$ has its forward H-orbit bounded on the right or every $\tilde{z} \in W$ has its forward H-orbit bounded on the left. Since any $\tilde{z} \in \widetilde{\mathbb{A}}$ has a translate $\tau^n(\tilde{z})$ in W and since $H \circ \tau = \tau \circ H$, the last sentence remains true if one replaces "$\tilde{z} \in W$" with "$\tilde{z} \in \widetilde{\mathbb{A}}$". The first part of the theorem is proved. Suppose now that the Nielsen class $N_H = \Pi(\mathrm{Fix}(H))$ contains at most one point. Given a positive integer n, we define

$\breve{\mathbb{A}} = \mathbb{R}/n\mathbb{Z} \times [-1,1]$ and we consider the two natural covering maps $\breve{\Pi} : \widetilde{\mathbb{A}} \to \breve{\mathbb{A}}$ and $q : \breve{\mathbb{A}} \to \mathbb{A}$. Of course $\breve{\mathbb{A}}$ is homeomorphic to $\mathbb{A}$ with boundary components $\mathrm{Bd}^{\pm}(\breve{\mathbb{A}}) = \breve{\Pi}(\mathrm{Bd}^{\pm}(\widetilde{\mathbb{A}})) = q^{-1}(\mathrm{Bd}^{\pm}(\mathbb{A}))$ and we write $\breve{h}, \breve{\tau}$ for the homeomorphisms of $\breve{\mathbb{A}}$ induced by respectively H, τ. Thus $\breve{h}$ is lifted by H and $\breve{h}$ is a lift of h, that means $\breve{h} \circ \breve{\Pi} = \breve{\Pi} \circ H$ and $h \circ q = q \circ \breve{h}$. Moreover one can choose n so large that $\breve{\Pi}(\mathrm{Fix}(H)) = \mathrm{Fix}(\breve{h})$ because the map $\tilde{z} \mapsto p_1(H(\tilde{z})) - p_1(\tilde{z})$ ($\tilde{z} \in \widetilde{\mathbb{A}}$) is bounded. It is well known that if a Nielsen class of h has finitely many fixed points then the sum of their Lefschetz indexes equals $\chi(\mathbb{A}) = 0$ so N_H is either empty or contains a single point z_0 with Lefschetz index $\mathrm{Ind}_h(z_0) = 0$ [2]. Consequently $\mathrm{Fix}(\breve{h})$ consists in a single Nielsen class of $\breve{h}$ which is either empty or contains exactly n points; in the latter case, these n fixed points have index 0 w.r.t. $\breve{h}$ and they all lie either in the same boundary component of $\breve{\mathbb{A}}$ or in $\breve{\mathbb{A}} \setminus \mathrm{Bd}(\breve{\mathbb{A}})$. Hence we can apply Theorem 3.2 to $\breve{h} : \breve{\mathbb{A}} \to \breve{\mathbb{A}}$. If (2') occurs we get in particular an arc $\breve{\alpha}$ crossing $\breve{\mathbb{A}}$ which has no point of transverse intersection with $\breve{h}(\breve{\alpha})$ and we conclude as above by lifting $\breve{\alpha}$ to $\widetilde{\mathbb{A}}$. We end by showing that the existence of a subannulus $\breve{\mathbb{B}} \subset \breve{\mathbb{A}}$ given by (1') contradicts our hypotheses. We assume that $\breve{h}(\breve{\mathbb{B}}) \subsetneq \breve{\mathbb{B}}$ and that the boundary of $\breve{\mathbb{B}}$ is the union of $\mathrm{Bd}^-(\breve{\mathbb{A}})$ together with a Jordan curve $\breve{J} \subset \breve{\mathbb{A}} \setminus \mathrm{Bd}^-(\breve{\mathbb{A}})$, the other cases being similar. We consider the annulus $\breve{\mathbb{A}}' = \mathbb{R}/n\mathbb{Z} \times [-1,2]$ and we let U be the connected component of $\breve{\mathbb{A}}' \setminus \bigcup_{0 \le i \le n-1} \breve{\tau}^i(\breve{\mathbb{B}})$ containing the upper boundary $\mathbb{R}/n\mathbb{Z} \times \{2\}$. According to Kerékjártó's result mentioned in the proof of Lemma 4.5, the set $\partial_{\breve{\mathbb{A}}'} U$ is a Jordan curve contained in $\bigcup_{0 \le i \le n-1} \breve{\tau}^i(\breve{J}) \subset \breve{\mathbb{A}} \setminus \mathrm{Bd}^-(\breve{\mathbb{A}})$. For any $\breve{z} \in \breve{J} \setminus \mathrm{Fix}(\breve{h})$ and any $i = 0, \cdots, n-1$ one has $\breve{h} \circ \breve{\tau}^i(\breve{z}) = \breve{\tau}^i \circ \breve{h}(\breve{z}) \in \breve{\tau}^i(\mathrm{Int}_{\breve{\mathbb{A}}}(\breve{\mathbb{B}}))$. Moreover $\bigcup_{i=0}^{n-1} \breve{\tau}^i(\mathrm{Int}_{\breve{\mathbb{A}}}(\breve{\mathbb{B}}))$ is connected, disjoint from $\partial_{\breve{\mathbb{A}}'} U$ and contains $\mathrm{Bd}^-(\breve{\mathbb{A}})$ hence $\breve{h}(\partial_{\breve{\mathbb{A}}'} U \setminus \mathrm{Fix}(\breve{h})) \subset \breve{\mathbb{A}}' \setminus \mathrm{Cl}_{\breve{\mathbb{A}}'}(U) \subset \breve{\mathbb{A}}$. Finally $\breve{\tau}(\partial_{\breve{\mathbb{A}}'} U) = \partial_{\breve{\mathbb{A}}'} U$ so $J = q(\partial_{\breve{\mathbb{A}}'} U)$ is an essential Jordan curve in $\mathbb{A} \setminus \mathrm{Bd}^-(\mathbb{A})$. Letting $\mathbb{B}$ be the subannulus of $\mathbb{A}$ bounded by $J \cup \mathrm{Bd}^-(\mathbb{A})$, one deduces that $h(\mathbb{B}) \subsetneq \mathbb{B}$, a contradiction. ■

Remark 5.3
- *The same argument could be used with [13][Théorème 5.1] to find one fixed point of h under the assumptions of Theorem 5.2. Another result by Franks ([10][Corollary 2.3]) with a somewhat different generalized conservative assumption (h is assumed to be chain transitive) also ensures the existence of one fixed point for H if there exist $\tilde{x}, \tilde{y}$ as in Theorem 5.2.*
- *Observe that in the proof of Theorem 5.2 it was enough to find an arc $\tilde{\alpha}$ crossing $\widetilde{\mathbb{A}}$ which has no point of transverse intersection with $H(\tilde{\alpha})$, even if it does not project onto an arc of $\mathbb{A}$. In contrast, the full properties of α are needed for Theorem 5.7 below.*

Question 5.4 *I do not know an example of conservative homeomorphism $h : \mathbb{A} \to \mathbb{A}$ isotopic to the identity with a lift H satisfying the generalized twist property of Theorem 5.2 and not the one in Theorem 5.1, even if the word conservative is understood in a generalized sense. Such an example is easily constructed if one drops the conservative assumption. One can formulate this problem by using the rotation set of H. We recall this notion is a well-known adaptation of a similar concept introduced by M. Misiurewicz and K. Ziemian for maps of tori ([22]). Given a lift $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of h, the rotation set $\rho(H) \subset \mathbb{R}$ may be defined by deciding that $r \in \rho(H)$ if there exist a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers and a sequence $(\tilde{z}_k)_{k\in\mathbb{N}}$ of points of $\widetilde{\mathbb{A}}$ such that*

$$\lim_{k\to+\infty} n_k = +\infty \text{ and } r = \lim_{k\to+\infty} \frac{p_1(H^{n_k}(\tilde{z}_k)) - p_1(\tilde{z}_k)}{n_k}.$$

It is a compact interval, say $\rho(H) = [a,b]$ with possibly $a = b$. Equivalently, one has $r \in \rho(H)$ if $r = \int_{\mathbb{A}} D_H(z) d\mu(z)$ for some h-invariant Borel probability measure μ on $\mathbb{A}$. It follows that the endpoints of $\rho(H)$ are the rotation number of some points, that means

$$a = \lim_{n\to+\infty} \frac{p_1(H^n(\tilde{z})) - p_1(\tilde{z})}{n}, \quad b = \lim_{n\to+\infty} \frac{p_1(H^n(\tilde{z}')) - p_1(\tilde{z}')}{n}$$

[2] A direct argument is the following. Pick $\tilde{z}_0 \in \Pi^{-1}(\{z_0\})$. We first assume that $z_0 \notin \mathrm{Bd}(\mathbb{A})$ and we choose a rectangle $R = [a, a+1] \times [-1,1]$ containing $\tilde{z}_0$ in its interior. We have then $R \cap \mathrm{Fix}(H) = \{\tilde{z}_0\}$ and it is easily seen that the index of the Jordan curve $J = \partial_{\mathbb{R}^2} R$ w.r.t H is equal to 0. It follows that $0 = \mathrm{Ind}_H(\tilde{z}_0) = \mathrm{Ind}_h(z_0)$. If z_0 lies on the boundary of $\mathbb{A}$, say $z_0 \in \mathrm{Bd}^+(\mathbb{A})$, extend H to a homeomorphism H_* of the "double strip" $\mathbb{R} \times [-1,3]$ by letting $H_* = S \circ H \circ S$ on $\mathbb{R} \times [1,3]$ where $S(\theta, r) = (\theta, 2-r)$. One gets as above $0 = \mathrm{Ind}_{H_*}(\tilde{z}_0) = 2\mathrm{Ind}_h(z_0)$.

for some $\tilde{z}, \tilde{z}' \in \widetilde{\mathbb{A}}$. Hence there exist $\tilde{x}, \tilde{y}$ as in Theorem 5.1 if and only if $\rho(H)$ contains 0 in its interior. Moreover one checks that if $0 \notin \rho(H)$ then all the forward orbits of H go to the same end of $\widetilde{\mathbb{A}}$ hence our initial question is essentially the same as the following. Can a conservative homeomorphism h of $\mathbb{A}$ be lifted by $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ having a forward orbit unbounded on the left and such that $\rho(H) = [0, b]$, $b \geq 0$?

5.2 Link with Conley-Zehnder theorem in the annulus

We write λ for the Lebesgue measure on $\mathbb{A}$ and $\tilde{\lambda}$ for the lift of λ to $\widetilde{\mathbb{A}}$. Recall that the measure $\tilde{\lambda}$ is invariant by τ and that $\tilde{\lambda}(X) = \lambda(\Pi(X))$ for any Borelian set X contained in a fundamental domain of Π. If h is supposed to preserve λ then $\tilde{\lambda}$ is also invariant by any lift $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ of h. We first state a slight complement to Theorem 3.2.

Proposition 5.5 *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity. We suppose that h satisfies the assumptions (i)-(iii) in Theorem 3.2 (in particular this holds if h has at most one fixed point) and furthermore that h preserves the measure λ. Then the alternative (2') in Theorem 3.2 occurs and (1') does not; moreover the arc α in (2') can be chosen such that $\lambda(\alpha) = 0$.*

Proof of Proposition 5.5. There is no subannulus $\mathbb{B} \subset \mathbb{A}$ as in Theorem 3.2 since otherwise h would have a nonempty wandering open set, contradicting $\lambda(\mathbb{A}) < \infty$. Let us remark now that our constructions can be improved in order to get $\lambda(\alpha) = 0$ in Theorem 3.2. First, the brick decomposition $\widetilde{\mathcal{D}}_H = \{B_i\}_{i \in I}$ given by Lemma 4.2 can be chosen in such a way $\tilde{\lambda}(\bigcup_{i \in I} \partial B_i) = 0$; it suffices for example to deal only with bricks decompositions of S having polygonal bricks. Using now the notation from Lemma 4.5 and its proof, observe that the arc $\tilde{\beta}$ is a subset of $\bigcup_{n \in \mathbb{N}} G^n(\tilde{\alpha})$ where $\tilde{\alpha}$ is itself a subset of $(\bigcup_{i \in I} \partial B_i) \cup F$. Since F is either empty or a countable set and since G preserves the measure $\tilde{\lambda}$ one obtains $\tilde{\lambda}(\tilde{\beta}) = 0$. Keeping finally the notation from Lemma 4.6 (2) and its proof, it remains to see that the modification of $\tilde{\beta}$ into $\tilde{\gamma}$ (if needed, i.e. if $N \geq 2$) can be performed in such a way that $\tilde{\lambda}(\tilde{\gamma}) = 0$. It is enough to observe that the arcs $\tilde{\gamma}_i$, $i = 1, \cdots, n$, can be chosen such that $\tilde{\lambda}(\tilde{\gamma}_i) = 0$ because, for a given i, there are uncountably many pairwise disjoint arcs $\tilde{\gamma}_i$ with the required properties while only countably many of them can have positive $\tilde{\lambda}$-measure (as an alternative argument, one can get $\tilde{\lambda}(\tilde{\gamma}_i) = 0$ by constructing $\tilde{\gamma}_i$ piecewise linear; this is possible by using that densely many points in $\tilde{\alpha}_i$ are accessible from W_l by a straight segment). After $N - 1$ such modifications one reduces to the case $\tilde{\beta} \cap G(\tilde{\beta}) = \emptyset$ with $\tilde{\lambda}(\tilde{\beta}) = 0$, so $\alpha = \Pi(\tilde{\beta})$ is an arc as described in Theorem 3.2 with furthermore $\lambda(\alpha) = 0$. ■

The next lemma gives an interesting interpretation for the mean horizontal displacement of H. This seems to be well known although I found it explicitly only in [1].

Lemma 5.6 *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity, preserving the measure λ, and let $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be a lift of h. Consider an arc α crossing $\mathbb{A}$ such that $\lambda(\alpha) = 0$ and a connected component $\tilde{\alpha}$ of $\Pi^{-1}(\alpha)$. Then the mean horizontal displacement $\int_{\mathbb{A}} D_H(z)\, d\lambda(z)$ is equal to the algebraic area (w.r.t the measure $\tilde{\lambda}$) between $\tilde{\alpha}$ and its image $H(\tilde{\alpha})$.*

Under the assumptions of the above lemma, recall from Birkhoff's ergodic theorem that λ-almost every $z \in \mathbb{A}$ has a rotation number $\rho_H(z) \in \mathbb{R}$ defined by $\rho_H(z) = \lim_{n \to +\infty} \frac{1}{n}\big(p_1(H^n(\tilde{z})) - p_1(\tilde{z})\big)$ for any $\tilde{z} \in \Pi^{-1}(\{z\})$ and that the integrable map $z \mapsto \rho_H(z)$ satisfies $\int_{\mathbb{A}} \rho_H(z)\, d\lambda(z) = \int_{\mathbb{A}} D_H(z)\, d\lambda(z)$. Hence Lemma 5.6 is given by Proposition 5.2 of [1] when the arc α is the projection of a vertical segment $\{\theta\} \times [-1, 1] \subset \widetilde{\mathbb{A}}$ (the boundedness assumption on D_H appearing in [1] is needed only when working in the open annulus and is always satisfied in our compact framework). To obtain exactly the above statement, use the Oxtoby-Ulam theorem on homeomorphic measures ([23]) to construct a homeomorphism $g : \mathbb{A} \to \mathbb{A}$ isotopic to the identity, preserving the measure λ and mapping α onto a radial segment $\alpha' = \{e^{2i\pi\theta}\} \times [-1, 1]$. Let $G : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ be the lift of g mapping $\tilde{\alpha}$ onto $\tilde{\alpha}' = \{\theta\} \times [-1, 1]$ and define $H' = G \circ H \circ G^{-1}$ which is a lift of $h' = g \circ h \circ g^{-1}$.

Of course the algebraic area between $\tilde{\alpha}$ and $H(\tilde{\alpha})$ is the same as the one between $G(\tilde{\alpha}) = \tilde{\alpha}'$ and $G(H(\tilde{\alpha})) = H'(\tilde{\alpha}')$. One concludes because

$$\int_{\mathbb{A}} D_H(z)\, d\lambda(z) = \int_{\mathbb{A}} D_{H'}(z)\, d\lambda(z).$$

The next result is essentially Theorem 2 in Flucher's paper [8] restricted to the case of the Lebesgue measure.

Theorem 5.7 *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism isotopic to the identity and preserving the measure λ. If h admits a lift $H : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ with vanishing mean horizontal displacement, i.e. with $\int_{\mathbb{A}} D_H(z)\, d\lambda(z) = 0$, then h possesses at least two fixed points; more precisely the Nielsen class $\Pi(\mathrm{Fix}(H))$ contains at least two points.*

Proof of Theorem 5.7. Suppose first that h has at most one fixed point. Consider an arc α given by Proposition 5.5 and a connected component $\tilde{\alpha}$ of $\Pi^{-1}(\alpha)$. For any lift H of h, the arc $H(\tilde{\alpha})$ lies entirely in a connected component of $\widetilde{\mathbb{A}} \setminus \tilde{\alpha}$ except possibly for one point in $\tilde{\alpha}$. The algebraic area between and $\tilde{\alpha}$ and $H(\tilde{\alpha})$ is then nonzero and so is $\int_{\mathbb{A}} D_H(z)\, d\lambda(z)$ by Lemma 5.6. We suppose now that H is a lift of h such that $\Pi(\mathrm{Fix}(H))$ contains at most one fixed point. We consider a n-folded covering $\breve{\mathbb{A}}$ of $\mathbb{A}$ and $\breve{h} : \breve{\mathbb{A}} \to \breve{\mathbb{A}}$ as in the proof of Theorem 5.2. The homeomorphism $\breve{h}$ preserves the measure $\breve{\lambda}$ obtained by lifting λ to the annulus $\breve{\mathbb{A}}$. Observe that Proposition 5.5 applies with $\breve{h}, \breve{\lambda}$ instead of h, λ (with the same proof since $H, \tilde{\lambda}$ are also the lifts of $\breve{h}, \breve{\lambda}$) and consider an arc $\breve{\alpha} \subset \breve{\mathbb{A}}$ obtained in this way. Defining the horizontal displacement function on $\breve{\mathbb{A}}$ by $\breve{D}_H(\breve{z}) = \frac{1}{n}\big(p_1(H(\tilde{z})) - p_1(\tilde{z})\big)$ (where $\breve{z} \in \breve{\mathbb{A}}$ and $\tilde{z} \in \breve{\Pi}^{-1}(\{\breve{z}\})$) one checks that Lemma 5.6 is still valid with $\breve{h}, \breve{\alpha}, \breve{\lambda}, \breve{D}_H$ instead of h, α, λ, D_H and we get as above

$$0 \neq \int_{\breve{\mathbb{A}}} \breve{D}_H(\breve{z})\, d\breve{\lambda}(\breve{z}) = \int_{\mathbb{A}} D_H(z)\, d\lambda(z).$$

■

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