Lefschetz index for orientation reversing planar homeomorphisms.

Marc Bonino
Université Paris 13, Institut Galilée, Département de Mathématiques
Avenue J.B. Clément 93430 Villetaneuse (France)
e-mail: : bonino@math.univ-paris13.fr

ABSTRACT: we prove that an isolated fixed point of an orientation reversing homeomorphism of the plane always has Lefschetz index 0 or ± 1.

1 Introduction

If U is an open subset of $\mathbb{R}^2$ and $p \in U$ an isolated fixed point of a continuous map $h : U \to \mathbb{R}^2$, the Lefschetz index of h at p, denoted by $\mathrm{Ind}(h, p)$, is the winding number of the vector field $h(z) - z$ on any simple closed curve surrounding p and close enough to p. There is plentiful literature where this fixed-point index plays an important role, for example to detect other fixed points, or to obtain local and/or global dynamical properties for surfaces homeomorphisms.

Throughout this paper, the map h is a (local) homeomorphism and we will focus on the set of all the possible values for the Lefschetz index. It is well known that, for every integer $n \in \mathbb{Z}$, there exists an orientation preserving planar homeomorphism h_n which has the origin o as an isolated fixed point and such that $\mathrm{Ind}(h_n, o) = n$ (see for example [1]). Surprisingly, M. Brown announced in the same paper that for an orientation reversing homeomorphism of the plane, the Lefschetz index is one of the three values -1,0 or +1, but no proof of this result has been given until now. This motivated the writing of this paper, where we will show precisely the following:

Theorem: *Let V, W be two connected open subsets of $\mathbb{R}^2$ containing the origin o and $h : V \to W = h(V)$ be an orientation reversing homeomorphism which possesses o as an isolated fixed point.*

Then $\mathrm{Ind}(h, o) \in \{-1, 0, +1\}$.

[0] 2000 MSC: 55M20 (54H20)

2 Index on a Jordan curve

For completeness, we recall briefly in this section some classical results and definitions.

If C is a Jordan curve (i.e. a simple closed curve in $\mathbb{R}^2$, i.e. a subset of $\mathbb{R}^2$ homeomorphic to the unit circle $\mathbb{S}^1$), then $\mathbb{R}^2 \setminus C$ has exactly two connected components and C is their common frontier. The bounded one (resp. the unbounded one) is named *the interior domain* (resp. *the exterior domain*) of C and is denoted by $\mathrm{int}(C)$ (resp. by $\mathrm{ext}(C)$). A subset of $\mathbb{R}^2$ which is the interior domain of a Jordan curve is said to be *a Jordan domain*. In this paper, a Jordan curve C is always counter-clockwise oriented. This yields an ordering relation (defined up to circular permutation) on C. If x, y are two points on C, $[x, y]_C$ (resp. $(x, y)_C$) denotes the closed arc (resp. the open arc) on C from x to y for this orientation of C.

Let $u : \mathbb{S}^1 \to C$ be a homeomorphism which endows the Jordan curve $C = u(\mathbb{S}^1)$ with its counter-clockwise orientation. If X is a subset of $\mathbb{R}^2$ containing C and $f : X \to \mathbb{R}^2$ a continuous map without fixed point on C, the degree of the map $z \mapsto (f(u(z)) - u(z))/\|f(u(z)) - u(z)\|$ $(z \in \mathbb{S}^1)$ does not depend on the choice of u. It is named *the index of f on C* and is denoted by $\mathrm{Ind}(f, C)$. Now, if X is open and $p \in X$ is an isolated fixed point of f, choose a disk neighborhood D of p, so small that $D \subset X$ and f has no other fixed point in D. Then the index of f is the same on any Jordan curve C lying in D and such that $p \in \mathrm{int}(C)$, and this common value defines *the Lefschetz index* $\mathrm{Ind}(f, p)$ of f at the point p.

3 Proof of the Theorem

In the following, we will write respectively $\overline{X}$ and ∂X for the closure and the frontier of a set $X \subset \mathbb{R}^2$.

Choose once and for all a circle C around o, so that the closed disk $D = \overline{\mathrm{int}(C)}$ is contained in $V \cap W$ and o is the only fixed point of h in D. First, observe there are two cases where the index can be easily computed:

• if $h(D) \subset D$ then $\mathrm{Ind}(h, o) = 1$,

• if $D \subset h(D)$ then, since h reverses the orientation, $\mathrm{Ind}(h, o) = -\mathrm{Ind}(h^{-1}, o)$ $= -1$.

From now on, we exclude these two simple situations, that is we suppose $h(\mathrm{int}(C)) \not\subset \mathrm{int}(C)$ and $\mathrm{int}(C) \not\subset h(\mathrm{int}(C))$.

It is well known since Kerékjártó that every connected component of the intersection of two Jordan domains is again a Jordan domain. We need a more precise description, which can be stated as follows. All the assertions in Proposition 3.1 and their proofs are contained in the first section of [2].

Proposition 3.1 *Let U,U' be two Jordan domains containing the origin o, such that $U \not\subset U'$ and $U' \not\subset U$. Denote by $U \wedge U'$ the connected component* *of $U \cap U'$ which contains o and by $\partial U \wedge \partial U'$ the frontier of $U \wedge U'$.*

(1) We have a partition

$$(P) \quad \partial U \wedge \partial U' = ((\partial U \wedge \partial U') \cap \partial U \cap \partial U') \bigcup_{i \in I} \alpha_i \bigcup_{j \in J} \beta_j$$

where

- *I, J are non-empty and at most countable sets,*
- *for every $i \in I$, $\alpha_i = (a_i, b_i)_{\partial U}$ is a connected component of $\partial U \cap U'$,*
- *for every $j \in J$, $\beta_j = (c_j, d_j)_{\partial U'}$ is a connected component of $\partial U' \cap U$.*

(2) for every $j \in J$, $U \wedge U'$ is contained in the Jordan domain bounded by $\beta_j \cup [d_j, c_j]_{\partial U}$.

(3) $\partial U \wedge \partial U'$ is homeomorphic to ∂U. In particular, it is a Jordan curve.

(4) Three points a, b, c of $(\partial U \wedge \partial U') \cap \partial U$ (resp. of $(\partial U \wedge \partial U') \cap \partial U'$) are met in this order on ∂U (resp on $\partial U'$) if and only if they are met in the same order on $\partial U \wedge \partial U'$.

Let us consider the partition (P_h) and the arcs α_i and β_j that we obtain when we apply Proposition 3.1 with $U = \text{int}(C)$ and $U' = \text{int}(h^{-1}(C)) = h^{-1}(U)$.

Let ϕ be the inversion in the circle C and $\Gamma = C \wedge h^{-1}(C)$ the Jordan curve given by Proposition 3.1. Since ϕ is a homeomorphism of $\mathbb{R}^2 \setminus \{o\}$, $\phi(\Gamma)$ is also a Jordan curve.

Now, define a map H from $\Gamma \cup \phi(\Gamma)$ to $\mathbb{R}^2$ by setting

$$H(z) = \begin{cases} h(z) & \text{if } z \in \Gamma \\ h(\phi(z)) & \text{if } z \in \phi(\Gamma) \end{cases}$$

Clearly, H is well-defined, continuous and fixed-point free. Since $H(\phi(\Gamma)) = h(\Gamma) \subset D \subset \overline{\text{int}(\phi(\Gamma))}$, we have Ind$(H, \phi(\Gamma))$= 1. We have also $H = h$ on Γ and consequently Ind(H, Γ)= Ind(h, Γ) = Ind(h, o).

Observe that, for every $j \in J$, the set $C_j = \overline{\beta_j} \cup \phi(\overline{\beta_j})$ is a Jordan curve such that $(c_j, d_j)_C \subset \text{int}(C_j)$ and $(d_j, c_j)_C \subset \text{ext}(C_j)$. The orientations induced on β_j by C_j and Γ are opposite (see Figure 1). Furthermore, the β_j are pairwise disjoint open arcs in $h^{-1}(C)$ and then, for a given $\epsilon > 0$, there is only a finite number of indexes $j \in J$ such that the diameter of β_j is superior to ϵ. It follows that there are only finitely many indexes $j \in J$ such that Ind$(H, C_j) \neq 0$.

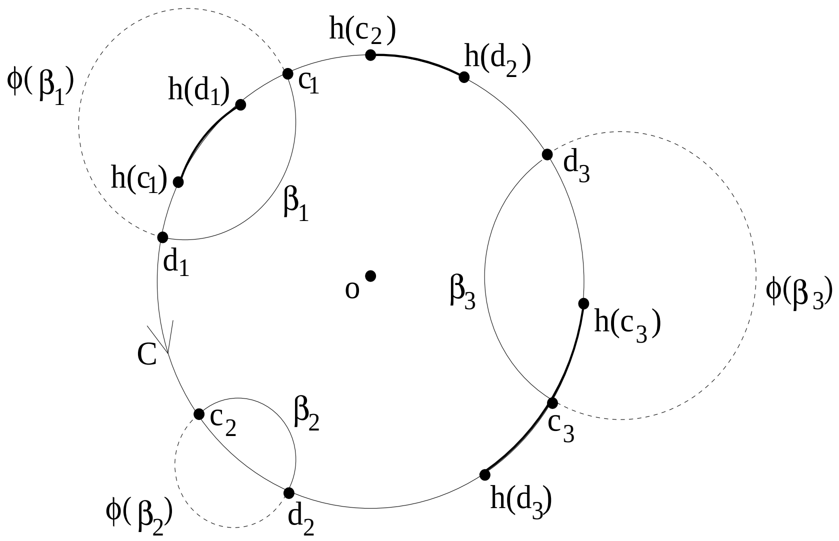

Figure 1: The curves Γ and $\phi(\Gamma)$

Bringing together the above remarks , we obtain the formula:

$$(*) \qquad 1 = \mathrm{Ind}(H, \phi(\Gamma)) = \mathrm{Ind}(h, 0) + \sum_{j \in J} \mathrm{Ind}(H, C_j).$$

Keeping in mind that, for every $j \in J$, we have $H(C_j) = h(\overline{\beta_j}) = [h(d_j), h(c_j)]_C$, we now state three basic lemmas:

Lemma 3.2 *If $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C = \emptyset$, then Ind$(H, C_j)$ = 0.*

Proof: It is enough to remark that $H(C_j) = [h(d_j), h(c_j)]_C \subset (d_j, c_j)_C \subset \mathrm{ext}(C_j)$.

Lemma 3.3 *If the set $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$ is non empty and connected, then Ind(H, C_j) = 1.*

Proof: This is clear if $[h(d_j), h(c_j)]_C \subset [c_j, d_j]_C$ because we have then $H(C_j) \subset \overline{\mathrm{int}(C_j)}$. The following argument allows us to be reduced to this case; Denote $[z_1, z_2]_C = [h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$. Let $(r_t)_{0 \le t \le 1}$ be a strong retracting deformation of the arc $[h(d_j), h(c_j)]_C$ onto its subarc $[z_1, z_2]_C$. The maps $r_t \circ H|_{C_j} : C_j \to \mathbb{R}^2$ $(0 \le t \le 1)$ are fixed-point free because $C_j \cap r_t(H(C_j)) \subset \{c_j, d_j\}$ and, if $H(c_j) = h(c_j) \notin [z_1, z_2]_C$ (resp. if $H(d_j) = h(d_j) \notin [z_1, z_2]_C$) then $[z_2, h(c_j)]_C = [d_j, h(c_j)]_C$ does not contain the point c_j (resp. $[h(d_j), z_1]_C = [h(d_j), c_j]_C$ does not contain the point d_j). Since the map $t \mapsto \mathrm{Ind}(r_t \circ H|_{C_j}, C_j)$ is continuous and $r_1(H(C_j)) = [z_1, z_2]_C \subset [c_j, d_j]_C \subset \overline{\mathrm{int}(C_j)}$, we obtain $\mathrm{Ind}(H, C_j) = \mathrm{Ind}(r_1 \circ H|_{C_j}, C_j)$ =1.

Lemma 3.4 *There are at most two indexes $j \in J$ such that $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$ is non empty.*

Proof: Suppose that we can find three distinct arcs β_{j_k} ($k \in \{1,2,3\}$) among the β_j such that $[h(d_{j_k}), h(c_{j_k})]_C \cap [c_{j_k}, d_{j_k}]_C \neq \emptyset$. Since h has no fixed point on C, it is the same to write $(h(d_{j_k}), h(c_{j_k}))_C \cap (c_{j_k}, d_{j_k})_C \neq \emptyset$ and then one can choose a point $x_k \in \beta j_k$ such that $h(x_k) \in (c_{j_k}, d_{j_k})_C$. Renaming the indexes j_k if necessary, one can suppose that $\beta_{j_1}, \beta_{j_2}, \beta_{j_3}$ are met in this order on $h^{-1}(C)$.

On one hand, since $x_k \in \beta j_k$ and h reverses the orientation, we must have $h(x_3), h(x_2), h(x_1)$ in this order on C. On the other hand, we know from Proposition 3.1 (4) that the points $c_{j_1}, d_{j_1}, c_{j_2}, d_{j_2}, c_{j_3}, d_{j_3}$ are met in this order on $h^{-1}(C)$, on Γ and on C. With $h(x_k) \in (c_{j_k}, d_{j_k})_C$, we obtain now that $h(x_1), h(x_2), h(x_3)$ are in this order on C. This is not possible because the points $h(x_k)$ ($k \in \{1,2,3\}$) are pairwise distinct.

We can now complete the proof of the Theorem: If, for every $j \in J$, the set $[h(d_j), h(c_j)]_C \cap [c_j, d_j]_C$ is either empty or connected, the result is an obvious consequence of formula $(*)$ and of Lemmas (3.2)-(3.4).

Otherwise, there exists an arc $\beta = (c,d)_{h^{-1}(C)}$ among the β_j such that $c, h(c), h(d), d$ are met in this order on C. According to Proposition 3.1 (2) and (4), the points d, a_i, b_i, c are met in this order on C and on $h^{-1}(C)$ for every $i \in I$. It follows that $c, h(c), h(b_i), h(a_i), h(d), d$ are in this order on C. Thus, we obtain $\overline{\alpha_i} \cap [h(b_i), h(a_i)]_C = \emptyset$. For convenience, let us define now $g = h^{-1}$. Of course, one can use Proposition 3.1 with $U = \mathrm{int}(C)$ and $U' = \mathrm{int}(g^{-1}(C))$ and then obtain a corresponding partition (P_g) for $C \wedge g^{-1}(C)$. But $C \wedge g^{-1}(C)$ is nothing but $h(C \wedge h^{-1}(C))$ and therefore the partition (P_g) can be written using the arcs α_i and β_j of (P_h). Precisely:

$$(P_g) \quad C \wedge g^{-1}(C) = \Big((C \wedge g^{-1}(C)) \cap C \cap g^{-1}(C) \Big) \bigcup_{j \in J} h(\beta_j) \bigcup_{i \in I} h(\alpha_i)$$

Observe that the arcs $h(\alpha_i) = (h(b_i), h(a_i))_{g^{-1}(C)}$ play the same role in (P_g) as the arcs β_j in (P_h). Since $\overline{\alpha_i} = g(h(\overline{\alpha_i}))$ is disjoint from $[h(b_i), h(a_i)]_C$ for every $i \in I$, we see that we are reduced to the previous situation if we replace h by g. Then we obtain $\mathrm{Ind}(h,o) = -\mathrm{Ind}(g,o) = -1$. The Theorem is proved.

Acknowledgements: I would like to thank P. Le Calvez, who improved a preliminary version of this paper.

References

[1] M. Brown, *On the fixed points index of iterates of planar homeomorphisms*, Proc. AMS **108** (1990), 1109-1114.

[2] P. Le Calvez and J.C. Yoccoz, *Un théorème d'indice pour les homéomorphismes du plan au voisinage d'un point fixe*, Annals of Mathematics **146** (1997), 241-293.