A dynamical property for planar homeomorphisms and an application to the problem of canonical position around an isolated fixed point

Marc BONINO

1 Introduction

The first result of this paper (Theorem 2.1) is a fixed point theorem for planar homeomorphisms and describes a situation where we can answer the following question: given such a homeomorphism h and a compact disc $D \subset \mathbb{R}^2$ disjoint from the fixed point set $\mathrm{Fix}(h)$, does this disc contain the whole positive orbit $\mathcal{O}_+ = \{h^n(x) \mid n \in \mathbb{N}\}$ of a point $x \in D$? The proof depends mainly on Bell's Theorem ([Be]), which asserts that h has a fixed point in every non-separating invariant continuum of the plane.

Section 3 provides an application of Theorem 2.1 to the study of an orientation preserving homeomorphism h of $\mathbb{R}^2$ near an isolated fixed point p. We will show that, up to small and compactly supported perturbations (which do not alter the fixed point set $\mathrm{Fix}(h)$), one can suppose that h is conjugated on a suitable circle around p with a "canonical homeomorphism" which depends only on the Lefschetz index $\mathrm{ind}(h, p)$ of h at the point p (Theorem 3.4). As a matter of fact, this lemma, which has several applications (see Section 3), is asserted in a paper of Schmitt ([Sc]) but the proof is there very difficult to follow and seems to contain a gap. Another proof, due to Slaminka ([Sl]), strongly uses the fact that every point $x \notin \mathrm{Fix}(h)$ is a wandering point. In particular, it does not apply when $\mathrm{ind}(h, p) = 1$. We will see that our arguments, in contrast with the earlier ones, are valid for every value of the fixed point index. Then we will be able to confirm Schmitt's Theorem: for every $n \in \mathbb{Z}$, the space $\mathcal{H}_n$ of all orientation preserving homeomorphisms h of $\mathbb{R}^2$ such that the origin o is the only fixed point and $\mathrm{ind}(h, o) = n$, endowed with the compact-open topology, is path connected (Theorem 3.7).

2 A dynamical property

2.1 Notations and conventions

$\mathbb{R}^2$ is equipped with the metric $d(m, m') = \|m - m'\|$, where $\|\cdot\|$ is the euclidian norm. The topology of $\mathbb{R}^2$ induces a topology on every subset $X \subset \mathbb{R}^2$.

For $Y \subset X \subset \mathbb{R}^2$, $Cl_X(Y)$, $Int_X(Y)$ and $Fr_X(Y) = Cl_X(Y) \setminus Int_X(Y)$ denote respectively the closure, the interior and the frontier of Y relative to X. More briefly, we will write $Cl_{\mathbb{R}^2}(Y) = \overline{Y}$, $Int_{\mathbb{R}^2}(Y) = \overset{\circ}{Y}$ and $Fr_{\mathbb{R}^2}(Y) = Fr(Y)$.

Unless the contrary is stated, a simple closed curve $C \subset \mathbb{R}^2$ is positively oriented. If x, y are two points on C, $[x, y]_C$ (resp. $(x, y)_C$) denotes the closed arc (resp. the open arc) from x to y for the choosen orientation on C.

Finally, we set $B = \{m \in \mathbb{R}^2 \mid \|m\| \leq 1\}$ and $rB = \{m \in \mathbb{R}^2 \mid \|m\| \leq r\}$.

2.2 Statement of theorem

Theorem 2.1 *Let h be a homeomorphism of $\mathbb{R}^2$ (preserving or reversing the orientation) and $D \subset \mathbb{R}^2$ a topological closed disc bounded by the simple closed curve $C = Fr(D)$.*

Assume we can find two arcs $\alpha = [a, b]_C$ and $\beta = [b, a]_C$ such that $D \cap h^{-1}(\beta) = \emptyset$ and $h^{-1}(D) \cap \alpha = \emptyset$ (see fig.1).

Then, if $\bigcap_{n \in \mathbb{N}} h^{-n}(D) \neq \emptyset$, there exists a point $m \in D$ such that $h(m) \in D$ and $h^2(m) = m$.

Furthermore, if h preserves the orientation, we can choose m to be a fixed point (that is, $h(m) = m$).

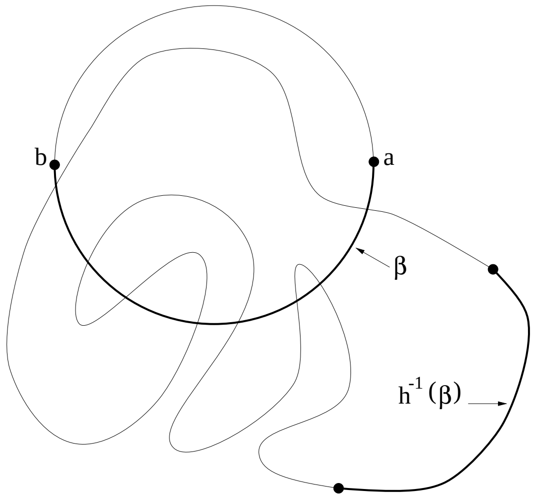

fig. 1

2.3 Plane topology

This paragraph is devoted to more or less well known results, which will be basic tools in the proof of Theorem 2.1. We give some of their proofs when they do not appear in the literature.

Proposition 2.2 *Let K_1 and K_2 be two connected compact sets included in a topological closed disc $D \subset \mathbb{R}^2$. Suppose there exist two points a, c in $K_1 \cap Fr(D)$ and two points b, d in $K_2 \cap Fr(D)$ such that a, b, c, d are met in this order on $C = Fr(D)$.*

Then $K_1 \cap K_2 \neq \emptyset$.

The proof is left as an exercice to the reader.

Proposition 2.3 (see for example [WD, exercice 9 page 113]) *Let K be a connected compact set in $\mathbb{R}^2$ and X a connected component of $\mathbb{R}^2 \setminus K$.*

Then $Fr(X)$ is a connected set.

Corollary 2.4 *Let K be a connected compact set included in a topological closed disc $D \subset \mathbb{R}^2$, $K \neq D$, and X a connected component of $D \setminus K$.*

Then the frontier $Fr_D(X) = \overline{X} \setminus X$ of X relative to D is a connected set.

Proof : If $K \subset \overset{\circ}{D}$, this is an obvious consequence of Proposition 2.3. Then we can suppose $K \cap Fr(D) \neq \emptyset$. Using the Schoenflies Theorem, we can also assume that $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $0 \notin K$. Define a homeomorphism f of the sphere $S^2 = \mathbb{C} \cup \{\infty\}$ by the formula

$$f(z) = \frac{z}{|z|^2} \text{ if } z \in \mathbb{C} \setminus \{0\},\ f(0) = \infty,\ f(\infty) = 0.$$

Clearly, the set $K' = K \cup f(K) \subset \mathbb{R}^2$ is compact and connected. If $X \subset \overset{\circ}{D}$ then X is also a connected component of $\mathbb{R}^2 \setminus K'$ and then $Fr_D(X) = Fr(X)$ is connected by Proposition 2.3. If $X \cap Fr(D) \neq \emptyset$, then $(X \cup f(X)) \setminus \{\infty\}$ is a connected component of $\mathbb{R}^2 \setminus K'$ and then $Fr\left((X \cup f(X)) \setminus \{\infty\}\right)$ is connected. One can check that

$$Fr\left((X \cup f(X)) \setminus \{\infty\}\right) = Fr_D(X) \cup f(Fr_D(X))$$

and this implies the connectedness of $Fr_D(X)$. □

Definition 2.5 *Let $Y \subset X \subset \mathbb{R}^2$ and $m_1 \neq m_2$ be two points in $X \setminus Y$. It is said that Y* separates m_1 and m_2 in X *if m_1 and m_2 do not belong to the same connected component of $X \setminus Y$. For $X = \mathbb{R}^2$, we will simply write "Y* separates m_1 and m_2*".*

Definition 2.6 *A set $X \subset \mathbb{R}^2$ is* simply connected *if its complement $S^2 \setminus X$ in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ is connected.*

If X is a bounded set, this is clearly equivalent to the connectedness of $\mathbb{R}^2 \setminus X$.

Properties 2.7 *(1) If $X_i \subset \mathbb{R}^2$, $i \in I$, is a family of simply connected sets, then $\bigcap_{i \in I} X_i$ is simply connected.*

(2) A compact set $K \subset \mathbb{R}^2$ separates two points m_1 and m_2 if, and only if one of its connected component separates m_1 and m_2. Consequently, K is simply connected if and only if each of its connected component is simply connected.

Proof : (1) Since $\infty \in S^2 \setminus X_i$ for all i in I, $S^2 \setminus \left(\bigcap_{i \in I} X_i\right) = \bigcup_{i \in I} S^2 \setminus X_i$ is connected.

(2) : If a connected component L of K separates m_1 and m_2, so does K. Conversely, suppose there is no connected component of K separating m_1 and m_2. We recall the following result (known as the Zoretti Theorem, see [Wh, page 35]) :

If L is a connected component of a compact set $M \subset \mathbb{R}^2$ and ϵ is any positive number, then there exists a simple closed curve J which encloses L and is such that $J \cap M = \emptyset$, and every point of J is at distance less than ϵ from some point of L.

Let g be a homography of S^2 such that $g(m_2) = \infty$ and consider the compact set $K_1 = g(K) \subset \mathbb{R}^2$. Choose $r > 0$ such that $K_1 \subset rB$ and $m_0 \notin 2rB$. It is sufficient to prove that K_1 does not separate m_0 and $g(m_1)$.

Let L_i be a connected component of K_1. Since the connected components of $\mathbb{R}^2 \setminus L_i$ are path connected, there exists a path $\alpha_i \subset \mathbb{R}^2 \setminus L_i$ from $g(m_1)$ to m_0. Note that L_i is also a connected component of the compact set $M_i = K_1 \cup \alpha_i$ and applying the Zoretti Theorem with $L = L_i$, $M = M_i$ and $\epsilon = r$ gives a simple closed curve J_i bounding a topological closed disc D_i such that $L_i \subset \overset{\circ}{D}_i$, $J_i \cap K_1 \subset J_i \cap M_i = \emptyset$ and $D_i \cap \alpha_i = \emptyset$. Repeating this construction for every connected component of K_1, we obtain a finite open covering

$$K_1 \subset \overset{\circ}{D}_{i_1} \cup \ldots \cup \overset{\circ}{D}_{i_n}$$

where we can suppose $D_{i_j} \not\subset D_{i_k}$ for $j \neq k$. Then one can check that the set

$$\left(\mathbb{R}^2 \setminus \left(\bigcup_{j=1}^{n} D_{i_j} \right) \right) \cup \bigcup_{j=1}^{n} J_{i_j}$$

is connected, disjoint from K_1 and contains both $g(m_1)$ and m_0. □

Corollary 2.8 *Let K be a compact set in a topological closed disc $D \subset \mathbb{R}^2$, $K \neq D$, and two points $m_i \in D \setminus K$ $(i = 1, 2)$.*

Then K separates m_1 and m_2 in D if, and only if one of its connected component separates m_1 and m_2 in D.

Proof : By the Schoenflies Theorem, we can assume that $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $0 \notin K$. Suppose that K separates m_1 and m_2 in D and consider the homeomorphism f of S^2 defined in the proof of Corollary 2.4. It is easily seen that

$$K \text{ separates } m_1 \text{ and } m_2 \text{ in } D \Longleftrightarrow K \cup f(K) \text{ separates } m_1 \text{ and } m_2$$

then we obtain with Proposition 2.7 *(2)* a connected component L' of $K \cup f(K)$ which separates m_1 and m_2. It is not difficult to deduce that a connected component L of K separates m_1 and m_2 in D. $\square$

Theorem 2.9 (Cartwright-Littlewood-Bell Theorem) *Let h be a homeomorphism of $\mathbb{R}^2$ (preserving or reversing the orientation) and $K \subset \mathbb{R}^2$ a connected and simply connected compact set.*

If $h(K) = K$ then h possesses a fixed point in K.

When h preserves the orientation, this result is known as the Cartwright-Littlewood Theorem (see [CL] or [Br1]). The general case is proved in [Be].

Corollary 2.10 *Let h and K be as in Theorem 2.9.*

If we suppose only $h(K) \subset K$, then h possesses also a fixed point in K.

Proof : Define $K' = \bigcap_{n\in\mathbb{N}} h^n(K)$. Then K' is a connected compact set (as a decreasing intersection of connected compact sets) and is simply connected by Property 2.7 *(1)*. Since $h(K) \subset K$ we have $h(K') = K'$ and Theorem 2.9 gives a fixed point in $K' \subset K$. $\square$

2.4 Proof of Theorem 2.1

From now on, we suppose $\bigcap_{k\in\mathbb{N}} h^{-k}(D) \neq \emptyset$.

2.4.1 The sets $D_n,\ n \in \mathbb{N} \cup \{\infty\}$

Notations 2.11 *$\forall n \in \mathbb{N}$, set $D_n = \bigcap_{k=0}^{n} h^{-k}(D)$ and $D_\infty = \bigcap_{k\in\mathbb{N}} h^{-k}(D)$.*

Lemma 2.12 *$Fr(D_n) \subset \beta \cup h^{-n}(\alpha)\ \forall n \in \mathbb{N}$.*

Proof (by induction on n) : The result is obvious for $n = 0$. If we suppose it is true for n we obtain :

$$\begin{aligned} Fr(D_{n+1}) = Fr\left(h^{-1}(D_n) \cap D_n\right) &\subset h^{-1}\left(Fr(D_n)\right) \cup Fr(D_n) \\ &\subset h^{-1}(\beta) \cup h^{-(n+1)}(\alpha) \cup \beta \cup h^{-n}(\alpha). \end{aligned}$$

But $Fr(D_{n+1}) \cap h^{-n}(\alpha) \subset D_{n+1} \cap h^{-n}(\alpha) \subset h^{-n}(h^{-1}(D) \cap \alpha) = \emptyset$ and $Fr(D_{n+1}) \cap h^{-1}(\beta) \subset D \cap h^{-1}(\beta) = \emptyset$, so we have

$$Fr(D_{n+1}) \subset \beta \cup h^{-(n+1)}(\alpha). \ \Box$$

Proposition 2.13 *Every connected component of D_∞ is simply connected and meets the arc β.*

Proof : First we prove, for $n \in \mathbb{N}$, that every connected component of D_n meets the arc β : recall the intersection $U_1 \cap U_2$ of two Jordan domains is a disjoint and countable union $\coprod_i V_i$, where each V_i is a Jordan domain such that $Fr(V_i) \subset Fr(U_1) \cup Fr(U_2)$ (see for example [Ke]). It follows that

$$\forall n \in \mathbb{N},\ \overset{\circ}{D}_n = \bigcap_{k=0}^{n} h^{-k}(\overset{\circ}{D}) = \coprod_i V_{n,i}$$

where each $V_{n,i}$ is a Jordan domain such that $Fr(V_{n,i}) \subset \bigcup_{k=0}^{n} h^{-k}(Fr(D))$ and precisely, with Lemma 2.12, $Fr(V_{n,i}) \subset Fr(D_n) \subset \beta \cup h^{-n}(\alpha)$. Then every topological closed disc $\overline{V}_{n,i} = V_{n,i} \cup Fr(V_{n,i})$ meets the arc β. Now note that we have $Fr(D_n) \setminus \beta \subset \bigcup_i Fr(V_{n,i})$; there is nothing to prove for $n = 0$; let $n \geq 1$ and $m \in Fr(D_n) \setminus \beta \subset h^{-n}(\alpha)$. Since $D_n \subset D_{n-1}$ and $h^{-n}(\alpha) \cap h^{-n+1}(\alpha) \subset h^{-n+1}(h^{-1}(D) \cap \alpha) = \emptyset$ we have also

$$m \in D_{n-1} \setminus (\beta \cup h^{-n+1}) \subset \overset{\circ}{D}_{n-1}$$

and then there exists a neighbourghood V of m in $\mathbb{R}^2$ such that $V \subset \overset{\circ}{D}_{n-1}$ and $h^{-n}(\alpha)$ divides V into exactly two connected components A and B, with $A \subset h^{-n}(\overset{\circ}{D})$ and $B \subset h^{-n}(\mathbb{R}^2 \setminus D)$ (see fig. 2).

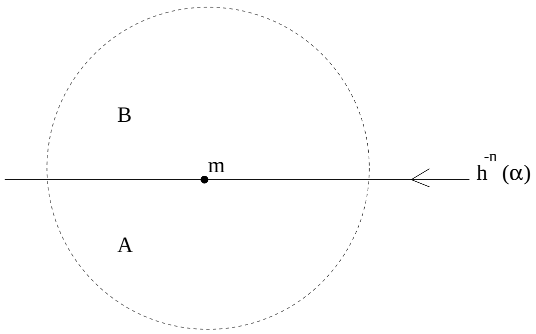

fig. 2

We obtain

$$A \subset h^{-n}(\overset{\circ}{D}) \cap \overset{\circ}{D}_{n-1} = \overset{\circ}{D}_n = \coprod_i V_{n,i} \ ,$$

then there exists an index i_0 such that $A \subset V_{n,i_0}$ and $m \in Fr(V_{n,i_0})$. It follows that $D_n \subset \beta \cup \bigcup_i \overline{V}_{n,i}$ and then, for $n \in \mathbb{N}$, every connected component of D_n meets β.

Let c_∞ be a connected component of D_∞ and, for $n \in \mathbb{N}$, c_n the connected component of D_n which contains c_∞. We have clearly

$$c_\infty \subset \bigcap_{n \in \mathbb{N}} c_n \subset D_\infty.$$

Furthermore, $\bigcap_{n \in \mathbb{N}} c_n$ is connected (as a decreasing intersection of connected compact sets) and then $c_\infty = \bigcap_{n \in \mathbb{N}} c_n$ meets the arc β.

Finally, the simple connectedness of c_∞ is an obvious consequence of Properties 2.7. $\square$

Notation 2.14 $\pi_0(D_\infty)$ *denotes the set of all connected components of* D_∞*.*

2.4.2 Two order relations on $\pi_0(D_\infty)$

Remark that $D_\infty \subset D_1 \subset D \setminus \alpha$. Then for every $c \in \pi_0(D_\infty)$ there exists a (unique) connected component of $D \setminus c$ which contains α. We define a relation $\ll$ on $\pi_0(D_\infty)$ in the following way :

For c_1 and c_2 in $\pi_0(D_\infty)$, we will write $c_1 \ll c_2$ if either $c_1 = c_2$ or c_1 is contained in a connected component of $D \setminus c_2$ which does not contain α (see fig. 3).

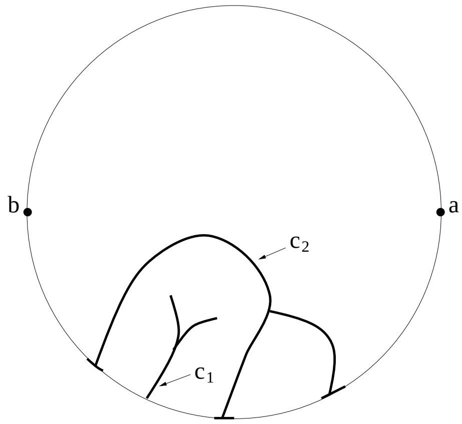

fig. 3

Proposition 2.15 *(1) The relation* $\ll$ *is a (partial) ordering of* $\pi_0(D_\infty)$.

(2) (existence of maximal elements for $\ll$)

$$\forall c_1 \in \pi_0(D_\infty)\ \exists c_2 \in \pi_0(D_\infty)\ \textit{such that} \begin{cases} c_1 \ll c_2\,, \\ c_2 \ll c_3 \Rightarrow c_2 = c_3\,. \end{cases}$$

Proof : (1) (i) Suppose $c_1 \ll c_2$ and $c_1 \neq c_2$. Let A be the connected component of $D \setminus c_2$ which contains α. Since $\emptyset \neq Fr_D(A) = \overline{A} \setminus A \subset c_2$, the set $A \cup c_2 = \overline{A} \cup c_2$ is connected and satisfies

$$\alpha \cup c_2 \subset A \cup c_2 = \overline{A} \cup c_2 \subset D \setminus c_1\,.$$

This excludes the situation $c_2 \ll c_1$.

(ii) Let $c_1 \ll c_2$, $c_2 \ll c_3$ with $c_1 \neq c_2$ and $c_2 \neq c_3$. Suppose $c_1 \cup \alpha$ is contained in a connected component A of $D \setminus c_3$. Since $c_1 \ll c_2$ and $c_1 \neq c_2$ we have $A \not\subset D \setminus c_2$, that is $A \cap c_2 \neq \emptyset$, and then $c_2 \subset A$ which gives a contradiction with $c_2 \ll c_3$.

(2) For $c \in \pi_0(D_\infty)$, define

$$Maj(c) = \{c' \in \pi_0(D_\infty) \mid c \ll c'\}.$$

We need the following lemma :

Lemma 2.16 *The set* $Maj(c)$ *is totally ordered by* $\ll$.

Proof of Lemma 2.16 : Suppose $Maj(c)$ is not reduced to $\{c\}$. Let $c' \neq c''$ be in $Maj(c)$. If $c' = c$ (resp. $c'' = c$) we have obviously $c' \ll c''$ (resp. $c'' \ll c'$). Then we can assume $c' \neq c$ and $c'' \neq c$. We note A' (resp. A'') the connected component of $D \setminus c'$ (resp. of $D \setminus c''$) which contains c. If $c'' \subset A'$ then $c'' \ll c'$. If $c'' \cap A' = \emptyset$ then we have

$$c \cup c' \subset A' \cup c' = \overline{A'} \cup c' \subset D \setminus c'' .$$

Since $A' \cup c' = \overline{A'} \cup c'$ is connected, it is contained in A'' and then $c' \ll c''$. $\square$

continuation of the proof of Proposition 2.15 (2) : If $Maj(c_1) = \{c_1\}$ just set $c_2 = c_1$. If $c \in Maj(c_1)$ and $c \neq c_1$ we let $A(c)$ be the connected component of $D \setminus c$ which contains c_1. For convenience, set $A(c_1) = \emptyset$ and then define

$$A = \bigcup_{c \in Maj(c_1)} A(c) \subset D, \; K = Fr_D(A).$$

Thus A is an open set in D disjoint from the arc α and $K = \overline{A} \setminus A$.

(i) We have $K \subset D_\infty$:

Let $m \in K$ and U a connected neighbourhood of m in D. Since $m \in \overline{A}$ there exists $c \in Maj(c_1)$ such that $U \cap A(c) \neq \emptyset$. On the other hand, $m \notin A(c)$ so $U \cap (D \setminus A(c)) \neq \emptyset$. It follows from the connectedness of U that $\emptyset \neq U \cap Fr_D(A(c)) \subset c \subset D_\infty$. The assertion is proved because U is arbitrary small and D_∞ is closed.

(ii) α and c_1 are not in the same connected component of $D \setminus K$:

We have the partition $D \setminus K = \left(D \setminus \overline{A}\right) \coprod A$. The set $D \setminus \overline{A}$ is an open set in D and contains α (because $\overline{A} = A \cup K \subset A \cup D_\infty \subset D \setminus \alpha$) whereas A is an open set in D and contains c_1.

(iii) K is a connected set :

Suppose there exists a partition $K = K_1 \coprod K_2$ where K_i is a non empty closed set in K $(i = 1, 2)$. Then there exist two open sets Ω_1 and Ω_2 in $\mathbb{R}^2$ such that $K_i \subset \Omega_i$ $(i = 1, 2)$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Let $m \in K_1$ and $U \subset \Omega_1$ a connected neighbourhood of m in D. The proof of *(i)* above gives $c_{1,1} \in Maj(c_1)$ such that

$$\emptyset \neq Fr_D(A(c_{1,1})) \cap U \subset Fr_D(A(c_{1,1})) \cap \Omega_1 .$$

Furthermore, it is easily seen that

$$c_1 \ll c \ll c' \Rightarrow A(c) \subset A(c')$$

and we obtain precisely $Fr_D(A(c)) \cap \Omega_1 \neq \emptyset$ for every $c \gg c_{1,1}$. In the same way, there exists $c_{1,2} \in Maj(c_1)$ such that $Fr_D(A(c)) \cap \Omega_2 \neq \emptyset$ for every $c \gg c_{1,2}$. According to Lemma 2.16, we can suppose $c_{1,2} \ll c_{1,1}$ and $c_1 \neq c_{1,1}$. For every $c \in Maj(c_{1,1}) \subset Maj(c_1)$, $Fr_D(A(c)) = \overline{A(c)} \setminus A(c)$ is a connected set (see Corollary 2.4) then there exists $m(c) \in Fr_D(A(c)) \setminus (\Omega_1 \coprod \Omega_2)$ and one can define

$$X(c) = \{m(c') \mid c \ll c'\}.$$

It follows from Lemma 2.16 that the set

$$\mathcal{B} = \{X(c) \mid c \in Maj(c_{1,1})\}$$

is the basis of a filter $\mathcal{F}$ on D and by compactness of D there exists a filter $\mathcal{F}_1$ finer than $\mathcal{F}$ which converges to a point $m \in D$.

Let U be an open neighbourhood of m in D. Then

$$X(c_{1,1}) \in \mathcal{B} \subset \mathcal{F} \subset \mathcal{F}_1$$

and $U \in \mathcal{F}_1$ so $X(c_{1,1}) \cap U \neq \emptyset$, that is there exists $c \gg c_{1,1}$ such that $m(c) \in U$. Since $m(c) \in Fr_D(A(c)) \subset \overline{A(c)}$ we obtain

$$m \in \overline{\bigcup_{c \in Maj(c_1)} A(c)} = \overline{A}\,.$$

On the other hand, we have $m \notin A$: indeed $m \in A(c)$, $c \in Maj(c_1)$, would imply $A(c) \in \mathcal{F}_1$ and as above one could find $c' \gg c$ such that $m(c') \in A(c)$. But $c \ll c'$ implies $A(c) \subset A(c')$ then we would have $m(c') \in A(c')$ which is absurd. Thus we have obtained

$$m \in K = K_1 \coprod K_2 \subset \Omega_1 \coprod \Omega_2\,.$$

Suppose for example that Ω_1 is a neighbourhood of m in $\mathbb{R}^2$. Then

$$D \cap \Omega_1 \in \mathcal{F}_1 \text{ and } X(c) \cap (D \cap \Omega_1) \neq \emptyset \text{ for every } c \in Maj(c_{1,1})$$

which gives a contradiction with $m(c') \notin \Omega_1 \coprod \Omega_2$ for all $c' \in Maj(c_{1,1})$.

(iv) It follows from *(i)* and *(iii)* that K is contained in a connected component c_2 of D_∞. It remains to be checked that c_2 possesses the required properties. Let c be in $Maj(c_1)$. If $c \neq c_2$, we have by the construction $K \cap A(c) = \emptyset$ and therefore $c_2 \cap (A(c) \cup c) = \emptyset$. Furthermore, the set $A(c) \cup c = \overline{A(c)} \cup c$ is connected and satisfies

$$c \cup c_1 \subset A(c) \cup c = \overline{A(c)} \cup c \subset D \setminus c_2$$

so it is contained in the connected component $A(c_2)$ of $D \setminus c_2$ which contains c_1. According to *(ii)*, $A(c_2) \cap \alpha = \emptyset$.This proves simultaneously $c_1 \ll c_2$ and $c \ll c_2$. $\square$

Notation 2.17 $\mathcal{E} = \{c \in \pi_0(D_\infty) \mid c \text{ is maximal for the ordering } \ll\}$

Now consider the natural ordering $\leq$ of β provided by the positive orientation of C (for x and y in β, $x \leq y$ simply means that b, x, y, a are met in this order on C). Clearly, the ordered set $(\beta, \leq)$ has the properties of the interval $[0, 1]$ ordered as usual. Especially, every set $X \subset \beta$ admits a least upper bound $\mathrm{Sup}(X) \in \beta$ and a greater lower bound $\mathrm{Inf}(X) \in \beta$ which belongs to X if X is closed.

Keeping in mind Proposition 2.13, we have the following result which will be a convenient criterion to use with the order $\ll$:

Proposition 2.18 *Let $c_1 \neq c_2$ be in $\pi_0(D_\infty)$. The following properties are equivalent :*

(1) $c_1 \ll c_2$

(2) $\forall x \in c_1 \cap \beta,\ \mathit{Inf}(c_2 \cap \beta) < x < \mathit{Sup}(c_2 \cap \beta)$

(3) $\exists\, x \in c_1 \cap \beta$ *such that* $\mathit{Inf}(c_2 \cap \beta) < x < \mathit{Sup}(c_2 \cap \beta)$.

Proof : (1) $\Rightarrow$ *(2)* Let $x \in c_1 \cap \beta$. If $x \notin [\mathrm{Inf}(c_2 \cap \beta), \mathrm{Sup}(c_2 \cap \beta)]_C$ then either the arc $[b, x]_C$ or the arc $[x, a]_C$ joins α and c_1 in $D \setminus c_2$, which contradicts $c_1 \ll c_2$.

(2) $\Rightarrow$ *(3)* is obvious.

(3) $\Rightarrow$ *(1)* Suppose $\alpha \cup c_1$ is included in a connected component A of $D \setminus c_2$. Since A is also path connected, there exists a path $\gamma \subset A$ from x to a point $y \in \alpha$. We obtain with Proposition 2.2

$$\emptyset \neq \gamma \cap c_2 \subset A \cap c_2 = \emptyset$$

which is absurd. $\square$

Proposition and definition 2.19 *The set $\pi_0(D_\infty)$ is totally ordered by the relation $\preceq$ defined as follows :*

$$c_1 \preceq c_2 \Leftrightarrow \mathit{Inf}(c_1 \cap \beta) \leq \mathit{Inf}(c_2 \cap \beta).$$

The verification is easy and left to the reader. As usual, $c_1 \prec c_2$ will mean $c_1 \preceq c_2$ and $c_1 \neq c_2$.

Proposition 2.20 *In the (totally) ordered set $(\mathcal{E}, \preceq)$, every non empty subset $\mathcal{F} \subset \mathcal{E}$ admits a least upper bound.*

Proof : Set $x_0 = \mathrm{Sup}\left(\bigcup_{c \in \mathcal{F}} c \cap \beta\right)$. Since D_∞ is a closed set, there exists $c_0 \in \pi_0(D_\infty)$ such that $x_0 \in c_0$.

(i) $c_0 \in \mathcal{E}$: suppose there exists $c_1 \in \pi_0(D_\infty)$ such that $c_1 \neq c_0$ and $c_0 \ll c_1$. According to Proposition 2.18 we have

$$\mathrm{Inf}(c_1 \cap \beta) < x_0 < \mathrm{Sup}(c_1 \cap \beta)$$

and, by the definition of x_0, we obtain an element $c \in \mathcal{F}$ and a point $x \in c \cap \beta$ such that

$$\mathrm{Inf}(c_1 \cap \beta) < x \leq x_0 < \mathrm{Sup}(c_1 \cap \beta).$$

If $c \neq c_1$, we deduce again from Proposition 2.18 that $c \ll c_1$, which is not possible because $c \in \mathcal{F} \subset \mathcal{E}$. We obtain therefore $c = c_1 \in \mathcal{F}$ and then

$$\mathrm{Sup}(c_1 \cap \beta) \leq x_0 < \mathrm{Sup}(c_1 \cap \beta)$$

which is absurd.

(ii) c_0 is a upper bound for $\mathcal{F}$: suppose there exists $c_1 \in \mathcal{F}$ such that $c_0 \prec c_1$. Then

$$\mathrm{Inf}(c_0 \cap \beta) < \mathrm{Inf}(c_1 \cap \beta) \leq \mathrm{Sup}(c_1 \cap \beta) \leq x_0 \leq \mathrm{Sup}(c_0 \cap \beta)$$

and it follows from Proposition 2.18 that $c_1 \ll c_0$ which gives a contradiction with $c_1 \in \mathcal{E}$.

(iii) c_0 is the least upper bound of $\mathcal{F}$: let $c_1 \in \mathcal{E}$ such that $c_1 \prec c_0$. Then $\mathrm{Inf}(c_1 \cap \beta) < \mathrm{Inf}(c_0 \cap \beta) \leq x_0$. Since $c_0 \in \mathcal{E}$ and using Proposition 2.18, we obtain also

$$\mathrm{Sup}(c_1 \cap \beta) < \mathrm{Inf}(c_0 \cap \beta) \leq x_0$$

then there exist $c \in \mathcal{F}$ and $x \in c \cap \beta$ such that

$$\mathrm{Sup}(c_1 \cap \beta) < x \leq x_0.$$

Since $c_1 \in \mathcal{E}$, we obtain with Proposition 2.18

$$\mathrm{Inf}(c_1 \cap \beta) \leq \mathrm{Sup}(c_1 \cap \beta) < \mathrm{Inf}(c \cap \beta) \leq x$$

and then $c_1 \prec c$ is not an upper bound for $\mathcal{F}$. $\square$

2.4.3 A map φ from $\mathcal{E}$ to $\mathcal{E}$

Proposition and definition 2.21 *For every $c \in \pi_0(D_\infty)$ there exists a unique $c' \in \pi_0(D_\infty)$ such that $c \subset h^{-1}(c')$.*
Furthermore, we have : $c \in \mathcal{E} \Rightarrow c' \in \mathcal{E}$.
We obtain a well-defined map φ from $\mathcal{E}$ to $\mathcal{E}$ by setting $\varphi(c) = c'$.

Proof : For the first part of the proposition, just note that $D_\infty \subset h^{-1}(D_\infty)$. Then, for every $c \in \pi_0(D_\infty)$, there exists a unique $c' \in \pi_0(D_\infty)$ such that $h^{-1}(c')$ is the connected component of $h^{-1}(D_\infty)$ containing c. Let ${c_1}' \neq {c_2}'$ be in $\pi_0(D_\infty)$ such that ${c_1}' \ll {c_2}'$.

It is enough to prove :

$$\Big(c_1 \in \pi_0(D_\infty),\, c_1 \subset h^{-1}({c_1}')\Big) \Rightarrow (c_1 \notin \mathcal{E})\ .$$

Let $\mathcal{C}_2 \subset \pi_0(D_\infty)$ be the family of all connected components of $D \cap h^{-1}(c_2')$ and, for every $c \in \mathcal{C}_2$, $Maj(c, \mathcal{C}_2) = \{c' \in \mathcal{C}_2 \mid c \ll c'\}$.

Since $D \cap h^{-1}(c_2')$ is a closed set, one can replace $Maj(c_1)$ by $Maj(c_1, \mathcal{C}_2)$ in the proof of Proposition 2.15 *(2)* and obtain the following result :

$$\forall c_1 \in \mathcal{C}_2\ \exists c_2 \in \mathcal{C}_2 \text{ such that } \begin{cases} c_1 \ll c_2\,, \\ (c_2 \ll c_3\,, c_3 \in \mathcal{C}_2) \Rightarrow (c_2 = c_3). \end{cases}$$

We denote $\mathcal{C}_{max} = \{c \in \mathcal{C}_2 \mid (c \ll c'\,, c' \in \mathcal{C}_2) \Rightarrow (c = c')\}$ and, for every $c \in \mathcal{C}_{max}$

- $B(c)$ the connected component of $D \setminus c$ which contains α ,
- $\mathcal{A}_c$ the set of the connected components of $D \setminus c$ which are disjoint from α.

Since $\emptyset \neq \overline{A} \setminus A \subset c$ for every connected component A of $D \setminus c$, the set

$$K(c) = D \setminus B(c) = \left(\bigcup_{A \in \mathcal{A}_c} A\right) \cup c$$

is connected.

Note that the set $K = \bigcup_{c \in \mathcal{C}_{max}} K(c)$ is closed : suppose there exists a sequence $(m_k)_{k \in \mathbb{N}}$ which converges to a point m and satisfies

$$\forall k \in \mathbb{N}\,, \exists\, c_k \in \mathcal{C}_{max} \text{ such that } m_k \in K(c_k), \text{ with } K(c_k) \neq K(c_l) \text{ for } k \neq l.$$

By the construction, the $K(c)$, $c \in \mathcal{C}_{max}$, are pairwise disjoint therefore

$$\lim_k d\,(m_k,\, Fr_D(K(c_k))) = 0.$$

Since $Fr_D(K(c_k)) \subset c_k \subset D \cap h^{-1}(c_2')$ there exists $c \in \mathcal{C}_2$ such that $m \in c$ and we obtain $m \in K(c')$, where $c' \in \mathcal{C}_{max}$ satisfies $c \ll c'$.

Furthermore, one can check that the $K(c)$, $c \in \mathcal{C}_{max}$, are exactly the connected components of K. We deduce from the connectedness of $D \setminus K(c)$ and from Corollary 2.8 that $D \setminus K$ is connected.

Suppose $c_1 \subset D \setminus K$. Since $\alpha \cap h^{-1}(D) = \emptyset$ and $c_1 \subset D \cap h^{-1}(D)$, we have

$$\emptyset \neq (D \setminus K) \cap Fr_D(D \cap h^{-1}(D)) \subset (D \setminus K) \cap h^{-1}(\alpha)$$

and there exists a connected component Y of $(D \setminus K) \cap h^{-1}(D)$ which meets both c_1 and $h^{-1}(\alpha)$. Then $h(Y) \subset D$ is connected, disjoint from c_2', and joins $h(c_1) \subset c_1'$ and α. This contradicts $c_1' \ll c_2'$ and we obtain therefore $c \in \mathcal{C}_{max}$ such that $c_1 \cap K(c) \neq \emptyset$. Since $c_1 \cap c \subset h^{-1}(c_1' \cap c_2') = \emptyset$ there exists $A \in \mathcal{A}_c$ containing c_1. Thus $c_1 \ll c$ and $c_1 \notin \mathcal{E}$. □

Proposition 2.22 *If h preserves (resp. reverses) the orientation then φ preserves (resp. reverses) the order $\preceq$ of $\mathcal{E}$.*

Proof : Let c_i be in $\mathcal{E}$ and $c_i{}' = \varphi(c_i)$ $(i = 1, 2)$. Suppose we have $c_1 \preceq c_2$ and $c_1{}' \neq c_2{}'$. Since $c_2 \in \mathcal{E}$, we obtain with Proposition 2.18

$$\mathrm{Inf}(c_1 \cap \beta) \leq \mathrm{Sup}(c_1 \cap \beta) < \mathrm{Inf}(c_2 \cap \beta)$$

Furthermore, $c_1 \cap h^{-1}(c_2{}') \subset h^{-1}(c_1{}' \cap c_2{}') = \emptyset$ then there exists a point $x \in \beta$ such that

$$\mathrm{Sup}(c_1 \cap \beta) < x < \mathrm{Inf}(c_2 \cap \beta) \text{ and } [\mathrm{Sup}(c_1 \cap \beta), x]_C \cap h^{-1}(c_2{}') = \emptyset.$$

Now choose a point $y \in (a, b)_C$. As in the proof of Proposition 2.21, we denote by $\mathcal{C}_i \subset \pi_0(D_\infty)$ $(i = 1, 2)$ the set of all connected components of $D \cap h^{-1}(c_i{}')$. Since $c_1 \cup [\mathrm{Sup}(c_1 \cap \beta), x]_C$ is connected and disjoint from $h^{-1}(c_2{}')$, observe that if a connected component $c \in \mathcal{C}_2$ separates x and y in D then c_1 and α are not contained in the same connected component of $D \setminus c$, which is not possible because $c_1 \in \mathcal{E}$. On the other hand, it is clear that c_1 does not separate x and y in D. The set $\{c_1\} \cup \mathcal{C}_2$ is exactly the set of all connected components of $c_1 \cup (h^{-1}(c_2{}') \cap D))$ and it follows from Corollary 2.8 that there exists a path

$$\gamma \subset D \setminus \Big(c_1 \cup \Big(D \cap h^{-1}(c_2{}')\Big)\Big)$$

from x to y. Furthermore, γ can be chosen to be a simple arc which intersects $Fr(D)$ only in its endpoints x and y. Thus c_1 (resp. c_2) is included in the topological closed disc D_1 (resp. D_2) bounded by the simple closed curve $C_1 = [y, x]_C \cup \gamma$ (resp. $C_2 = [x, y]_C \cup \gamma$) and $D_1 \cap D_2 = \gamma$. Let $z_2 \in (a, y)_C$ and $m_2 \in c_2$. Since $c_2 \in \mathcal{E}$, we obtain again with Corollary 2.8 that $D \cap h^{-1}(c_1{}')$ does not separate m_2 and z_2 in D and then there exists a path

$$\gamma_2 \subset D \setminus \left(D \cap h^{-1}(c_1{}')\right)$$

from m_2 to z_2. If necessary, we can modify γ_2 and obtain $\gamma_2 \subset D_2 \setminus \gamma$, $a \notin \gamma_2$.

With the Schoenflies Theorem, we can assume that $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, $a = 1$, $b = -1$ and $h^{-1}(\beta)$ is a horizontal segment below D (say for example $h^{-1}(\beta) = [-1, 1] \times \{-2\}$) such that $h^{-1}(\mathring{D})$ is above $h^{-1}(\beta)$. Choose two simple arcs μ_i $(i = 1, 2)$ in such a way that

- the endpoints of μ_1 (resp. of μ_2) are b and $(-1, -2)$ (resp. a and $(1, -2)$),
- $C_\star = \alpha \cup \mu_1 \cup h^{-1}(\beta) \cup \mu_2$ forms a simple closed curve,
- the topological closed disc $D_\star$ bounded by $C_\star$ contains $D \cup h^{-1}(D)$

(see fig. 4).

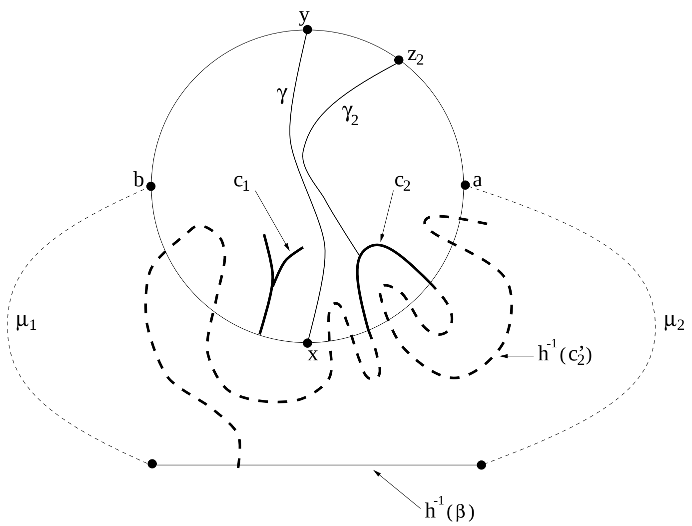

fig. 4

Define $K_2 = h^{-1}({c_2}') \cup \gamma_2 \supset c_2 \cup \gamma_2$. Thus K_2 is a connected compact set included in $D_\star \setminus h^{-1}({c_1}')$ and contains neither a nor b. Denote X_a (resp. X_b) the connected component of $D_\star \setminus K_2$ which contains a (resp. b). It follows from Proposition 2.2 that $X_a \neq X_b$. Since

$$c_1 \cup [\mathrm{Sup}(c_1 \cap \beta), x]_C \cup \gamma \cup [y, b]_C$$

joins $c_1 \subset h^{-1}({c_1}')$ and b in $D_\star \setminus K_2$, we have $h^{-1}({c_1}') \subset X_b$ and then $X_a \cap h^{-1}({c_1}') = \emptyset$.
• If h preserves the orientation, this implies ${c_1}' \preceq {c_2}'$: otherwise, since ${c_1}' \in \mathcal{E}$ (see Proposition 2.21), we obtain with Proposition 2.18

$$\mathrm{Inf}({c_2}' \cap \beta) \leq \mathrm{Sup}({c_2}' \cap \beta) < \mathrm{Inf}({c_1}' \cap \beta)\,.$$

As h preserves orientation, $h^{-1}(\mathring{D})$ is (locally) on the left of $h^{-1}(C)$ and therefore $h^{-1}(b) = (-1, -2)$, $h^{-1}(a) = (1, -2)$. Thus the connected set $h^{-1}\left([\mathrm{Inf}({c_1}' \cap \beta), a]_C\right) \cup \mu_2$ joins $h^{-1}({c_1}')$ and a in $D_\star \setminus K_2$, a contradiction.
• If h reverses the orientation, we obtain ${c_2}' \preceq {c_1}'$: otherwise, since ${c_2}' \in \mathcal{E}$, we have with Proposition 2.18

$$\mathrm{Inf}({c_1}' \cap \beta) \leq \mathrm{Sup}({c_1}' \cap \beta) < \mathrm{Inf}({c_2}' \cap \beta)\,.$$

Realizing that $h^{-1}(\mathring{D})$ is now (locally) on the right of $h^{-1}(C)$, we can see that $h^{-1}\left([b, \mathrm{Inf}({c_1}' \cap \beta)]_C\right) \cup \mu_2$ is connected and joins $h^{-1}({c_1}')$ and a in $D_\star \setminus K_2$, a contradiction. □

We can now prove Theorem 2.1 :

Let $x_0 = \mathrm{Inf}(D_\infty \cap \beta)$. Since D_∞ is closed, x_0 belongs to a connected component c_l (l for "left") of D_∞ and c_l is clearly the minimal element of the ordered set $(\pi_0(D_\infty), \preceq)$. Furthermore, it is an obvious consequence of Proposition 2.18 that $c_l \in \mathcal{E}$. If h preserves (resp. reverses) the orientation, define $\Phi = \varphi$ (resp. $\Phi = \varphi \circ \varphi$). According to Proposition 2.22, Φ maps $\mathcal{E}$ into $\mathcal{E}$ and preserves the order $\preceq$.

Assertion : Φ possesses a fixed point $c_\star \in \mathcal{E}$.

We can suppose $c_l \prec \Phi(c_l)$. Then consider $\mathcal{F} = \{c \in \mathcal{E} \mid c \prec \Phi(c)\}$ and $c_0 \in \mathcal{E}$ the least upper bound of $\mathcal{F}$ (see Proposition 2.20). If $\Phi(c_0) \prec c_0$, there exists $c_1 \in \mathcal{F}$ such that $\Phi(c_0) \prec c_1 \preceq c_0$ and we obtain both

$\Phi(c_1) \preceq \Phi(c_0) \prec c_1$ (since Φ preserves the order $\preceq$) and $c_1 \prec \Phi(c_1)$, which is absurd. Thus we have $c_0 \preceq \Phi(c_0)$, and we can assume $c_0 \prec \Phi(c_0)$. Consequently, we obtain $\Phi(c_0) \notin \mathcal{F}$, that is $\Phi \circ \Phi(c_0) \preceq \Phi(c_0)$. On the other hand,

$$c_0 \preceq \Phi(c_0) \Rightarrow \Phi(c_0) \preceq \Phi \circ \Phi(c_0)$$

and then $\Phi(c_0) = \Phi \circ \Phi(c_0)$. The assertion is proved.

We have obtain the following :

• if h is an orientation preserving homeomorphism, then

$$c_\star \subset h^{-1}\left(\varphi(c_\star)\right) = h^{-1}\left(\Phi(c_\star)\right) = h^{-1}(c_\star)$$

and Corollary 2.10 gives a fixed point for h in $c_\star$.

• if h is an orientation reversing homeomorphism, then $\varphi(c_\star) \subset h^{-1}\left(\varphi \circ \varphi(c_\star)\right)$ implies

$$c_\star \subset h^{-1}\left(\varphi(c_\star)\right) \subset h^{-2}\left(\varphi \circ \varphi(c_\star)\right) = h^{-2}(\Phi(c_\star)) = h^{-2}(c_\star)$$

and Corollary 2.10 gives a fixed point for h^2 in $c_\star$. Furthermore, $h(c_\star) \subset \varphi(c_\star) \subset D$. Theorem 2.1 is proved. $\square$

3 Canonical position

We begin with two definitions :

Definition 3.1 *Let h and h' be two homeomorphisms of $\mathbb{R}^2$. It is said that* h' is a free modification of h *if there exists a finite sequence $h_1, \ldots, h_n$ which satisfies*

$$\begin{array}{ll} h_1 = h, \quad h_n = h' & \\ \forall i = 1, \ldots, n-1 & h_{i+1} = \varphi_i \circ h_i \ (\text{or } h_{i+1} = h_i \circ \varphi_i\ ) \end{array}$$

where each φ_i is a planar homeomorphism supported on a countable union $\coprod_j D_{i,j}$ of pairwise disjoint topological closed discs $D_{i,j}$ such that $h_i(D_{i,j}) \cap D_{i,j} = \emptyset$ for every pair (i,j).

Definition 3.2 *Let h be a planar homeomorphism and $C \subset \mathbb{R}^2$ a simple closed curve disjoint from the fixed point set Fix(h), bounding the topological closed disc D.*

It is said that h is in a canonical position on C *if the following conditions are satisfied :*

(C1) $h(C)$ intersects C transversely and only finitely often,

(C2) $D \cap h(D)$ is connected,

(C3) Card $(C \cap h(C)) = 2|n-1|$

where n is the index ind(h, C) of C with respect to h (see [Sl, page 430]*); furthermore, if x belongs to the intersection $\mathcal{I} = C \cap h(C)$, then either the arc $(x, h(x))_{h(C)}$ or the arc $(h(x), x)_{h(C)}$ is disjoint from $\mathcal{I} \cup h(\mathcal{I})$.*

See examples in figure 5, where h is supposed to preserve the orientation and where C', a', b' denote respectively $h(C)$, $h(a)$, $h(b)$.

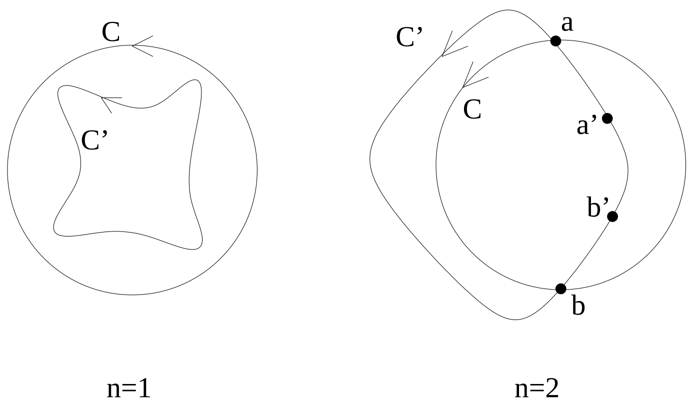

fig. 5

The following properties are well known and easily checked :

Properties 3.3 *Let h' be a free modification of a planar homeomorphism h and $C \subset \mathbb{R}^2$ a simple closed curve such that $C \cap$ Fix$(h) = \emptyset$. Then there exists an isotopy $(h_t)_{0 \leq t \leq 1}$ from $h_0 = h$ to $h_1 = h'$ such that Fix(h_t) = Fix(h) for every t. Consequently, we have ind(h, C) = ind(h', C).*

When C surrounds an isolated fixed point p of h (and is close enough to p), $n = \text{ind}(h, C)$ is nothing but the Lefschetz index $\text{ind}(h, p)$ of p and the purpose of this section is to find a free modification of h which is in a canonical position on C. As mentioned in the introduction, this idea is due to Schmitt and is a central point in several papers (see [Sc], but also [Bo], [Br2], [PS] and [Sl]). Nevertheless, it is not possible to obtain the condition *(C2)* with the arguments in [Sc] (see [Bo, section 5]). To avoid this difficulty, Slaminka ([Sl]) modifies both the homeomorphism h and the simple closed curve C. We follow his strategy and adapt his proof in such a way it works for every value of the Lefschetz index of p.

Theorem 3.4 *Let p be an isolated fixed point of an orientation preserving planar homeomorphism h and a circle C bounding a closed disc D such that $p \in \overset{\circ}{D}$, $D \cap$ Fix(h) $= \{p\}$.*

Then there exist a free modification h' of h and a simple closed curve C' bounding a topological closed disc $D' \subset D$ such that

(1) $p \in \overset{\circ}{D}{}'$,

(2) h' is in a canonical position on C'.

Proof : The condition *(C1)* can be obtained adapting the proof of theorem A1 in [Ep, Appendix]. These details are left to the reader. Then we follow the proof in [Sl] from "Reduction to a connected component", page 434. We consider the disc F_0 bounded by μ and γ, as described page 436. Observe that F_0 satisfies the hypothesis of Theorem 2.1 (see [Sl, figure 8]) and $F_0 \cap$ Fix$(h) = \emptyset$ then (as asserted page 437) there exists an integer $n \geq 1$ such that

$$(\star) \quad h^{-n}(F_0) \cap h^{-n+1}(F_0) \cap \ldots \cap F_0 = \emptyset$$

but this property depends no more on Brown's lemma and is now valid for every value of the fixed point index. Furthermore, it is true but not obvious that property $(\star)$ is sufficient to remove the intersection of γ with C : since [Sl] contains no proof, the reader is referred to [Bo, lemme 5.4]. Performing a finite number of removals, we are reduced to the case where *(C1)* and *(C2)* are satisfied. Afterwards, proceed as explained in [Sl]. One can also obtain *(C3)* following [PS] from "reduction to canonical form", page 471. □

Remark 3.5 The reader can observe that the removal of the arcs such as γ above has to be inserted in [PS] to complete the proof.

Remark 3.6 Given a neighbourhood U of p, we can choose the disc D small enough to have $h|_{\mathbb{R}^2 \setminus U} = h'|_{\mathbb{R}^2 \setminus U}$. Furthermore, there exists a neighbourhood V of p such that $h|_V = h'|_V$.

Using Theorem 3.4, we can confirm the following result:

Theorem 3.7 (Schmitt's Theorem, see [Sc]) *For every $n \in \mathbb{Z}$, the space $\mathcal{H}_n$ is path connected.*

Proof : Just replace in [Sc] the process to obtain a canonical position by Theorem 3.4. □

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Université Paris 13
Institut Galilée, Dépt de Mathématiques
Avenue J.B. Clément
93430 Villetaneuse, France

bonino@math.univ-paris13.fr