Classification of upper motives of algebraic groups of inner type A_n

Charles De Clercq

Abstract

Let A, A' be two central simple algebras over a field F and $\mathbb{F}$ be a finite field of characteristic p. We prove that the upper indecomposable direct summands of the motives of two anisotropic varieties of flags of right ideals $X(d_1,...,d_k;A)$ and $X(d'_1,...,d'_{k'};A')$ with coefficients in $\mathbb{F}$ are isomorphic if and only if the p-adic valuations of $\gcd(d_1,...,d_k)$ and $\gcd(d'_1,..,d'_{k'})$ are equal and the classes of the p-primary components A_p and A'_p of A and A' generate the same group in the Brauer group of F. This result leads to a surprising dichotomy between upper motives of absolutely simple adjoint algebraic groups of inner type A_n.

Résumé

Soient A, A' deux algèbres centrales simples sur un corps F et $\mathbb{F}$ un corps fini de caractéristique p. Nous prouvons que les facteurs directs indécomposables supérieurs des motifs de deux variétés anisotropes de drapeaux d'idaux droite $X(d_1,...,d_k;A)$ et $X(d'_1,...,d'_{k'};A')$ à coefficients dans $\mathbb{F}$ sont isomorphes si et seulement si les valuations p-adiques de $\mathrm{pgcd}(d_1,...,d_k)$ et $\mathrm{pgcd}(d'_1,..,d'_{k'})$ sont égales et les classes des composantes p-primaires A_p et A'_p de A et A' engendrent le même sous-groupe dans le groupe de Brauer de F. Ce résultat mène à une surprenante dichotomie entre les motifs supérieurs des groupes algébriques absolument simples, adjoints et intérieurs de type A_n.

1. Introduction

Throughout this note p will be a prime and $\mathbb{F}$ will be a finite field of characteristic p. Let F be a field and $\mathrm{CM}(F;\mathbb{F})$ be the category of Grothendieck Chow motives with coefficients in $\mathbb{F}$. Motivic properties of projective homogeneous F-varieties and their relations with classical discrete invariants have been intensively studied recently (see for example [7], [11], [12], [13], [14], [15]). In this article we deal with the particular case of projective homogeneous F-varieties under the action of an absolutely simple affine adjoint algebraic group of inner type A_n. More precisely we prove the following result:

Theorem 1.1 *Let A and A' be two central simple F-algebras. The upper indecomposable direct summands of the motives of two anisotropic varieties of flags of right ideals $X(d_1,...,d_k;A)$ and $X(d'_1,...,d'_{k'};A')$ in $\mathrm{CM}(F;\mathbb{F})$ are isomorphic if and only if $v_p(\gcd(d_1,...,d_k)) = v_p(\gcd(d'_1,..,d'_{k'}))$ and the p-primary components A_p and A'_p of A and A' generate the same subgroup of $\mathrm{Br}(F)$.*

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In §1 we recall classical discrete invariants and varieties associated to central simple F-algebras, while §2 is devoted to the theory of upper motives. Finally we prove theorem 1.1 in §3, using an index reduction formula due to Merkurjev, Panin and Wadsworth and the theory of upper motives. Theorem 1.1 gives a purely algebraic criterion to compare upper direct summands of varieties of flags of ideals, and leads to a quite unexpected dichotomy between upper motives of absolutely simple adjoint algebraic groups of inner type A_n.

2. Generalities on central simple algebras

Our reference for classical notions on central simple F-algebras is [9]. A finite-dimensional F-algebra A is a central simple F-algebra if its center $Z(A)$ is equal to F and if A has no non-trivial two-sided ideals. The square root of the F-dimension of A is the *degree* of A, denoted by $\deg(A)$. Two central simple F-algebras A and B are *Brauer-equivalent* if $M_n(A)$ and $M_m(B)$ are isomorphic for some integers n and m, and the *Schur index* $\mathrm{ind}(A)$ of a central simple F-algebra A is the degree of the (uniquely determined up to isomorphism) central division F-algebra Brauer-equivalent to A. The tensor product endows the set $\mathrm{Br}(F)$ of equivalence classes of central simple F-algebras under the Brauer equivalence with a structure of a torsion abelian group. The exponent of A, denoted by $\exp(A)$, is the order of the class of A in $\mathrm{Br}(F)$ and divides $\mathrm{ind}(A)$.

Let A be a central simple F-algebra and $0 \leq d_1 < ... < d_k \leq \deg(A)$ be a sequence of integers. A convenient way to define the variety of flags of right ideals of reduced dimension d_1,..., d_k in A is to use the language of functor of points. For any commutative F-algebra R, the set of R-points $\mathrm{Mor}_F\,(\mathrm{Spec}(R), X(d_1, ..., d_k; A))$ consists of the sequences $(I_1, ..., I_k)$ of right ideals of the Azumaya R-algebra $A \otimes_F R$ such that $I_1 \subset ... \subset I_k$, the injection of A_R modules $I_s \to A_R$ splits and the rank of the R-module I_s is equal to $d_s \cdot \deg(A)$ for any $1 \leq s \leq k$. For any morphism $R \to S$ of F-algebras the corresponding map from R-points to S-points is given by $(I_1, ..., I_k) \mapsto (I_1 \otimes_R S, ..., I_k \otimes_R S)$. Two important particular cases of varieties of flags of right ideals are the classical Severi-Brauer variety $X(1; A)$, and the generalized Severi Brauer varieties $X(i; A)$.

If G is an algebraic group and X a projective G-homogeneous F-variety, we say that X is *isotropic* if X has a zero-cycle of degree coprime to p, and X is *anisotropic* if X is not isotropic. If $X = X(d_1, ..., d_k; A)$ is a variety of flags of right ideals, X is isotropic if and only if $v_p(\gcd(d_1, ..., d_k)) \geq v_p(\mathrm{ind}(A))$. Note that if the Schur index of A is a power of p, X is isotropic if and only if X has a rational point.

3. The theory upper motives

Our basic references for the definitions and the main properties of Chow groups with coefficients and the category $\mathrm{CM}(F; \Lambda)$ of Grothendieck Chow motives with coefficients in a commutative ring Λ are [2] and [5]. In the sequel G will be a semisimple affine adjoint algebraic group of inner type, X a projective G-homogeneous F-variety and Λ will be assumed to be a finite and connected ring. By [3] (see also [8]) the motive of X decomposes in a unique way (up to isomorphism) as a direct sum of indecomposable motives under these assumptions. Among all the indecomposable direct summands in the complete motivic decomposition of X, the (uniquely determined up to isomorphism) indecomposable direct summand M such that the 0-codimensional Chow group of M is non-zero is the *upper motive* of X.

Upper motives are essential : any indecomposable direct summand in the complete motivic decomposition of X is the upper motive of another projective G-homogeneous F-variety by [8, Theorem 3.5]. This structural result implies that the study of the motivic decomposition of a projective G-homogeneous F-variety X is reduced to the case $\Lambda = \mathbb{F}_p$. Indeed by [16, Corollary 2.6] the complete motivic decomposition of X with coefficients in Λ remains the same when passing to the residue field of Λ, and is also the same as if the ring of coefficients is $\mathbb{F}_p$ by [4, Theorem 2.1], where p is the characteristic of the residue field of Λ. These results motivate the study of the set $\mathfrak{X}_G$ of *upper p-motives* of the algebraic group G, which consists of the isomorphism classes of upper motives of projective G-homogeneous F-varieties in $\mathrm{CM}(F;\mathbb{F}_p)$. Furthermore the dimension of the upper motive of X in $\mathrm{CM}(F;\mathbb{F}_p)$ is equal to the canonical p-dimension of X by [6, Theorem 5.1], hence upper motives encode important information on the underlying varieties. Upper motives also have good properties : the upper motives of two projective G-homogeneous F-varieties X and X' in $\mathrm{CM}(F;\mathbb{F})$ are isomorphic if and only if both $X_{F(X')}$ and $X'_{F(X)}$ are isotropic by [8, Corollary 2.15]. The variety X is isotropic if and only if the upper motive of X is isomorphic to the *Tate motive* (that is to say the motive of $\mathrm{Spec}(F)$) and this is why we focus in this note on the case of anisotropic varieties of flags of right ideals.

If G is absolutely simple adjoint of inner type A_n, G is isomorphic to $\mathrm{PGL}_1(A)$, where A is a central simple F-algebra of degree $n+1$. Any projective G-homogeneous F-variety is then isomorphic to a variety $X(d_1,...,d_k;A)$ of flags of right ideals in A (see [10]) thus theorem 1.1 classifies upper motives of absolutely simple affine adjoint algebraic groups of inner type A_n. In the particular case of classical Severi-Brauer varieties theorem 1.1 corresponds to [1, Theorem 9.3], since for any field extension E/F a central simple F-algebra becomes split over E if and only if the Severi-Brauer variety $X(1;A_E)$ has a rational point.

4. Main results

Let D be a central division F-algebra of degree p^n. For any $0 \le k \le n$, $M_{k,D}$ will denote the upper indecomposable direct summand of $X(p^k;D)$ in $\mathrm{CM}(F;\mathbb{F})$. If D' is another central division F-algebra of degree p^n and j satisfies $1 \le j \le p^n$, we denote the integer $\frac{p^k}{\gcd(j,p^k)} \cdot \mathrm{ind}(D \otimes D'^{-j})$ by $\mu_{k,j}^{D,D'}$. In the sequel the following index reduction formula (see [10, p. 565]) will be of constant use :

$$\mathrm{ind}(D_{F(X(p^k;D'))}) = \underset{1\le j\le p^n}{\gcd}\, \mu_{k,j}^{D,D'} = \min_{1\le j\le p^n} \mu_{k,j}^{D,D'}$$

Proposition 4.1 *Let D and D' be two central division F-algebras of degree p^n. Assume that $\exp(D) \ge \exp(D')$ and that $X(p^k;D)_{F(X(p^k;D'))}$ is isotropic for some integer $0 \le k < n$. If $\mathrm{ind}(D_{F(X(k;D'))}) = \mu_{k,j_0}^{D,D'}$, j_0 is coprime to p.*

Proof. Suppose that p divides j_0 and $\mathrm{ind}(D_{F(X(k;D'))}) = \mu_{k,j_0}^{D,D'}$. By assumption $X(k;D)_{F(X(k;D'))}$ has a rational point, hence the integer $\mu_{k,j_0}^{D,D'}$ divides p^k by [9, Proposition 1.17] and $\mathrm{ind}(D \otimes D'^{-j_0}) \mid \gcd(j_0,p^k)$. Since p divides j_0, $\exp(D'^{-j_0}) < \exp(D')$, therefore $\exp(D'^{-j_0}) < \exp(D)$ and $\exp(D) = \exp(D \otimes D'^{-j_0})$. It follows that $\exp(D)$ divides j_0, thus $\exp(D')$ also divides j_0. The central simple F-algebra D'^{j_0} is therefore split and $D \otimes D'^{-j_0}$ is Brauer-equivalent to D so that $\mathrm{ind}(D)$ divides p^k, a contradiction. □

Theorem 4.2 *Let $\mathbb{F}$ be a finite field of characteristic p and D, D' be two central division F-algebras of degree p^n. The following assertions are equivalent :*

1) for some integer $0 \leq k < n$*,* $M_{k,D}$ *and* $M_{k,D'}$ *are isomorphic in* $\mathrm{CM}(F;\mathbb{F})$*;*
2) the classes of D *and* D' *generate the same subgroup of* $\mathrm{Br}(F)$*;*
3) for any $0 \leq k < n$*,* $M_{k,D}$ *is isomorphic to* $M_{k,D'}$ *in* $\mathrm{CM}(F;\mathbb{F})$*.*

Proof. We first show that 1) $\Rightarrow$ 2). We may exchange D by D' and thus assume that $\exp(D)$ is greater than $\exp(D')$. Since $M_{k,D}$ is isomorphic to $M_{k,D'}$, there is an integer j_0 coprime to p such that the Schur index of $D \otimes D'^{-j_0}$ is equal to 1 by [9, Proposition 1.17] and proposition 4.1, hence $D \otimes D'^{-j_0}$ is split and the class of D is equal to the class of D'^{j_0} in $\mathrm{Br}(F)$. Furthermore since j_0 is coprime to p the class of D in $\mathrm{Br}(F)$ is also a generator of the subgroup of $\mathrm{Br}(F)$ generated by $[D']$.

Now we show that 2) $\Rightarrow$ 3) : if $[D]$ and $[D']$ generate the same group in $\mathrm{Br}(F)$, $\mathrm{ind}(D_E) = \mathrm{ind}(D'_E)$ for any field extension E/F. Given an integer $0 \leq k < n$, since $X(p^k;D)$ has a rational point over $F(X(p^k;D))$, $\mathrm{ind}(D'_{F(X(p^k;D))}) = \mathrm{ind}(D_{F(X(p^k;D))})$ divides p^k. The variety $X(p^k;D')$ then also has a rational point over $F(X(p^k;D))$ by [9, Proposition 1.17]. The same procedure replacing D by D' shows that $X(p^k;D)$ has a rational point over $F(X(p^k;D'))$, hence $M_{k,D}$ is isomorphic to $M_{k,D'}$.

Finally 3) $\Rightarrow$ 1) is obvious. □

Corollary 4.3 *Let* D *and* D' *be two central division* F*-algebras of degree* p^n *and* $p^{n'}$*. The upper summands* $M_{k,D}$ *and* $M_{k',D'}$ *are isomorphic for some integers* $0 \leq k < n$ *and* $0 \leq k' < n'$ *if and only if* $k = k'$ *and the classes of* D *and* D' *generate the same subgroup of* $\mathrm{Br}(F)$*.*

Proof. Since by [8, Theorem 4.1] the generalized Severi-Brauer varieties $X(p^k;D)$ and $X(p^{k'};D')$ are p-incompressible, if $M_{k,D}$ and $M_{k',D'}$ are isomorphic, the dimension of $X(p^k;D)$ (which is $p^k(p^n - p^k)$) is equal to the dimension of $X(p^{k'};D')$. The equality $p^k(p^n - p^k) = p^{k'}(p^{n'} - p^{k'})$ implies that $k = k'$, $n = n'$ and it remains to apply theorem 4.2. The converse is clear by theorem 4.2. □

Proof of theorem 1.1. Set $X = X(d_1, ..., d_k; A)$, $X' = X(d_1, ..., d_{k'}; A')$, and also $v = v_p(\gcd(d_1, ..., d_k))$ and $v' = v_p(\gcd(d'_1, .., d'_{k'}))$. If D and D' are two central division F-algebras Brauer-equivalent to A_p and A'_p, the upper indecomposable direct summand of X (resp. of X') is isomorphic to $M_{v,D}$ (resp. to $M_{v',D'}$) by [8, Theorem 3.8]. By corollary 4.3 these summands are isomorphic if and only if $v = v'$ (since X and X' are anisotropic) and the classes of A_p and A'_p generate the same subgroup of $\mathrm{Br}(F)$. □

Theorem 4.4 *Let* G *and* G' *be two absolutely simple affine adjoint algebraic groups of inner type* A_n *and* $A_{n'}$*. Then either* $\mathfrak{X}_G \cap \mathfrak{X}_{G'}$ *is reduced to the class of the Tate motive or* $\mathfrak{X}_G = \mathfrak{X}_{G'}$*.*

Proof. If $\mathfrak{X}_{\mathrm{PGL}_1(A)} \cap \mathfrak{X}_{\mathrm{PGL}_1(A')}$ is not reduced to the class of the Tate motive, there are two anisotropic varieties of flags of right ideals $X = X(d_1, ..., d_k; A)$ and $X' = X(d'_1, ..., d'_{k'}; A')$ whose upper motives are isomorphic. By theorem 1.1 this implies that the upper p-motive of any anisotropic $\mathrm{PGL}_1(A)$-homogeneous F-variety $X(d_1, ..., d_{\tilde{k}}; A)$ is isomorphic to, say, the upper p-motive of $X(d_1, ..., d_{\tilde{k}}; A')$. □

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Charles De Clercq declercq@math.jussieu.fr
Université Paris VI, 4 place Jussieu, 75252 Paris CEDEX 5