CRITICAL VARIETIES AND MOTIVIC EQUIVALENCE FOR ALGEBRAS WITH INVOLUTION

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Abstract. Motivic equivalence for algebraic groups was recently introduced in [6], where a characterization of motivic equivalent groups in terms of higher Tits indices is given. As a consequence, if the quadrics associated to two quadratic forms have the same Chow motives with coefficients in $\mathbb{F}_2$, this remains true for any two projective homogeneous varieties of the same type under the orthogonal groups of those two quadratic forms. In the first part of the paper, we extend this result to all groups of classical type, introducing a notion of critical variety. On the way, we prove that motivic equivalence of the automorphism groups of two involutions on a given algebra can be checked after extending scalars to an index reduction field, which depends on the type of the involutions. The second part of the paper describes conditions on the base field which guarantee that motivic equivalent involutions actually are isomorphic, extending a result of Hoffmann on quadratic forms.

1. Introduction and notations

André Weil proved in the 60's that classical algebraic groups can be described in terms of automorphism groups of central simple algebras with involution [25]. This is a key tool for important results both on algebraic groups, and on related algebraic objects, such as algebras with involution, or quadratic and hermitian forms. As a major example, we mention Bayer and Parimala's proof of Serre's conjecture II [2] and of the Hasse principle conjecture II [3], which provide a description of the first cohomology group of a simply connected classical group when the base field has cohomological dimension ≤ 2, or virtual cohomological dimension ≤ 2, respectively. Besides Galois cohomology, another important tool must be added to this picture, namely the study of projective homogeneous varieties under a semisimple algebraic group. This led, for instance, to the so-called index reduction formulae, due to Merkurjev, Panin and Wadsworth [19] [20].

In this paper, we use a scalar extension to an index reduction field to study the Chow motives of projective homogeneous varieties under an algebraic group of classical type, and the relation to cohomological invariants of associated algebraic objects. More precisely, recall the notion of motivic equivalence modulo 2 for algebraic groups introduced by the first named author in [6] : two algebraic groups which are inner twisted forms of the same quasi-split group are called motivic equivalent modulo 2 if the projective homogeneous varieties of the same type have the same Chow motives with coefficients in $\mathbb{F}_2$. A characterization of motivic equivalent groups is also provided in the same paper, in terms of higher Tits indices, which consist of the collection of Tits indices of the given group over all field extensions of the base field. Combining this result with the anisotropy results of Karpenko

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and Karpenko-Zhykhovich ([14], [15]), which can be rephrased in terms of 2-Witt indices of involutions (see Lemma 2.5), we prove that motivic equivalence modulo 2 for classical groups can be checked after an index reduction process, thus reducing the question to quadratic form theory (see Proposition 3.9). This is the main tool of our paper, from which we derive two consequences.

If the quadrics associated to two quadratic forms have the same Chow motives with coefficients in $\mathbb{F}_2$, this remains true for all projective homogeneous varieties under the orthogonal groups of those two quadratic forms by [6]. As a first consequence of Proposition 3.9, we extend this result to algebraic groups of classical type. More precisely, we introduce a projective homogeneous variety, called a critical variety, and depending only on the type of the group, which plays the same role as a quadric for an orthogonal group, that is which is enough to track down motivic equivalence (see Def 3.5 and Thm 3.6). Hence, given two involutions of the same type on a central simple algebra, if the corresponding critical varieties have the same Chow motive with coefficients in $\mathbb{F}_2$, then this remains true for all projective homogeneous varieties under the automorphism groups of both involutions. As a consequence, we also prove a generalization of Vishik's criterion for motivic equivalence of quadrics (see Corollary 3.8).

The relation between motivic equivalence and isomorphism has already been explored by several authors. Izhboldin proved in [10] that motivic equivalent odd dimensional quadratic forms are similar, so that the corresponding orthogonal groups are isomorphic. He also provided examples, in even dimension, where this is not the case anymore. Therefore, one may ask under which conditions on the base field one can guarantee that motivic equivalent algebraic groups are isomorphic, or, equivalently, motivic equivalent involutions are isomorphic. In the split orthogonal case, that is for quadratic forms, this question was solved by Hoffmann in [9]. In the second part of this paper, we extend his results to arbitrary involutions. The proof uses the fact that motivic equivalent involutions have the same cohomological invariants of degree 1 and 2, which we easily derive from Proposition 3.9, combined with some classification theorems for involutions due to Lewis and Tignol [18].

Notations. Throughout this paper, we work over a base field of characteristic different from 2. Given an involution σ on a central simple algebra A of degree n over a field K, we denote by F the subfield of K fixed by σ, hence $K = F$ if σ is orthogonal or symplectic, and $K = F(\sqrt{\delta})$ is a quadratic field extension of F if σ is unitary. We also allow A to be a direct product of two central simple algebras over F, so that its center is the quadratic étale algebra $K = F \times F$, endowed with an involution which acts on $F \times F$ as the unique non-trivial F-automorphism. If so, the involution induces an anti-automorphism between the two components of A. In all cases, A admits no non trivial σ-stable two-sided ideal, and we will say that (A, σ) is a central simple algebra with involution over F, or an algebra with involution over F, for short. With this convention, for any field extension L/F, the pair (A_L, σ_L) defined by $A_L = A \otimes_F L$ and $\sigma_L = \sigma \otimes \mathrm{Id}$ is an algebra with involution over L. We refer the reader to [16] for more details on algebras with involution and on the corresponding algebraic groups. In particular, we let $\mathrm{PSim}^+(A, \sigma)$ be the connected component of the identity in the automorphism group of (A, σ). It is a semisimple F-algebraic group, which is of type A, B, C or D depending on the degree of the algebra A and on the type of the involution σ. Conversely, by [25] (see also [16, §26]), any algebraic group of classical type over F is isogeneous to

$\mathrm{PSim}^+(A,\sigma)$ for some algebra with involution (A,σ) over F. If (A,σ) and (A,τ) both are algebras with involution over F, and σ and τ are of the same type, we will say that (A,σ,τ) is an algebra with two involutions of the same type over F. Assuming in addition for orthogonal involutions that their respective discriminants d_σ and d_τ coincide, the corresponding groups $\mathrm{PSim}^+(A,\sigma)$ and $\mathrm{PSim}^+(A,\tau)$ are inner twisted forms of the same quasi-split group.

All the quadratic forms we consider are supposed to be non degenerate. The invariants are as defined in [23], [17]. In particular, the discriminant $d_\varphi \in F^\times/F^{\times 2}$ of a quadratic form φ over F is a signed discriminant, and the Clifford invariant $c(\varphi)\in \mathrm{Br}(F)$, called the Witt invariant in Lam's book, see [17, Chap.3, (3.12)], is the Brauer class of the full Clifford algebra $\mathcal{C}(\varphi)$ if φ has even dimension, and of its even part $\mathcal{C}_0(\varphi)$ if φ has odd dimension. The Witt index of q is denoted by $i_w(q)$.

Given an algebra with involution (A,σ) over F, we call Witt-index of σ, and we denote by $i_w(\sigma)$, the reduced dimension of the maximal totally isotropic right ideals in (A,σ). Therefore, $i_w(\sigma)$ is the largest element in the index of (A,σ) as defined in [16, §6.A]. In addition, we call Witt 2-index of (A,σ), denoted by $i_{w,2}(\sigma)$, the maximal value of $i_w(\sigma_{F'})$, where F' runs over all odd degree field extensions of F. As explained in [7], the Tits-index (respectively the 2-Tits index) of $\mathrm{PSim}^+(A,\sigma)$ is uniquely determined by the Schur index of the algebra A and by $i_w(\sigma)$ (respectively by the 2-primary part of the Schur index of A and $i_{w,2}(\sigma)$).

If A has even degree, and σ has orthogonal type, we let $d_\sigma \in F^\times/F^{\times 2}$ be the discriminant of σ [16, (7.2)], and $\mathcal{C}(\sigma)$ be the Clifford algebra of (A,σ) [16, (8.7)]. We denote by K_σ/F the quadratic étale extension associated to d_σ, which also is the center of $\mathcal{C}(\sigma)$ [16, (8.10)]. If $d_\sigma = 1\in F^\times/F^{\times 2}$, then $\mathcal{C}(\sigma)$ is a direct product of two central simple algebras over F, denoted by $\mathcal{C}_+(\sigma)$ and $\mathcal{C}_-(\sigma)$. By the so-called fundamental relations [16], we have $[\mathcal{C}_+(\sigma)]-[\mathcal{C}_-(\sigma)]=[A]\in \mathrm{Br}(F)$. The Clifford invariant of σ is

$$c(\sigma)=[\mathcal{C}_+(\sigma)]=[\mathcal{C}_-(\sigma)]\in \mathrm{Br}(F)/\langle [A]\rangle.$$

If A has even degree, and σ is unitary, we let $\mathcal{D}(\sigma)$ be the discriminant algebra of σ, which is a central simple algebra over F [16, (10.28)].

Assume now that the field F is formally real, and denote by X_F the space of orderings of F. For all $P\in X_F$, we let $\mathrm{sign}_P(\sigma)$ be the signature of σ at the ordering P (see [16, (11.10)(11.25)] and [1, §4]). By definition, $\mathrm{sign}_P(\sigma)$ is a positive integer, whose square is equal to the signature at P of the trace form of (A,σ). If σ is K/F-unitary, with $K=F(\sqrt{\alpha})$, then

$$\mathrm{sign}_P(\sigma)=0 \text{ for all } P\in X_F \text{ such that } \alpha >_P 0. \tag{1}$$

Indeed, over a real closure F_P of F at such an ordering, we have $K\otimes_F F_P = F_P\times F_P$, so that all hermitian forms with values in (K,ι) become hyperbolic over F_P (see [1, 3.1(d)]). This applies in particular to the trace form of (A,σ). Therefore, as noticed in [1], one may extend the definition given in [16, § 11] to the orderings of F that do not extend to K, with the convention above.

The motives considered in this paper are Chow motives with coefficients in a ring Λ, see for instance [8]. In particular, if X is a variety over F, we denote in the sequel by $M(X)$ the motive associated to X with coefficient ring $\Lambda=\mathbb{F}_2$.

2. Involutions, low index algebras and index reduction fields

Assume $A = \mathrm{End}_F(V)$ is a split algebra. Any orthogonal involution on A is adjoint to some quadratic form $q : V \to F$. Conversely, two quadratic forms give rise to the same involution if and only if they are isomorphic up to a scalar factor. Hence, the study of orthogonal involutions in this case boils down to quadratic form theory, or more precisely to the study of quadratic forms up to similarities. In particular, all invariants of an orthogonal involution on a split algebra can be computed in terms of invariants of quadratic forms. As we now proceed to explain, this is also true for symplectic involutions on an index at most 2 algebra and unitary involutions on a split algebra. Moreover, most invariants of involutions can be computed after an index reduction process, so that they can be expressed in terms of some quadratic form invariants.

2.1. **Invariants of involutions.** Let us first recall precisely the situation in the split orthogonal case. For any quadratic form $q : V \to F$, we denote by $\mathrm{Ad}_q = (\mathrm{End}_F(V), \mathrm{ad}_q)$ the associated algebra with involution. It is well known that invariants of orthogonal involutions extend invariants of quadratic forms. More precisely, we have :

Lemma 2.1. *Let q be an even-dimensional quadratic form over F, and let $\sigma = \mathrm{ad}_q$ be the corresponding adjoint involution.*

(a) $i_w(\sigma)= i_{w,2}(\sigma) = i_w(q)$;
(b) $d_\sigma = d_q \in F^\times/F^{\times 2}$;
(c) $\mathcal{C}(\sigma)$ *and* $\mathcal{C}_0(q)$ *are canonically isomorphic;*
(d) $c(\sigma_K) = c(q_K)$, *where K is the discriminant quadratic extension;*
(e) For all $P \in X_F$, $\mathrm{sign}_P(\sigma) = |\,\mathrm{sign}_P(q)|$.

Remark 2.2. The Clifford algebra of an involution, denoted here by $\mathcal{C}(\sigma)$, corresponds in the split case to the even part $\mathcal{C}_0(q)$ of the Clifford algebra of q. The full Clifford algebra $\mathcal{C}(q)$ of the quadratic form q is not an invariant up to similarities; therefore, it does not give rise to an invariant of σ. Likewise, $\mathrm{sign}(q)$ is not an invariant up to similarities, hence the absolute value is required in (e).

Proof. In the situation considered here, the underlying division algebra is $D = F$, therefore by [16, p.73], the maximal element of the index of (A, σ) is the Witt-index $i_w(q)$, hence $i_w(\sigma) = i_w(q)$. Moreover, for any odd degree field extension F'/F, by Springer's theorem for quadratic forms (see e.g. [17, Chap. VII, Thm. 2.7]), we have

$$i_w(\sigma_{F'}) = i_w(q_{F'}) = i_w(q) = i_w(\sigma).$$

Hence $i_{w,2}(\sigma) = i_w(\sigma)$ and (a) is proved. Assertions (b) and (c) are given in [16, (7.3)(3) and (8.8)]. Since K is the discriminant quadratic extension, in order to prove (d), it is enough to check that $c(\sigma) = c(q)$ when the discriminants are trivial. Under this assumption, we have $\mathcal{C}(q) \simeq M_2(C)$ for some central simple algebra C over F, and $\mathcal{C}_0(q) \simeq C \times C$ (see e.g. [17, Chap. V, Thm 2.5(3)]). Therefore $[\mathcal{C}_+(\sigma)] = [\mathcal{C}_-(\sigma)] = [C] = [\mathcal{C}(q)] \in \mathrm{Br}\, F$, which gives the required equality. Finally, if P is any ordering of F, we have $\mathrm{sign}_P(\sigma) = |\,\mathrm{sign}_P(q_\sigma)|$ by [16, (11.10) and (11.7)]. □

Let us assume now that either σ is unitary and $A = M_n(K)$, or σ is symplectic and $A = M_m(H)$ for some quaternion algebra H over F. Let n be the degree of

A, that is $n = 2m$ if $A = M_m(H)$. In both cases, there exists a $2n$-dimensional quadratic form q_σ over F, which is unique up to a scalar factor, and whose invariants are related to the invariants of σ, as we now proceed to show.

A unitary involution σ on the split algebra $M_n(K)$ is adjoint to a rank n hermitian form, denoted by h_σ, with values in $(K, \bar{\ })$, where $\bar{\ }$ denotes the non-trivial F-automorphism of the quadratic extension K/F. Similarly, a symplectic involution σ of the algebra $M_m(H)$ is adjoint to a rank m hermitian form, still denoted by h_σ, with values in $(H, \bar{\ })$, where $\bar{\ }$ stands for the canonical involution of H. Let q_σ be the trace form of h_σ, that is the quadratic form defined on K^n (respectively H^m), now viewed as an F vector space, by $q_\sigma(x) = h_\sigma(x, x)$. It has dimension $2n$ in the unitary case, and $4m = 2n$ in the symplectic case. Moreover, as explained in [23, Chap.10], the hermitian form h_σ is uniquely determined by its trace form q_σ. Invariants of σ and invariants of q_σ are related as follows:

Proposition 2.3. *Let σ be either a unitary involution of a split algebra $M_n(K)$ or a symplectic involution of an algebra $M_m(H)$ of index at most 2, and consider as above a trace form q_σ of the underlying hermitian form. This form is well defined up to a scalar factor, and its invariants relate to those of σ as follows :*

- *(a)* $i_w(\sigma) = i_{w,2}(\sigma) = \frac{1}{2} i_w(q_\sigma)$;
- *(b) If A has even degree and σ is unitary,*

$$d_{q_\sigma} = 1 \in F^\times/F^{\times 2} \text{ and } [\mathcal{D}(\sigma)] = c(q_\sigma) \in \mathrm{Br}(F);$$

- *(c) For all $P \in X_F$,* $\mathrm{sign}_P(\sigma) = \frac{1}{2}|\,\mathrm{sign}_P(q_\sigma)|$.

Proof. By the same argument as in the orthogonal case, based on Springer's theorem for quadratic forms, to prove (a), it is enough to prove the equality $i_w(\sigma) = \frac{1}{2} i_w(q_\sigma)$. Pick an arbitrary diagonalisation $h_\sigma = \langle \alpha_1, \ldots, \alpha_r \rangle$ of h_σ, where $r = n$ in the unitary case, and $r = m = \frac{n}{2}$ in the symplectic case. The elements α_i are symmetric elements of $(K, \bar{\ })$ or $(H, \bar{\ })$ depending on the type of σ; hence in both cases, they belong to F. An easy computation now shows that

$$q_\sigma \simeq \begin{cases} \langle 1, -\delta \rangle \otimes \langle \alpha_1, \ldots, \alpha_n \rangle & \text{if } \sigma \text{ is unitary;} \\ n_H \otimes \langle \alpha_1, \ldots, \alpha_m \rangle & \text{if } \sigma \text{ is symplectic,} \end{cases}$$

where n_H denotes the norm form of the quaternion algebra H. Since h_σ is uniquely defined up to a scalar factor, the same holds for q_σ.

If $K = F \times F$, that is $\delta = 1$, or H is split, then the involution σ is hyperbolic and $i_w(\sigma) = \frac{1}{2}\deg(A)$. On the other hand, under those assumptions, $\langle 1, -\delta \rangle$ and n_H are hyperbolic, hence q_σ also is, and $i_w(q_\sigma) = \frac{1}{2}\dim(q_\sigma) = \deg(A)$. Therefore (a) holds in this case. Moreover, for any ordering P of F, we have $1 >_P 0$, hence $\mathrm{sign}_P(\sigma) = 0$ by (1) in the unitary case and [16, (11.7)] in the symplectic case. Therefore, since q_σ is hyperbolic, (c) also holds in this case.

Assume now that K is a field or H is division, depending on the type of the involution. Combining the explicit description of q_σ above with [23, Chap.10, Thm1.1 and 1.7], one may observe that any Witt decomposition of the hermitian form h_σ gives rise to a Witt decomposition of q_σ, so that

$$i_w(q_\sigma) = \begin{cases} 2 i_w(h_\sigma) & \text{if } \sigma \text{ is unitary;} \\ 4 i_w(h_\sigma) & \text{if } \sigma \text{ is symplectic.} \end{cases}$$

On the other hand, from the description of the index of an algebra with involution given in [16, p.73], $i_w(\sigma)$ is equal to $i_w(h_\sigma)$ in the unitary case, and $2 i_w(h_\sigma)$ in the symplectic case, since the index of the underlying algebra is 1 or 2, accordingly. Therefore (a) holds.

In order to prove (c), let us consider an ordering P of the field F, and pick a real closure F_P of F at this ordering. Assume first that σ is unitary. If $\delta >_P 0$, we have $\mathrm{sign}_P(\sigma) = 0$ by (1), hence (1) holds since q_σ is hyperbolic. If $\delta <_P 0$, the diagonalisation above gives $\mathrm{sign}_P(\sigma) = |\,\mathrm{sign}_P(h_\sigma)| = \frac{1}{2}|\,\mathrm{sign}_P(q_\sigma)|$ as required. Assume now σ is symplectic. The quaternion algebra H is split over F_P if and only if its norm form n_H has signature 0. When these conditions hold, we have by [16, (11.11)(2)(b)] $\mathrm{sign}_P(\sigma) = 0 = \mathrm{sign}_P(q_\sigma)$. Otherwise, n_H is positive definite, and $\mathrm{sign}_P(\sigma) = 2|\,\mathrm{sign}_P(h_\sigma)| = \frac{1}{2}|\,\mathrm{sign}_P(q_\sigma)|$. Hence, (c) is now proved.

It only remains to prove assertion (b). Assume A has even degree, $n = 2m$, so that the discriminant algebra $\mathcal{D}(\sigma)$ is well defined. The diagonalisation of q_σ given above shows that $d(q_\sigma) = (-\delta)^{2m} = 1 \in F^\times/F^{\times 2}$. Moreover, by [16, (10.35)], $\mathcal{D}(\sigma)$ is Brauer equivalent to $(\delta, (-1)^m\alpha_1 \dots \alpha_n)$, which in nothing but the Clifford invariant of q_σ. This concludes the proof of the proposition. □

2.2. **Index reduction function fields.** Let A be a central simple algebra of arbitrary index. The discriminant and the Clifford invariant in the even degree orthogonal case, the discriminant algebra in the even degree unitary case, and the signatures of two involutions can be compared after a scalar extension to a well chosen function field, which reduces the index of A. More precisely, given an algebra with involution (A, σ) over F of type t, where $t = o$ (respectively s,u) stands for orthogonal (respectively symplectic, unitary), we consider the field $\mathcal{F}_{A,t}$ and the quadratic form $\mathcal{Q}_\sigma$ over $\mathcal{F}_{A,t}$ defined as follows. We let $\mathcal{F}_{A,o}$ be the function field of the Severi-Brauer variety of A in the orthogonal case, $\mathcal{F}_{A,s}$ be the function field of the generalized Severi-Brauer variety $\mathrm{SB}_2(A)$ of right ideals of reduced dimension 2 in the symplectic case, and $\mathcal{F}_{A,u}$ be the function field of the K/F-Weil transfer of the Severi-Brauer variety of A in the unitary case. Hence, $\mathcal{F}_{A,t}$ is a field extension of F in all three cases, and the algebras $A \otimes_F \mathcal{F}_{A,o}$ and $A \otimes_F \mathcal{F}_{A,u}$ are split, while $A \otimes_F \mathcal{F}_{A,s}$ has index 2 (or is split if A already is). Therefore, by § 2.1, there exists a quadratic form $\mathcal{Q}_\sigma$, defined over $\mathcal{F}_{A,t}$, unique up to a scalar factor, and of dimension n in the orthogonal case, and $2n$ in the symplectic and unitary cases, which determines the involution $\sigma_{\mathcal{F}_{A,t}}$ and its invariants. Using the properties of the field $\mathcal{F}_{A,t}$, we get the following :

Proposition 2.4. *Let (A, σ, τ) be an algebra with two involutions of the same type t over F. Let $\mathcal{F}_{A,t}$ be the index reduction field defined above, and denote by $\mathcal{Q}_\sigma$ and $\mathcal{Q}_\tau$ the quadratic forms over $\mathcal{F}_{A,t}$ respectively associated to $\sigma_{\mathcal{F}_{A,t}}$ and $\tau_{\mathcal{F}_{A,t}}$, as in § 2.1.*

(a) Assume A has even degree and $\mathcal{Q}_\sigma$ and $\mathcal{Q}_\tau$ have the same discriminant. We denote by $\mathcal{K}/\mathcal{F}_{A,t}$ the corresponding quadratic étale algebra, and assume in addition that $c((\mathcal{Q}_\sigma)_\mathcal{K}) = c((\mathcal{Q}_\tau)_\mathcal{K}) \in \mathrm{Br}(\mathcal{K})$. Then, the following hold :

(i) If σ and τ are orthogonal, then

$$d_\sigma = d_\tau \in F^\times/F^{\times 2}$$

$$\textit{and } c(\sigma_K) = c(\tau_K) \in \mathrm{Br}(K)/\langle [A_K] \rangle,$$

where $K = F[X]/(X^2 - d_\sigma)$ is the discriminant quadratic extension.

(ii) If σ and τ are unitary, then

$$[\mathcal{D}(\sigma)] = [\mathcal{D}(\tau)] \in \mathrm{Br}(F).$$

(b) Assume that for all $Q \in X_{\mathcal{F}_{A,t}}$ we have $|\operatorname{sign}_Q(\mathcal{Q}_\sigma)| = |\operatorname{sign}_Q(\mathcal{Q}_\tau)|$. Then

$$\operatorname{sign}_P(\sigma) = \operatorname{sign}_P(\tau)$$

for all $P \in X_F$.

Proof. Let us first prove (a) in the orthogonal case. By lemma 2.1, the assumptions on $\mathcal{Q}_\sigma$ and $\mathcal{Q}_\tau$ guarantee that $\sigma_{\mathcal{F}_{A,o}}$ and $\tau_{\mathcal{F}_{A,o}}$ have the same discriminant, corresponding to the quadratic étale algebra $\mathcal{K}/\mathcal{F}_{A,o}$, and that $c(\sigma_\mathcal{K}) = c(\tau_\mathcal{K})$. Since the discriminant is a functorial invariant, and F is quadratically closed in $\mathcal{F}_{A,o}$, it follows that σ and τ have the same discriminant. Denote by K/F the corresponding quadratic étale algebra. If $d_\sigma = d_\tau = 1 \in F^\times/F^{\times 2}$, so that $K = F \times F$ and $\mathcal{K} = \mathcal{F}_{A,o} \times \mathcal{F}_{A,o}$, then $c(\sigma_\mathcal{K}) = (c(\sigma_{\mathcal{F}_{A,o}}), c(\sigma_{\mathcal{F}_{A,o}})) \in \operatorname{Br}(\mathcal{F}_{A,o}) \times \operatorname{Br}(\mathcal{F}_{A,o})$, and similarly for $\tau_\mathcal{K}$. Therefore the assumptions gives

$$c(\sigma_{\mathcal{F}_{A,o}}) = c(\tau_{\mathcal{F}_{A,o}}) \in \operatorname{Br}(\mathcal{F}_{A,o}).$$

By [21, Cor.2.7], the kernel of the restriction map $\operatorname{Br}(F) \to \operatorname{Br}(\mathcal{F}_{A,o})$ is the subgroup generated by $[A]$, therefore $\operatorname{Br}(F)/\langle [A] \rangle \to \operatorname{Br}(\mathcal{F}_{A,o})$ is injective. Hence, we get

$$c(\sigma) = c(\tau) \in \operatorname{Br}(F)/\langle [A] \rangle$$

which implies (i). Assume now that K is a field. The quadratic algebra $\mathcal{K}$ is the compositum of K and $\mathcal{F}_{A,o}$, or equivalently, the function field over K of the Severi-Brauer variety A_K. Hence, by the same argument as above, the map $\operatorname{Br} K/\langle [A_K] \rangle \to \operatorname{Br} \mathcal{K}$ is injective. Therefore, again in this case,

$$c(\sigma_\mathcal{K}) = c(\tau_\mathcal{K}) \in \operatorname{Br}(\mathcal{K})$$

implies

$$c(\sigma_K) = c(\tau_K) \in \operatorname{Br} K/\langle [A_K] \rangle.$$

Assume now that the assumptions of (a) hold and that σ and τ are unitary. By Proposition 2.3, $\mathcal{Q}_\sigma$ and $\mathcal{Q}_\tau$ have trivial discriminant. Hence $\mathcal{K}$ is the split quadratic étale algebra $\mathcal{F}_{A,u} \times \mathcal{F}_{A,u}$ and $c((\mathcal{Q}_\sigma)_\mathcal{K}) = c((\mathcal{Q}_\tau)_\mathcal{K}) \in \operatorname{Br}(\mathcal{K})$ implies

$$c(\mathcal{Q}_\sigma) = c(\mathcal{Q}_\tau) \in \operatorname{Br}(\mathcal{F}_{A,u}).$$

By Proposition 2.3, we get that $[\mathcal{D}(\sigma_{\mathcal{F}_{A,u}})] = [\mathcal{D}(\tau_{\mathcal{F}_{A,u}})] \in \operatorname{Br}(\mathcal{F}_{A,u})$. Since A admits unitary K/F involutions, its corestriction is split; therefore, by [21, Cor.2.7, Cor.2.12], the restriction map $\operatorname{Br}(F) \to \operatorname{Br}(\mathcal{F}_{A,u})$ is injective (see also [16, Proof of (10.36)]). Hence, we get

$$[\mathcal{D}(\sigma)] = [\mathcal{D}(\tau)] \in \operatorname{Br}(F)$$

as required.

Since any $Q \in X_{\mathcal{F}_{A,t}}$ restricts to an ordering $P \in X_F$, one implication in (b) follows immediately from Lemma 2.1 and Proposition 2.3. To prove the converse, assume $|\operatorname{sign}_Q(\mathcal{Q}_\sigma)| = |\operatorname{sign}_Q(\mathcal{Q}_\tau)|$ for all $Q \in X_{\mathcal{F}_{A,t}}$ and consider an ordering $P \in X_F$. Pick a real closure F_P of F at the ordering P. If σ and τ are orthogonal, then for all ordering P such that A_{F_P} is not split, we have $\operatorname{sign}_P(\sigma) = 0 = \operatorname{sign}_P(\tau)$ by [16, (11.11)]. Consider now $P \in X_F$ such that A_{F_P} is split. The compositum of $\mathcal{F}_{A,o}$ and F_P, which is the function field over F_P of the Severi-Brauer variety of A_{F_P} is a purely transcendental extension of F_P. Therefore, the ordering P extends to this field, and by restriction, there exists an ordering $Q \in \mathcal{F}_{A,o}$ which coincides with P over F. Therefore, $\operatorname{sign}_P(\sigma) = \operatorname{sign}_Q(\sigma_{\mathcal{F}_{A,o}}) = |\operatorname{sign}_Q(\mathcal{Q}_\sigma)|$, and similarly for τ. Hence σ and τ do have the same signature at P for all $P \in X_F$. The argument is similar in the symplectic and unitary cases. If σ and τ are symplectic, they both have trivial signature at P for all $P \in X_F$ such that A_{F_P} is split by [16, (11.11)]. Otherwise, A_{F_P} is Brauer equivalent to $(-1,-1)_{F_P}$, which is the only non split division algebra over F_P. Therefore, $\mathrm{SB}_2(A)$ has a rational point over F_P, its function field is purely transcendental, and the same argument as above applies. Finally, if σ and τ are unitary, their both have trivial signature at any ordering $P \in X_F$ such that $\delta >_P 0$ by (1). Consider now an ordering P such that $\delta <_P 0$. The compositum of F_P and K is the unique non trivial quadratic field extension of F_P, that is an algebraically closed field. Hence, $A \otimes_F F_P$ is split, so the Weil transfer of $\mathrm{SB}(A)$ has a rational point over F and again the same argument concludes the proof. □

As opposed to what happens for the invariants considered in the previous proposition, it is not known whether the Witt indices can be compared after scalar extension to $\mathcal{F}_{A,t}$. Nevertheless, we have the following, which is precisely what we need in the sequel, and which can be thought of as a reformulation of Thm 1, Thm A in [14] and Thm 6.1 in [15]:

Lemma 2.5. *For any involution σ of type t on the algebra A, we have*

$$i_{w,2}(\sigma) = i_{w,2}(\sigma_{\mathcal{F}_{A,t}}) = i_w(\sigma_{\mathcal{F}_{A,t}}),$$

where $\mathcal{F}_{A,t}$ is the function field defined above. Moreover, this index coincide with $i_w(\mathcal{Q}_\sigma)$ if σ is orthogonal, and $\frac{1}{2}i_w(\mathcal{Q}_\sigma)$ if it is symplectic or unitary.

Proof. By definition of $\mathcal{F}_{A,t}$, Lemma 2.1 or Proposition 2.3 apply to the involution $\sigma_{\mathcal{F}_{A,t}}$, depending on its type. Therefore we have $i_{w,2}(\sigma_{\mathcal{F}_{A,t}}) = i_w(\sigma_{\mathcal{F}_{A,t}})$, and this index coincides with $i_w(\mathcal{Q}_\sigma)$ or $\frac{1}{2}i_w(\mathcal{Q}_\sigma)$ depending on the type of the involution. Since the 2-Witt index can only increase under extension to $\mathcal{F}_{A,t}$, we get

$$i_{w,2}(\sigma) \leq i_{w,2}(\sigma_{\mathcal{F}_{A,t}}) = i_w(\sigma_{\mathcal{F}_{A,t}}).$$

By an easy induction argument, the converse inequality follows from Thm1 and Thm A in [14] and Thm 6.1 in [15], which precisely state that if σ is isotropic over $\mathcal{F}_{A,t}$, then it also is isotropic over some odd degree field extension of F (see also [4, Cor 5.6]). □

3. Motivic equivalence and critical varieties

Let G be a semi-simple algebraic group over F. It is an inner twisted form of a given quasi-split group G_0. Let T_0 be a maximal split torus in G_0. We choose a Borel subgroup of G_0 containing T_0, and we let Δ be the corresponding set of simple roots. We recall from [6, §VI], the definition of the standard motive of G of type Θ with coefficients in Λ, denoted by $M_{\Theta,G}$, where Θ is any subset of Δ. If Θ is invariant under the $\star$-action, $M_{\Theta,G}$ is the motive, with coefficients in Λ, of the variety $X_{\Theta,G}$ of parabolic subgroups of G of type Θ, as defined in [5]. In general, $M_{\Theta,G}$ is the motive of the corestriction from F_Θ to F of the variety $X_{\Theta,G_{F_\Theta}}$, where F_Θ/F is a minimal field extension over which Θ becomes invariant under the $\star$-action. Note that there are two opposite conventions for the parabolic subgroup of type Θ in the literature; in this paper, a Borel subgroup has type Δ. Let (A,σ) be an algebra with involution over F. If $\Theta \subset \Delta$ is invariant under the $\star$-action, we denote by $X_{\Theta,(A,\sigma)}$ the variety of parabolic subgroups of type Θ in $\mathrm{PSim}^+(A,\sigma)$, so that the standard motive of $\mathrm{PSim}^+(A,\sigma)$ of type Θ is the motive of $X_{\Theta,(A,\sigma)}$.

Assume now that G and G' are inner twisted forms of the same quasi-split group G_0. Recall from [6, Def 1] that they are called motivic equivalent with coefficients in Λ if the standard motives $M_{\Theta,G}$ and $M_{\Theta,G'}$ with coefficients in Λ are isomorphic for all $\Theta \subset \Delta$. When this holds for $\Lambda = \mathbb{F}_p$, G and G' are called motivic equivalent modulo p. For classical groups, as we now proceed to show, motivic equivalence modulo 2 actually follows from the isomorphism of standard motives for a given $\Theta \subset \Delta$.

Definition 3.1. Let (A, σ, τ) be an algebra with two involutions of the same type over F. The involutions σ and τ are called motivic equivalent, denoted by $\sigma \overset{m}{\sim} \tau$, if $\mathrm{PSim}^+(A, \sigma)$ and $\mathrm{PSim}^+(A, \tau)$ are inner twisted forms of the same quasi-split group, and are motivic equivalent modulo 2.

Note that $\mathrm{PSim}^+(A, \sigma)$ and $\mathrm{PSim}^+(A, \tau)$ always are inner twisted forms of the same quasi-split group in the symplectic and unitary cases. If the involutions are orthogonal, this is the case if and only if σ and τ have the same discriminant. Combining Lemma 2.5, [11, Lemma 2.6] and Proposition 2.4, one may easily check that two orthogonal involutions sharing same Witt 2-index over any field extension have the same discriminant. Hence, theorem [6, Thm. 15], which is a crucial tool in this paper, can be rephrased as follows, using the description of the Tits-indices given in [7] :

Proposition 3.2. *(Cf.* [6, Thm.15]*) Let (A, σ, τ) be an algebra with two involutions of the same type over F. The involutions σ and τ are motivic equivalent if and only if for all field extensions M/F we have*

$$i_{w,2}(\sigma_M) = i_{w,2}(\tau_M)$$

Example 3.3. Let q and q' be two quadratic forms defined on the same vector space V, and consider the involutions ad_q and $\mathrm{ad}_{q'}$ of the split algebra $A = \mathrm{End}_F(V)$. Recall that the quadratic forms q and q' are called motivic equivalent if the quadrics $X_q = X_{\{1\},\mathrm{Ad}_q}$ and $X_{q'} = X_{\{1\},\mathrm{Ad}_{q'}}$ have isomorphic motives modulo 2. Even though this is quite different from our definition above, we claim that q and q' are motivic equivalent if and only if ad_q and $\mathrm{ad}_{q'}$ also are. Indeed, if ad_q and $\mathrm{ad}_{q'}$ are motivic equivalent, by definition, q and q' have the same discriminant, and $\mathrm{PSim}^+(\mathrm{Ad}_q) = \mathrm{PGO}^+(q)$ and $\mathrm{PSim}^+(\mathrm{Ad}_{q'}) = \mathrm{PGO}^+(q')$ are motivic equivalent modulo 2. Hence in particular, $X_q = X_{\{1\},\mathrm{Ad}_q}$ and $X_{q'} = X_{\{1\},\mathrm{Ad}_{q'}}$ have isomorphic motives modulo 2, so q and q' are motivic equivalent. Let us now prove the converse. By [8, Corollary 72.6], for any $i < n/2$, where n is the degree of $\mathrm{End}_F(V)$, that is the dimension of q, we have $i_w(q) > i$ if and only if the complete motivic decomposition of X_q with coefficients in $\mathbb{F}_2$ contains the Tate motive $\mathbb{F}_2(i)$. Therefore, if X_q and $X_{q'}$ are motivic equivalent modulo 2, we have $i_w(q) = i_w(q')$, and by functoriality, $i_w(q_M) = i_w(q'_M)$ for all extensions M/F. By [11, Lemma 2.6], it follows that q and q', hence also ad_q and $\mathrm{ad}_{q'}$, have the same discriminant. In addition, lemma 2.1 applied over the field M, shows that $i_{w,2}\big((\mathrm{ad}_q)_M\big) = i_{w,2}\big((\mathrm{ad}_{q'})_M\big)$ for all extensions M/F. Hence, by Proposition 3.2, ad_q and $\mathrm{ad}_{q'}$ are motivic equivalent.

Remark 3.4. This example shows that if the quadrics X_q and $X_{q'}$ have isomorphic motives modulo 2, then the groups $\mathrm{PSim}^+(\mathrm{Ad}_q)$ and $\mathrm{PSim}^+(\mathrm{Ad}_{q'})$ are motivic equivalent modulo 2. So the standard motives of type Θ with coefficients in $\mathbb{F}_2$ for those two groups are isomorphic for all $\Theta \subset \Delta$.

The observation for quadrics leads to the following definition in general :

Definition 3.5. The variety $X_{\Theta,(A,\sigma)}$ is called critical for (A,σ) if, for all involution τ on A of the same type as σ, the following assertions are equivalent :

(1) $\sigma \overset{m}{\sim} \tau$
(2) The varieties $X_{\Theta,(A,\sigma)}$ and $X_{\Theta,(A,\tau)}$ have isomorphic motives modulo 2.

As we already noticed, the projective quadric X_q is critical for the split algebra with orthogonal involution Ad_q. The main result of this section asserts that any algebra with involution admits a critical variety:

Theorem 3.6. *Let (A,σ) be an algebra with involution over F. We let X_σ be the F-variety defined as follows, depending on the type of σ:*

(1) If σ is orthogonal, $X_\sigma = X_{\{1\},(A,\sigma)}$ is the so-called involution variety of (A,σ), which consists of isotropic right ideals in A of reduced dimension 1.
(2) If σ is symplectic, $X_\sigma = X_{\{2\},(A,\sigma)}$ is the variety of isotropic right ideals in A of reduced dimension 2.
(3) If σ is unitary, $X_\sigma = X_{\{1,\deg A-1\},(A,\sigma)}$ is the variety of isotropic right ideals in A of reduced dimension 1.

The variety X_σ is critical for the algebra with involution (A,σ).

Remark 3.7. Theorem 3.6 gives rise to critical varieties for all classical groups. The critical varieties obtained for orthogonal involutions correspond to the *involution varieties* considered previously by Tao [24].

Theorem 3.6 allows to generalize Vishik's celebrated criterion [8, Theorem 93.1] for motivic equivalence of quadrics.

Corollary 3.8. *Let (A,σ,τ) be an algebra with two involutions of the same type over F. The motives $M(X_\sigma)$ and $M(X_\tau)$ are isomorphic if and only if for any field extension M/F we have $i_{w,2}(\sigma_M) = i_{w,2}(\tau_M)$.*

Before proving the theorem, we reduce to low index cases, hence to quadratic forms, as follows :

Proposition 3.9. *Let (A,σ,τ) be an algebra with two involutions of the same type over F. Consider the associated quadratic forms $\mathcal{Q}_\sigma$ and $\mathcal{Q}_\tau$ over the function field $\mathcal{F}_{A,t}$, where t denotes the type of the involutions, as defined in § 2.2. The following assertions are equivalent*

(1) $\sigma \overset{m}{\sim} \tau$;
(2) $\sigma_{\mathcal{F}_{A,t}} \overset{m}{\sim} \tau_{\mathcal{F}_{A,t}}$;
(3) $\mathcal{Q}_\sigma \overset{m}{\sim} \mathcal{Q}_\tau$.

Proof. By scalar extension, (1) clearly implies (2). The converse can be proved using Proposition 3.2. Indeed, assume that (2) holds and fix a field extension M/F. We set $\mathcal{M}$ for the compositum of M and $\mathcal{F}_{A,t}$, that is the function field of the relevant variety for σ_M and τ_M. Applying Proposition 3.2 to $\sigma_{\mathcal{F}_{A,t}}$ and $\tau_{\mathcal{F}_{A,t}}$ and Lemma 2.5 to σ_M and τ_M, we get

$$i_{w,2}(\sigma_M) = i_{w,2}(\sigma_{\mathcal{M}}) = i_{w,2}(\tau_{\mathcal{M}}) = i_{w,2}(\tau_M).$$

It follows that the Witt 2-indices of σ_M and τ_M coincide for any field extension M/F, hence we get (1) applying Proposition 3.2 to σ and τ.

We now show that (2) and (3) are equivalent. If the involutions are orthogonal, it follows from Example 3.3 since $A_{\mathcal{F}_{A,o}}$ is split. Assume now that σ and τ are either symplectic or unitary. By Proposition 3.2, (2) holds if and only if σ and τ have the same 2-Witt indices over any extension $\mathcal{M}$ of $\mathcal{F}_{A,t}$. In view of Proposition 2.3 this is equivalent to equality of the Witt indices of the quadratic forms $\mathcal{Q}_{\sigma_\mathcal{M}}$ and $\mathcal{Q}_{\tau_\mathcal{M}}$ for all $\mathcal{M}$. As explained in Exemple 3.3, this in turn characterizes motivic equivalence of $\mathcal{Q}_\sigma$ and $\mathcal{Q}_\tau$. □

With this in hand, we can now prove the main theorem as follows :

Proof of Theorem 3.6. Let (A,σ,τ) be an algebra with two involutions of the same type t over F. If σ and τ are motivic equivalent, then X_σ and X_τ have isomorphic motives modulo 2 by definition. So we need to prove the converse. By Proposition 3.9 above, we may extend scalars to $\mathcal{F}_{A,t}$, or equivalently, we may assume that the algebra A is split in orthogonal and unitary type, and has index at most 2 in symplectic type. Therefore the orthogonal case follows from example 3.3.

Assume from now on that σ is either symplectic of unitary, A has index at most 2 or A is split, depending on the type of σ, and X_σ and X_τ are motivic equivalent modulo 2. By Propositions 3.2 and 2.3, we need to prove that $i_w(\sigma_M) = i_w(\tau_M)$ for all field extension M/F. So let M be a field extension of F. If the involutions are unitary and $K \otimes_F M$ is not a field, or the involutions are symplectic and $A \otimes_F M$ is split, then both involutions σ_M and τ_M are hyperbolic, so they have the same Witt indices. Otherwise, the result follows from the next proposition, which concludes the proof. □

Proposition 3.10. *Let (A,σ) be an algebra with unitary or symplectic involution. We assume A has index 1 in unitary type, and A has index 2 in symplectic type. If σ is unitary, we assume in addition that K/F is a field extension. Then the complete motivic decomposition of X_σ modulo 2 determines $i_w(\sigma)$.*

Proof. Assume first that σ is unitary and A is split. By [13, Lemma 3.1], for every $i < d/2$ the following holds : $i_w(\sigma) > i$ if and only if the complete motivic decomposition of X_σ contains the Tate motive $\mathbb{F}_2(2i)$. So the result is proved in this case.

Assume now σ is a symplectic involution of the algebra $M_m(H)$, for some quaternion division algebra H. The involution σ is adjoint to a rank m hermitian form, denoted by h_σ, defined on the H-module $V \simeq H^m$, and with values in $(H, \bar{\ })$, where $\bar{\ }$ stands for the canonical involution of H. Denote by $\mathrm{SB}(H)$ the Severi-Brauer variety of H. The variety X_σ, is isomorphic to the variety $X(2,(V,h_\sigma))$ of totally isotropic H-submodules of V of reduced dimension 2. Assume h_σ is isotropic, then we have the decomposition

$$(V,h_\sigma) = \mathbb{H}(H) \perp (W,h')\,.$$

Applying [12, Corollary 15.9] to the above decomposition, we obtain the motivic decomposition of X_σ consisting of the following direct summands (up to some shifts): two Tate motives, two motives of $\mathrm{SB}(D) \times X(1,(W,h'))$, motive of $\mathrm{SB}(D)$ and motive of $X(2,(W,h'))$. Note that, since $\operatorname{ind} D = 2$, the Tate motives do not appear in the complete motivic decompositions of $\mathrm{SB}(D) \times X(1,(W,h'))$ and $\mathrm{SB}(D)$. Therefore we have

$$N(X_\sigma) = N(X(2,(V,h_\sigma))) = 2 + N(X(2,(W,h')))\,,$$

where for a projective homogeneous variety X we denote by $N(X)$ the number of Tate motives in the complete motivic decomposition of X.

Also note that $i_w(h_\sigma) = 2+i_w(h')$, and $i_w(h_\sigma) = N(X_\sigma) = 0$ if h_σ is anisotropic. Therefore by induction on $i_w(h_\sigma)$ we get that $i_w(h_\sigma) = N(X_\sigma)$. Since $i_w(\sigma) = i_w(h_\sigma)$, we also have $i_w(\sigma) = N(X_\sigma)$. This concludes the proof. □

4. Motivic equivalence and isomorphism

Using Bayer and Parimala's proof of the Hasse principle conjecture II [3], Lewis and Tignol gave necessary and sufficient conditions on the base field F under which cohomological invariants and signatures are enough to classify involutions on a central simple algebra (see [18]). In this section, we prove that over a field F satisfying those conditions, motivic equivalent involutions are isomorphic. This extends a previous result on quadratic forms du to Hoffmann [9]. The key observation is that motivic equivalent involutions have the same invariants. This is proved by Hoffmann [9] in the split orthogonal case, and extends to other cases by our Proposition 3.9. More precisely, we have :

Proposition 4.1. *Let (A,σ,τ) be an algebra with two involutions of the same type over F. If σ and τ are motivic equivalent, then the following hold :*

(1) If A has even degree and the involutions are orthogonal,
$$d_\sigma = d_\tau \in F^\times/F^{\times 2} \text{ and } c(\sigma_K) = c(\tau_K) \in \mathrm{Br}(F)/\langle [A]\rangle,$$
where K/F is the discriminant quadratic extension;

(2) If A has even degree and the involutions are unitary,
$$[\mathcal{D}(\sigma)] = [\mathcal{D}(\tau)] \in \mathrm{Br}(F);$$

(3) In all three types, $\mathrm{sign}_P(\sigma) = \mathrm{sign}_P(\tau)$ for all $P \in X_F$.

Proof. If σ and τ are motivic equivalent, then by Proposition 3.9, the quadratic forms $\mathcal{Q}_\sigma$ and $\mathcal{Q}_\tau$ also are motivic equivalent. Hence, by [9, Cor.2.2, Lem.3.1], we have $d(\mathcal{Q}_\sigma) = d(\mathcal{Q}_\tau)$, $c\big((\mathcal{Q}_\sigma)_\mathcal{K}\big) = c\big((\mathcal{Q}_\tau)_\mathcal{K}\big)$, where $\mathcal{K}/\mathcal{F}_{A,t}$ is the discriminant quadratic extension, and $|\,\mathrm{sign}_Q(\mathcal{Q}_\sigma)| = |\,\mathrm{sign}_Q(\mathcal{Q}_\tau)|$ for all $Q \in X_{\mathcal{F}_{A,t}}$. The result follows by Proposition 2.4. □

From this proposition, we get that motivic equivalent involutions actually are isomorphic over any field F over which involutions are classified by their invariants. Let F be a formally real field. Recall that the space of orderings X_F is a topological space. Moreover, for all $a \in F^\times$, the so-called Harrison set
$$H(a) = \{P \in X_F, a >_P 0\}$$
is both closed and open in X_F (see eg [22]). The field F is called SAP if, conversely, each closed and open subset of X_F is a Harrison set, that is any prescription of signs at each ordering given by a partition of X_F into two closed and open subsets is attained by some $a \in F^\times$. If in addition all formally real quadratic extensions of F are SAP, the field F is called ED. Applying the classification theorems given by Lewis and Tignol in [18], we get :

Theorem 4.2. *Let F be either a non formally real field of cohomological dimension ≤ 2, or a formally real field with virtual cohomological dimension ≤ 2 and satisfying the ED property. Let (A,σ,τ) be an algebra with two involutions of the same type over F. If σ and τ are motivic equivalent, then they are isomorphic.*

Proof. The theorem follows immediately from [18, Thm.A,Thm.B], up to the following lemma. □

Lemma 4.3. *Let σ and τ be two orthogonal involutions on a central simple algebra A over F. Their Clifford algebras $\mathcal{C}(\sigma)$ and $\mathcal{C}(\tau)$ are isomorphic as F-algebras if and only of $d_\sigma = d_\tau$ and $c(\sigma_K) = c(\tau_K)$, where K/F is the discriminant quadratic extension.*

Proof. Assume first that $\mathcal{C}(\sigma)$ and $\mathcal{C}(\tau)$ are isomorphic as F-algebras. They have isomorphic centers, therefore $d_\sigma = d_\tau$. Moreover, extending scalars from F to K, we have $\mathcal{C}(\sigma_K) \simeq \mathcal{C}(\tau_K)$, where K/F is the quadratic discriminant extension. Hence $\mathcal{C}_+(\sigma_K) \times \mathcal{C}_-(\sigma_K)$ and $\mathcal{C}_+(\tau_K) \times \mathcal{C}_-(\tau_K)$ are isomorphic as K-algebras. It follows that $\mathcal{C}_+(\sigma_K)$ is isomorphic either to $\mathcal{C}_+(\tau_K)$ or to $\mathcal{C}_-(\tau_K)$, and in both cases, we get $c(\sigma_K) = c(\tau_K) \in \mathrm{Br}(K)/\langle [A_K] \rangle$.

Assume conversely that $d_\sigma = d_\tau$ and $c(\sigma_K) = c(\tau_K) \in \mathrm{Br}(K)/\langle [A_K] \rangle$. Since the center of $\mathcal{C}(\sigma)$ is K, the Clifford algebra of σ_K is

$$\mathcal{C}(\sigma_K) \simeq \mathcal{C}(\sigma) \otimes_F K \simeq \mathcal{C}(\sigma) \times \overline{\mathcal{C}(\sigma)},$$

where $\overline{\mathcal{C}(\sigma)}$ denotes the conjugate algebra, and similarly for $\mathcal{C}(\tau_K)$. Hence, the assumption $c(\sigma_K) = c(\tau_K) \in \mathrm{Br}(K)/\langle [A_K] \rangle$, says that $\mathcal{C}(\tau)$ is isomorphic either to $\mathcal{C}(\sigma)$ or to its conjugate, which precisely means that they are isomorphic as F-algebras. □

Remark 4.4. In the odd degree orthogonal case, Izhboldin has proved a much stronger result, namely that two odd dimensional motivic equivalent quadratic forms are similar. Hence, no assumption on the base field is required in this case. In some even degree cases, we can slightly weaken the assumption on the base field as follows, by [18, Thm.A, Thm.B] :

(1) For non formally real fields, $I^3(F) = 0$ is enough in symplectic and orthogonal types;
(2) For formally real fields, $I^3F(\sqrt{-1}) = 0$ and F SAP is enough for symplectic involutions;
(3) For formally real fields, $I^3F(\sqrt{-1}) = 0$ and F ED is enough for orthogonal involutions on even-degree algebras.

Remark 4.5. By Springer's theorem, a quadratic form is isotropic over an odd degree extension of the base field if and only if it is isotropic. Equivalently, we have $i_{w,2}(q) = i_w(q)$ for all quadratic form q. It is not known whether this also holds for involutions, even over a field satisfying the conditions of the theorem above. Under those conditions, though, we have the following weaker assertion : If (A, σ, τ) is an algebra with two involutions of the same type over a field F which is either a non formally real field of cohomological dimension ≤ 2, or a formally real field with virtual cohomological dimension ≤ 2 and satisfying the ED property, then

$$\forall M/F,\ i_{w,2}(\sigma_M) = i_{w,2}(\tau_M) \ \Rightarrow\ \forall M/F,\ i_w(\sigma_M) = i_w(\tau_M).$$

Indeed, the left condition guarantee that the involutions σ and τ are motivic equivalent by [6] (see Proposition 3.2). By the above theorem, this implies that the involutions are isomorphic, hence they do have the same Witt index over any extension of the base field.

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