Motivic decompositions of projective homogeneous varieties and change of coefficients

Charles De Clercq

Abstract

We prove that under some assumptions on an algebraic group G, indecomposable direct summands of the motive of a projective G-homogeneous variety with coefficients in $\mathbb{F}_p$ remain indecomposable if the ring of coefficients is any field of characteristic p. In particular for any projective G-homogeneous variety X, the decomposition of the motive of X in a direct sum of indecomposable motives with coefficients in any finite field of characteristic p corresponds to the decomposition of the motive of X with coefficients in $\mathbb{F}_p$. We also construct a counterexample to this result in the case where G is arbitrary.

Résumé

Nous prouvons que sous certaines hypothèses sur un groupe algébrique G, tout facteur direct indécomposable du motif associé à une variété projective G-homogène à coefficients dans $\mathbb{F}_p$ demeure indécomposable si l'anneau des coefficients est un corps de caractéristique p. En particulier pour toute variété projective G-homogène X, la décomposition du motif de X comme somme directe de motifs indécomposables à coefficients dans tout corps fini de caractéristique p correspond à la décomposition du motif de X à coefficients dans $\mathbb{F}_p$. Nous exhibons de plus un contre-exemple à ce résultat dans le cas où le groupe G est quelconque.

Introduction

Let F be a field, Λ be a commutative ring, $\mathrm{CM}(F;\Lambda)$ be the category of *Grothendieck Chow motives* with coefficients in Λ, G a semi-simple affine algebraic group and X a projective G-homogeneous F-variety. The purpose of this note is to study the behaviour of the complete motivic decomposition (in a direct sum of indecomposable motives) of $X \in \mathrm{CM}(F;\Lambda)$ when changing the ring of coefficients. In the first part we prove some very elementary results in non-commutative algebra and find sufficient conditions for the tensor product of two connected rings to be connected. In the second part we show that under some assumptions on G, indecomposable direct summands of X in $\mathrm{CM}(F;\mathbb{F}_p)$ remain indecomposable if the ring of coefficients is any field of characteristic p (Theorem 2.1), since these conditions hold for the *reduced endomorphism ring* of indecomposable direct summands of X. In particular theorem 2.1 implies that the complete decomposition of the motive of X with coefficients in any finite field of characteristic

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p corresponds to the complete decomposition of the motive of X with coefficients in $\mathbb{F}_p$. Finally we show that theorem 2.1 doesn't hold for arbitrary G by producing a counterexample.

Let Λ be a commutative ring. Given a field F, an F-variety will be understood as a separated scheme of finite type over F. Given such Λ and an F-variety X, we can consider $\mathrm{CH}_i(X;\Lambda)$, the Chow group of i-dimensional cycles on X modulo rational equivalence with coefficients in Λ, defined as $\mathrm{CH}_i(X)\otimes_{\mathbb{Z}}\Lambda$. These groups are the first step in the construction of the category $\mathrm{CM}(F;\Lambda)$ of *Grothendieck Chow motives* with coefficients in Λ. This category is constructed as the *pseudo-abelian envelope* of the category $\mathrm{CR}(F;\Lambda)$ of *correspondences* with coefficients in Λ. Our main reference for the construction and the main properties of these categories is [2]. For a field extension E/F and any correspondence $\alpha \in \mathrm{CH}(X\times Y;\Lambda)$ we denote by α_E the pull-back of α along the natural morphism $(X\times Y)_E \to X\times Y$. Considering a morphism of commutative rings $\varphi : \Lambda \longrightarrow \Lambda'$ we define the two following functors. The *change of base field* functor is the additive functor $\mathrm{res}_{E/F} : \mathrm{CM}(F;\Lambda) \longrightarrow \mathrm{CM}(E;\Lambda)$ which maps any summand $(X,\pi)[i] \in \mathrm{CM}(F;\Lambda)$ to $(X_E,\pi_E)[i]$ and any morphism $\alpha : (X,\pi)[i] \to (Y,\rho)[j]$ to α_E. The *change of coefficents* functor is the additive functor $\mathrm{coeff}_{\Lambda'/\Lambda} : \mathrm{CM}(F;\Lambda) \longrightarrow \mathrm{CM}(F;\Lambda')$ which maps any summand $(X,\pi)[i]$ to $(X,(id\otimes\varphi)(\pi))[i]$ and any morphism $\alpha : (X,\pi)[i] \to (Y,\rho)[j]$ to $(id\otimes\varphi)(\alpha)$.

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1. On the tensor product of connected rings

Recall that a ring A is *connected* if there are no idempotents in A besides 0 and 1.

Proposition 1.1 *Let A be a finite and connected ring. Then any element a in A is either nilpotent or invertible. The set $\mathcal{N}$ of nilpotent elements in A is a two-sided and nilpotent ideal.*

In order to prove Proposition 1.1 we will need the following elementary lemma.

Lemma 1.2 *Let A be a finite ring. An appropriate power of any element a of A is idempotent.*

Proof. For any $a \in A$, the set $\{a^n,\ n\in\mathbb{N}\}$ is finite, hence there is a couple $(p,k)\in\mathbb{N}^2$ (with k non-zero) such that $a^p = a^{p+k}$. The sequence $(a^n)_{n\geq p}$ is k-periodic and for example if s is the lowest integer such that $p < sk$, a^{sk} is idempotent. □

Proof of Proposition 1.1. For any $a\in A$, an appropriate power of a is an idempotent by lemma 1.2. Since A is connected, this power is either 0 or 1, that is to say a is either nilpotent or invertible.

We now show that the set $\mathcal{N}$ of nilpotent elements in A is a two-sided ideal. First if a is nilpotent in A, then for any b in A, ab and ba are not invertible, hence ab and ba belong to $\mathcal{N}$.

It remains to show that the sum of two nilpotent elements in A is nilpotent. Setting ν for the number of nilpotent elements in A, we claim that for any sequence $a_1,\ldots,\ a_\nu$ in $\mathcal{N}$, $a_1...a_\nu = 0$. Indeed if $a_{\nu+1}$ is any nilpotent in A the finite sequence $\Pi_1 = a_1$, $\Pi_2 = a_1a_2,\ldots$, $\Pi_{\nu+1} = a_1a_2...a_{\nu+1}$ consists of nilpotents and by the pigeon-hole principle $\Pi_k = \Pi_s$, for some k and s satisfying $1\leq k < s \leq \nu+1$. Therefore $\Pi_s = \Pi_k a_{k+1}...a_s = \Pi_k$ which implies that $\Pi_k(1-a_{k+1}...a_s) = 0$ and $\Pi_k = 0$ since $1-a_{k+1}...a_s$ is invertible. With this in hand it is clear that for any a and b in $\mathcal{N}$, $(a+b)^\nu = 0$. Furthermore $\mathcal{N}^\nu = 0$ and $\mathcal{N}$ is nilpotent. □

Corollary 1.3 *Let A be a finite and connected $\mathbb{F}_p$-algebra endowed with a ring morphism $\varphi : A \longrightarrow \mathbb{F}_p$. Then the set $\mathcal{N}$ of nilpotent elements in A is precisely* $\ker(\varphi)$. *Furthermore for any connected $\mathbb{F}_p$-algebra E, $A \otimes_{\mathbb{F}_p} E$ is connected.*

Proof. For any $a \in \mathcal{N}$ and $n \in \mathbb{N}^*$ such that $a^n = 0$, $0 = \varphi(a^n) = \varphi(a)^n$, hence a lies in the kernel of φ. On the other hand if $\varphi(a) = 0$, a is not invertible thus a is nilpotent and $\mathcal{N} = \ker(\varphi)$. Since $\mathcal{N}$ is nilpotent, $\mathcal{N} \otimes E$ is also nilpotent. The sequence

$$0 \longrightarrow \mathcal{N} \otimes E \longrightarrow A \otimes E \xrightarrow{\psi} E \longrightarrow 0$$

is exact and we want to show that any idempotent P in $A \otimes_{\mathbb{F}_p} E$ is either 0 or 1. Since E is connected, $\psi(P)$ is either 0 or 1. We may replace P by $1 - P$ and so assume that P lies in the kernel of ψ, which implies that the idempotent P is nilpotent, hence $P = 0$. $\square$

2. Application to motivic decompositions of projective homogeneous varieties

For any semi-simple affine algebraic group G, the full subcategory of $\mathrm{CM}(F;\Lambda)$ whose objects are finite direct sums of twists of direct summands of the motives of projective G-homogeneous F-varieties will be denoted $\mathrm{CM}_G(F;\Lambda)$. We now use corollary 1.3 to study how motivic decompositions in $\mathrm{CM}_G(F;\Lambda)$ behave when extending the ring of coefficients. A pseudo-abelian category $\mathcal{C}$ satisfies the *Krull-Schmidt principle* if the monoid $(\mathfrak{C}, \oplus)$ is free, where $\mathfrak{C}$ denotes the set of the isomorphism classes of objects of $\mathcal{C}$.

In the sequel Λ will be a connected ring and X an F-variety. A field extension E/F is a *splitting field* of X if the E-motive X_E is isomorphic to a finite direct sum of twists of Tate motives. The F-variety X is *geometrically split* if X splits over an extension of F, and X satisfies the *nilpotence principle*, if for any field extension E/F the kernel of the morphism $\mathrm{res}_{E/F} : \mathrm{End}(M(X)) \longrightarrow \mathrm{End}(M(X_E))$ consists of nilpotents. Any projective homogeneous variety (under the action of a semi-simple affine algebraic group) is geometrically split and satisfies the nilpotence principle (see [1]), therefore if Λ is finite the Krull-Schmidt principle holds for $\mathrm{CM}_G(F;\Lambda)$ by [5, Corollary 3.6], and we can serenely deal with motivic decompositions in $\mathrm{CM}_G(F;\Lambda)$.

Let G be a semi-simple affine algebraic group over F and p a prime. The absolute Galois group $Gal(F_{sep}/F)$ acts on the Dynkin diagram of G and we say that G is of *inner type* if this action is trivial. By [1] the subfield F_G of F_{sep} corresponding to the kernel of this action is a finite Galois extension of F, and we will say that G is *p-inner* if $[F_G : F]$ is a power of p. We now state the main result.

Theorem 2.1 *Let G be a semi-simple affine p-inner algebraic group and $M \in \mathrm{CM}_G(F;\mathbb{F}_p)$. For any field L of characteristic p, M is indecomposable if and only if $\mathrm{coeff}_{L/\mathbb{F}_p}(M)$ is indecomposable.*

If X is geometrically split the image of any correspondence $\alpha \in \mathrm{CH}_{\dim(X)}(X \times X;\Lambda)$ by the *change of base field* functor $\mathrm{res}_{E/F}$ to a splitting field E/F of X will be denoted $\overline{\alpha}$. The *reduced endomorphism ring* of any direct summand (X,π) is defined as $\mathrm{res}_{E/F}(\mathrm{End}_{\mathrm{CM}(F;\Lambda)}((X,\pi)))$ and denoted by $\overline{\mathrm{End}}((X,\pi))$.

Let X be a complete and irreducible F-variety. The pull-back of the natural morphism $\mathrm{Spec}(F(X)) \times X \longrightarrow X \times X$ gives rise to mult:$\mathrm{CH}_{\dim(X)}(X \times X;\Lambda) \longrightarrow \mathrm{CH}_0(X_{F(X)};\Lambda) \longrightarrow \Lambda$ (where the second map is the *degree* morphism). For any correspondence $\alpha \in \mathrm{CH}_{\dim(X)}(X \times X;\Lambda)$, $\mathrm{mult}(\alpha)$ is called the *multiplicity* of α and we say that a direct summand (X,π) given by a projector $\pi \in \mathrm{CH}_{\dim(X)}(X \times X; \Lambda)$ is *upper* if $\mathrm{mult}(\pi) = 1$. If (X, π) is an upper direct summand of a complete and irreducible F-variety, the multiplicity $\mathrm{mult} : \mathrm{End}_{\mathrm{CM}(F;\Lambda)}((X,\pi)) \longrightarrow \Lambda$ is a morphism of rings by [4, Corollary 1.7].

Proposition 2.2 *Let G be a semi-simple affine algebraic group and $M = (X, \pi) \in \mathrm{CM}(F; \mathbb{F}_p)$ the upper direct summand of the motive of an irreducible and projective G-homogeneous F-variety. Then for any field L of characteristic p, M is indecomposable if and only if* $\mathrm{coeff}_{L/\mathbb{F}_p}(M)$ *is indecomposable.*

Proof. Since the change of coefficients functor is additive and maps any non-zero projector to a non-zero projector, it is clear that if $\mathrm{coeff}_{L/\mathbb{F}_p}(M)$ is indecomposable, M is also indecomposable. Considering a splitting field E of X, the reduced endomorphism ring $\overline{\mathrm{End}}(M) := \overline{\pi} \circ \overline{\mathrm{End}}(X) \circ \overline{\pi}$ is connected since M is indecomposable and finite. Corollary 1.3, with $A = \overline{\mathrm{End}}(M)$, $E = L$ and $\varphi = mult$ implies that $\overline{\mathrm{End}}(M) \otimes L = \overline{\mathrm{End}}(coeff_{L/\mathbb{F}_p}(M))$ is connected, therefore by the nilpotence principle $\mathrm{End}(\mathrm{coeff}_{L/\mathbb{F}_p}(M))$ is also connected, that is to say $\mathrm{coeff}_{L/\mathbb{F}_p}(M)$ is indecomposable. □

Proof of theorem 2.1. Recall that G is a semi-simple affine p-inner algebraic group and consider a projective G-homogeneous F-variety X. By [6, Theorem 1.1], any indecomposable direct summand M of X is a twist of the upper summand of the motive of an irreducible and projective G-homogeneous F-variety Y, thus we can apply proposition 2.2 to each indecomposable direct summand of X. □

Remark 2.3 *If Λ is a finite and connected ring, complete motivic decompositions in* $\mathrm{CM}(F; \Lambda)$ *remain complete when the coefficients are extended to the residue field of Λ by [7, Corollary 2.6], hence the study of motivic decompositions in* $\mathrm{CM}_G(F; \Lambda)$*, where Λ is any finite connected ring whose residue field is of characteristic p, is reduced to the study motivic decompositions in* $\mathrm{CM}_G(F; \mathbb{F}_p)$*.*

We now produce a counterexample to Theorem 2.1 in the case where the algebraic group G doesn't satisfy the needed assumptions. Let L/F be a Galois extension of degree 3. By [1, Section 7], the endomorphism ring $\mathrm{End}(M(\mathrm{Spec}(L)))$ of the motive associated with the F-variety $\mathrm{Spec}(L)$ with coefficients in $\mathbb{F}_2$ is the $\mathbb{F}_2$-algebra of $Gal(L/F)$, i.e. $\frac{\mathbb{F}_2[X]}{(X^3-1)} \simeq \mathbb{F}_2 \times \mathbb{F}_4$, hence $M(\mathrm{Spec}(L)) = M \oplus N$, with $\mathrm{End}(N) = \mathbb{F}_4$ and both M and N are indecomposable. Now $\mathrm{End}(\mathrm{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)) = \mathbb{F}_4 \otimes \mathbb{F}_4$ is not connected since $1 \otimes \alpha + \alpha \otimes 1$ is a non-trivial idempotent for any $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$, hence $\mathrm{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)$ is decomposable.
Consider the $(\mathrm{PGL}_2)_L$-homogeneous L-variety $\mathbb{P}^1_L$. The *Weil restriction* $\mathcal{R}(\mathbb{P}^1_L)$ is a projective homogeneous F-variety under the action of the Weil restriction of $(\mathrm{PGL}_2)_L$, and the minimal extension such that $\mathcal{R}((PGL_2)_L)$ is of inner type is L. By [3, Example 4.8], the motive with coefficients in $\mathbb{F}_2$ of $\mathcal{R}(\mathbb{P}^1_L)$ contains two twists of $\mathrm{Spec}(L)$ as direct summands, therefore at least two indecomposable direct summands of $\mathcal{R}(\mathbb{P}^1_L)$ split off over $\mathbb{F}_4$.

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Charles De Clercq declercq@math.jussieu.fr
Université Paris VI, 4 place Jussieu, 75252 Paris CEDEX 5