The Quarterly Journal of Mathematics Advance Access published August 19, 2012

The Quarterly Journal of Mathematics
Quart. J. Math. 00 (2012), 1–8; doi:10.1093/qmath/has023

A GOING DOWN THEOREM FOR GROTHENDIECK CHOW MOTIVES

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[Received 2 December 2011. Revised 10 July 2012]

Abstract

Let X be a geometrically split, geometrically irreducible variety over a field F satisfying Rost nilpotence principle. Consider a field extension E/F and a finite field $\mathbb{F}$. We provide in this note a motivic tool giving sufficient conditions for so-called outer motives of direct summands of the Chow motive of X_E with coefficients in $\mathbb{F}$ to be lifted to the base field. This going down result has been used by S. Garibaldi, V. Petrov and N. Semenov to give a complete classification of the motivic decompositions of projective homogeneous varieties of inner type E_6, and to answer a conjecture of Rost and Springer.

1. Introduction

Throughout this note, F will be the base field and, by an F-variety, we will mean a smooth, projective scheme over F. Given an F-variety X, we denote by $\mathrm{Ch}(X)$ the Chow group $\mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{F}$ of cycles on X modulo rational equivalence with coefficients in a finite field $\mathbb{F}$. We write $\mathrm{Ch}(\bar{X})$ for the colimit of all $\mathrm{Ch}(X_K)$, where K runs through all field extensions K/F and, if X is integral, we denote by $F(X)$ its function field.

For any field extension L/F, an element lying in the image of the natural morphism of $\mathrm{Ch}(X_L) \longrightarrow \mathrm{Ch}(\bar{X})$ is called L-rational. The image of any correspondence $\alpha \in \mathrm{Ch}(X_L)$ under the canonical morphism $\mathrm{Ch}(X_L) \longrightarrow \mathrm{Ch}(\bar{X})$ is denoted by $\bar{\alpha}$. An F-variety X is geometrically split if the Grothendieck Chow motive of $X_{\bar{F}} = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(\bar{F})$ with coefficients in $\mathbb{F}$ is isomorphic to a finite direct sum of Tate motives, for an algebraic closure $\bar{F}/F$. The variety X satisfies the Rost nilpotence principle with coefficients in $\mathbb{F}$ if for any field extensions $L/E/F$ the kernel of the restriction map $\mathrm{res}_{L/E} : \mathrm{End}(M(X_E)) \longrightarrow \mathrm{End}(M(X_L))$ consists of nilpotents.

As shown in [2], any projective homogeneous F-variety under the action of a semisimple affine algebraic group is geometrically split and satisfies the Rost nilpotence principle. It follows by Chernousov *et al.* [1, Corollary 35] (see also [6, Corollary 2.6]) that the Grothendieck Chow motive of these varieties with coefficients in $\mathbb{F}$ decomposes in an essentially unique way as a direct sum of indecomposable motives. The study of these decompositions have already shown to be very fruitful (see [5, 6, 10]).

The notion of *upper* motives, previously defined by Vishik in the context of quadrics in [10], was further developed by Karpenko [6] to describe the indecomposable motives lying in the motivic decomposition of projective homogeneous varieties. If X is a homogeneous F-variety, E/F a field extension and the upper motive of $M(X_E)$ is a direct summand of another motive M_E, [10, Theorem 4.15; 5, Proposition 4.6] give sufficient conditions for the upper motive of X to be a direct

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summand of M. The purpose of the present note is to push these ideas further. We define the notions of *upper*, *lower* and *outer* direct summands of a direct summand N of the motive of a geometrically split F-variety. We then show some lifting property of outer summands of N_E to the base field with the following result.

THEOREM 1.1. *Let N be a direct summand of the motive (with coefficients in $\mathbb{F}$) of a geometrically split, geometrically integral F-variety X satisfying the Rost nilpotence principle, with coefficients in $\mathbb{F}$ and M a twisted direct summand of the motive of another F-variety Y. Assume that there is a field extension E/F such that*

(1) *every $E(X)$-rational cycle in $\mathrm{Ch}(\overline{X \times Y})$ is $F(X)$-rational;*
(2) *the motive N_E has an indecomposable outer direct summand which is also a direct summand of the motive M_E.*

Then the motive N has an outer direct summand which is also a direct summand of M.

Theorem 1.1 allows one to descend outer motives of direct summands projective homogeneous varieties which appear on some field extension E/F of the base field. This generalizes [5, Proposition 4.6], one of the key ingredients in the proof of Karpenko [5, Theorem 1.1], replacing the whole motive of a variety X by a direct summand. To replace X by an arbitrary direct summand, one needs to construct explicitly the rational cycles to get an outer summand defined over F, and thus Theorem 1.1 gives a new proof of Karpenko [5, Proposition 4.6]. Note that assumption (1) of Theorem 1.1 holds if the field extension $E(X)/F(X)$ is unirational, i.e. if there is a field extension $L/E(X)$ such that $L/F(X)$ is purely transcendental.

The following particular case of Theorem 1.1 was used by Garibaldi *et al.* [4] to both determine all the motivic decompositions of homogeneous F-varieties of inner type E_6 and prove a conjecture of Rost and Springer.

COROLLARY 1.2 [4, Proposition 3.2]. *Let X and Y be two projective homogeneous F-varieties for a semisimple affine algebraic group, and let M and N be direct summands of the motives of Y and X, respectively, with coefficients in $\mathbb{F}$. Assume that $N_{F(Y)}$ is an indecomposable direct summand of $M_{F(Y)}$ and Y has an $F(X)$-point. Then N is a direct summand of M.*

Proof Setting $E = F(Y)$, the field extension $E(X)/F(X)$ is purely transcendental, hence assumption (1) of Theorem 1.1 holds. □

2. Grothendieck Chow motives

Our main reference for the construction of the category of Grothendieck Chow motives over F with coefficients in $\mathbb{F}$ is [3, Sections 63–65].

Let X and Y be two F-varieties and $X = \coprod_{k=1}^{n} X_k$ be the decomposition of X as a disjoint union of irreducible components with respective dimension $d_1, \ldots, d_n$. For any integer i, the group of *correspondences* between X and Y of degree i with coefficients in $\mathbb{F}$ is defined by $\mathrm{Corr}_i(X, Y) = \coprod_{k=1}^{n} \mathrm{Ch}_{d_k+i}(X_k \times Y)$. We now consider the category $\mathrm{C}(F; \mathbb{F})$ whose objects are pairs $X[i]$, where X is an F-variety and i is an integer. Morphisms are defined in terms of correspondences by $\mathrm{Hom}_{\mathrm{C}(F;\mathbb{F})}(X[i], Y[j]) = \mathrm{Corr}_{i-j}(X, Y)$. For any correspondences $f : X[i] \rightsquigarrow Y[j]$ and

$g : Y[j] \rightsquigarrow Z[k]$ in Mor(C(F; $\mathbb{F}$)), the composite $g \circ f : X[i] \rightsquigarrow Z[k]$ is defined by

$$g \circ f = ({}^{X}p_{Y}^{Z})_{*}(({}^{X\times Y}p_{Z})^{*}(f) \cdot (p_{X}^{Y\times Z})^{*}(g)), \qquad (*)$$

where ${}^{U}p_{V}^{W} : U \times V \times W \to U \times W$ is the natural projection.

The category C(F; $\mathbb{F}$) is preadditive and its additive completion CR(F; $\mathbb{F}$) is the category of correspondences over F with coefficients in $\mathbb{F}$, which has a structure of tensor additive category given by $X[i] \otimes Y[j] = (X \times Y)[i + j]$. The category CM($F$; $\mathbb{F}$) of Grothendieck Chow motives with coefficients in $\mathbb{F}$ is the pseudo-abelian envelope of the category CR(F; $\mathbb{F}$). Its objects are couples (X, π), where X is an object of the category CR(F; Λ), and $\pi \in \mathrm{End}(X)$ is a projector (i.e. $\pi \circ \pi = \pi$). Morphisms are given by $\mathrm{Hom}_{\mathrm{CM}(F;\mathbb{F})}((X, \pi), (Y, \rho)) = \rho \circ \mathrm{Hom}_{\mathrm{CR}(F;\mathbb{F})}(X, Y) \circ \pi$ and the objects of CM(F; $\mathbb{F}$) are called *motives*. For any F-variety X, the motives $(X[i], \Gamma_{\mathrm{id}_X})$ (where Γ_{id_X} is the graph of the identity of X) will be denoted $X[i]$ and $X[0]$ is the motive of X. The motives $\mathbb{F}[i] = \mathrm{Spec}(F)[i]$ are the *Tate motives*.

LEMMA 2.1. *Let (X, π) be a direct summand of the motive of an F-variety X. A motive M is a direct summand of (X, π) if and only if M is isomorphic to (X, ρ), for some projector ρ satisfying $\pi \circ \rho \circ \pi = \rho$.*

Proof. Since $\mathrm{End}((X, \pi)) = \pi \circ \mathrm{Ch}_{\dim(X)}(X \times X) \circ \pi$, any projector ρ in $\mathrm{End}((X, \pi))$ satisfies $\pi \circ \rho \circ \pi = \rho$. □

DEFINITION 2.2. *Let $M \in$ CM(F; $\mathbb{F}$) be a motive and i an integer. The i-dimensional Chow group* $\mathrm{Ch}_i(M)$ *of M is defined by* $\mathrm{Hom}_{\mathrm{CM}(F;\mathbb{F})}(\mathbb{F}[i], M)$. *The i-codimensional Chow group* $\mathrm{Ch}^i(M)$ *of M is defined by* $\mathrm{Hom}_{\mathrm{CM}(F;\mathbb{F})}(M, \mathbb{F}[i])$.

For any field extension E/F and any correspondence $\alpha : X[i] \rightsquigarrow Y[j]$, the pull-back of α along the natural morphism $(X \times Y)_E \to X \times Y$ will be denoted α_E. If $N = (X, \pi)[i]$ is a twisted motivic direct summand of X, the motive $(X_E, \pi_E)[i]$ will be denoted N_E.

Finally, the category CM(F; $\mathbb{F}$) is endowed with a duality functor. If X and Y are two F-varieties and $\alpha \in \mathrm{Ch}(X \times Y)$ is a correspondence, the image of α under the exchange isomorphism $X \times Y \to Y \times X$ is denoted ${}^t\alpha$. The *duality functor* is the additive functor $\dagger : \mathrm{CM}(F; \Lambda)^{\mathrm{op}} \longrightarrow \mathrm{CM}(F; \Lambda)$ determined by the formula $M(X)[i]^{\dagger} = M(X)[-\dim(X) - i]$ and such that for any correspondence $\alpha : X[i] \rightsquigarrow Y[j]$, $\alpha^{\dagger} = {}^t\alpha$.

3. Direct summands of geometrically split F-varieties

Throughout this section, we consider a geometrically split F-variety X and E/F a splitting field of X. By Merkurjev [7, Proposition 1.5], the pairing

$$\Psi : \begin{array}{ccc} \mathrm{Ch}(X_E) \times \mathrm{Ch}(X_E) & \longrightarrow & \mathbb{F} \\ (\alpha, \beta) & \longmapsto & \deg(\alpha \cdot \beta) \end{array}$$

is non-degenerate, hence gives rise to an isomorphism of $\mathbb{F}$-modules between $\mathrm{Ch}(X_E)$ and its dual space $\mathrm{Hom}_{\mathbb{F}}(\mathrm{Ch}(X_E), \mathbb{F})$ given by $\alpha \mapsto \Psi(\alpha, \cdot)$. The dual basis of a homogeneous basis $(x_k)_{k=1}^{n}$ of $\mathrm{Ch}(X_E)$ with respect of Ψ is the basis $(x_k^*)_{k=1}^{n}$ of $\mathrm{Ch}(X_E)$ such that for any $1 \leq i, j \leq n$, $\Psi(x_i, x_j^*) =$

δ_{ij}, where δ_{ij} is the Kronecker symbol. By definition of the composition $(*)$ in $\mathrm{CM}(F;\mathbb{F})$, if y (respectively, y') lies in $\mathrm{Ch}(Y)$ (respectively, $\mathrm{Ch}(Y')$) for two other F-varieties Y and Y' and if (i,j) are two integers, the composition of the correspondences $x_i \times y \in \mathrm{Ch}(X_E \times Y)$ and $y' \times x_j^* \in \mathrm{Ch}(Y' \times X_E)$ is given by

$$(x_i \times y) \circ (y' \times x_j^*) = \delta_{ij}(y' \times y) \in \mathrm{Ch}(Y' \times Y). \tag{1}$$

Note that the Kunneth decomposition holds in $\mathrm{Ch}(X_E \times Y)$ and $\mathrm{Ch}(Y' \times X_E)$ in the view of Elman *et al*. [**3**, Proposition 64.3], since X_E is split, and thus the cycles of $\mathrm{Ch}(X_E \times Y)$ and $\mathrm{Ch}(Y' \times X_E)$ may always be written that way.

Upper, lower and outer motives. Let $\pi \in \mathrm{Ch}_{\dim(X)}(X \times X)$ be a non-zero projector and $N = (X,\pi)$ the associated summand of the motive of X. The *base* of N is the set $\mathcal{B}(N) = \{i \in \mathbb{Z},\ \ \mathrm{Ch}_i(N_E) \text{ is not trivial}\}$. The *bottom* of N (denoted by $b(N)$) is the least integer of $\mathcal{B}(N)$ and the *top* of N (denoted by $t(N)$) is the greatest integer of $\mathcal{B}(N)$. We now introduce the notion of upper and lower direct summands of N, previously introduced by Vishik in the context of the motives of quadrics in [**10**, Definition 4.6].

DEFINITION 3.1. *Let N be a direct summand of the twisted motive of a geometrically split F-variety and M a motivic direct summand of N. We say that*

(1) *M is* upper *in N if $b(M) = b(N)$;*
(2) *M is* lower *in N if $t(M) = t(N)$;*
(3) *M is* outer *in N if M is both lower and upper in N.*

REMARK 3.2. Keeping the same F-variety X and any direct summand $N = (X,\pi)$, consider a homogeneous basis $(x_k)_{k=1}^n$ of $\mathrm{Ch}(X_E)$ and its dual basis $(x_k^*)_{k=1}^n$. The base, bottom and top of N can be easily determined by the decomposition

$$\pi_E = \sum_{i,j=1}^{n} \pi_{i,j}(x_i \times x_j^*),$$

noticing that $\mathcal{B}(N) = \{\dim(x_i),\ \pi_{i,j} \neq 0 \text{ for some } j\}$.

LEMMA 3.3. *Let N be a motivic direct summand of a geometrically split F-variety and M a direct summand of N. Then M is lower in N (respectively, upper in N) if and only if the dual motive $M^\dagger$ is upper in $N^\dagger$ (respectively, $M^\dagger$ is lower in $N^\dagger$).*

Proof. For any motive O and for any integer i, $\mathrm{Ch}^i(O^\dagger) = \mathrm{Ch}_{-i}(O)$. It follows that $b(O^\dagger) = -t(O)$ and $t(O^\dagger) = -b(O)$. □

The Krull–Schmidt property. Let C be a pseudo-abelian category and $\mathfrak{C}$ be the set of the isomorphism classes of objects of C. We say that the category C satisfies the *Krull–Schmidt property* if the monoid $(\mathfrak{C}, \oplus)$ is free. The Krull–Schmidt property holds for the motives of geometrically split F-varieties satisfying the Rost nilpotence principle in $\mathrm{CM}(F;\mathbb{F})$ by Karpenko [**6**, Corollary 2.6].

Proof of the main result. In order to prove Theorem 1.1, we will need the following lemma, which will allow us to construct explicitly the rational cycles lifting outer motives to the base field.

LEMMA 3.4. *Let N be a motivic direct summand of a geometrically split, geometrically irreducible F-variety X satisfying the Rost nilpotence principle and M a twisted direct summand of an F-variety Y. Assume the existence of a field extension E/F such that*

(1) *any $E(X)$-rational cycle in $\mathrm{Ch}(\overline{X \times Y})$ is $F(X)$-rational;*
(2) *there are two correspondences $\alpha : N_E \rightsquigarrow M_E$ and $\beta : M_E \rightsquigarrow N_E$ such that $\beta \circ \alpha$ is a projector and $(X_E, \beta \circ \alpha)$ is a lower direct summand of N_E.*

Then there are two correspondences $\gamma : N \rightsquigarrow M$ and $\delta : M_E \rightsquigarrow N_E$ such that $(X_E, \delta \circ \gamma_E)$ is a direct summand of N_E, which contains all lower indecomposable direct summands of $(X_E, \beta \circ \alpha)$. Furthermore, if $\bar{\beta}$ is F-rational, then $\bar{\delta}$ is also F-rational.

Proof. We may assume by Lemma 2.1 that $M = (Y, \rho)[i]$ and $N = (X, \pi)$. We construct explicitly the two correspondences γ and δ. Since $E(X)$ is a field extension of E, $\bar{\alpha}$ is $E(X)$-rational, hence $F(X)$-rational by assumption (1). Let L/F be a field extension. Then the morphism $\mathrm{Spec}(F(X_L)) \longrightarrow X_L$ induces a pull-back morphism $\varepsilon^* : \mathrm{Ch}(\overline{X \times Y \times X}) \longrightarrow \mathrm{Ch}(\overline{(X \times Y)_{F(X)}})$ which maps F-rational cycles onto $F(X)$-rational cycles by Elman *et al*. [3, Corollary 57.11], and so there is a cycle $\alpha_1 \in \mathrm{Ch}(X \times Y \times X)$ such that $\varepsilon^*(\bar{\alpha}_1) = \bar{\alpha}$. Since ε^* maps any homogeneous cycle $\sum_i x_i \times y_i \times 1$ to $\sum_i x_i \times y_i$ and vanishes on homogeneous cycles whose codimension on the third factor is strictly positive, we have $\bar{\alpha}_1 = \bar{\alpha} \times 1 + \cdots$ where '$\cdots$' is a linear combination of homogeneous cycles in $\mathrm{Ch}(\overline{X \times Y \times X})$ with strictly positive codimension on the third factor.

We now look at α_1 as a correspondence $X \rightsquigarrow Y \times X$ and consider the cycle $\alpha_2 = \alpha_1 \circ \pi$. By formula (1), we have

$$\bar{\alpha}_2 = (\bar{\alpha} \times 1) \circ \bar{\pi} + \cdots ,$$

where '$\cdots$' is a linear combination of homogeneous cycles in $\mathrm{Ch}(\overline{X \times Y \times X})$ with dimension at most $t(N)$ on the first factor (since these terms come from the first factors of $\bar{\pi}$) and strictly positive codimension on the third factor (since these terms come from the third factors of $\bar{\alpha}_1 - \bar{\alpha} \times 1$). Finally, considering the pull-back of the morphism $\Delta : X \times Y \to X \times Y \times X$ induced by the diagonal embedding X and setting $\alpha_3 = \Delta^*(\alpha_2)$, we have

$$\bar{\alpha}_3 = \bar{\alpha} \circ \bar{\pi} + \cdots ,$$

where '$\cdots$' stands for a linear combination of homogeneous cycles in $\mathrm{Ch}(\overline{X \times Y})$ with dimension strictly lesser than $t(N)$ on the first factor.

Composing with $\bar{\pi} \circ \bar{\beta}$ on the left and $\bar{\pi}$ on the right, we get that

$$\bar{\pi} \circ \bar{\beta} \circ \bar{\alpha}_3 \circ \bar{\pi} = \bar{\pi} \circ \bar{\beta} \circ \bar{\alpha} \circ \bar{\pi} + \xi,$$

where ξ is a linear combination of homogeneous cycles of strictly lesser dimension than $t(N)$ on the first factor since they come from the first factors of $\bar{\alpha}_3 \circ \bar{\pi} - \bar{\alpha} \circ \bar{\pi}$. The correspondence $\beta \circ \alpha$ defines a direct summand of the motive N_E, and thus by Lemma 2.1

$$\bar{\pi} \circ \bar{\beta} \circ \bar{\alpha}_3 \circ \bar{\pi} = \bar{\beta} \circ \bar{\alpha} + \xi.$$

By formula (1), $\bar{\beta} \circ \bar{\alpha} \circ \xi$, $\xi \circ \xi$ and $\xi \circ \bar{\beta} \circ \bar{\alpha}$ are linear combinations of homogeneous cycles of dimension strictly lesser than $t(N)$ on the first factor. Repeating the same procedure and since $k \circ h$

is a projector, we see that for any integer n

$$(\bar{\pi} \circ \bar{\beta} \circ \bar{\alpha}_3 \circ \bar{\pi})^n = \bar{\beta} \circ \bar{\alpha} + \cdots , \tag{2}$$

where '$\cdots$' is a linear combination of homogeneous cycles in $\mathrm{Ch}(\overline{X \times X})$ with dimension on the first factor strictly lesser than $t(N)$. Since the direct summand $(X, \beta \circ \alpha)$ is lower, all these correspondences are non-zero and by Karpenko [6, Corollary 2.2] an appropriate power $(\pi_E \circ \beta \circ (\alpha_3)_E \circ \pi_E)^{\circ n_0}$ is a projector. If we set $\gamma = \rho \circ \alpha_3 \circ \pi$ and $\delta = (\pi_E \circ \beta \circ (\alpha_3)_E \circ \pi_E)^{\circ n_0 - 1} \circ \pi_E \circ \beta$, we see that $\bar{\delta}$ is F-rational if $\bar{\beta}$ is F-rational. The correspondence $\delta \circ \gamma_E$ is a projector which defines a direct summand of N_E by Lemma 2.1.

Consider the decomposition $\bar{\beta} \circ \bar{\alpha} = \sum_{i,j=1}^{s} p_{ij}(x_i \times x_j^*)$ of $\bar{\beta} \circ \bar{\alpha}$ with respect to a basis $(x_i)_{i=1}^{s}$ of $\mathrm{Ch}(\bar{X})$. By formula (2), the decomposition of $\bar{\delta} \circ \bar{\gamma}$ in $(x_i \times x_j^*)_{i,j=1}^{s}$ has a non-zero coefficient for any couple (i, j) such that p_{ij} is non-zero and $\dim(x_i) = t((X_E, \beta \circ \alpha))$. The Krull–Schmidt property and Remark 3.2 then imply that any lower indecomposable direct summand of $(X_E, \beta \circ \alpha)$ is a direct summand of $(X, \delta \circ \gamma_E)$. □

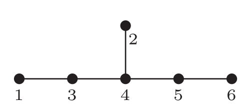
We now show how we can derive the proof of Theorem 1.1 from the rational cycles constructed in Lemma 3.4. To lift the outer motive to the base field F, we apply Lemma 3.4 and the duality functor twice in order to produce two correspondences which are defined on the base field.

Proof of Theorem 1.1. Let $O = (X_E, \kappa)$ be an outer indecomposable direct summand of N_E which is also a direct summand of M_E. We prove Theorem 1.1 by applying Lemma 3.4 once, then the duality functor and, finally, Lemma 3.4 another time to get all our correspondences defined over the base field F.

Since O is a direct summand of M_E, there are two correspondences $\alpha : N_E \rightsquigarrow M_E$ and $\beta : M_E \rightsquigarrow N_E$ such that $\beta \circ \alpha = \kappa$. Moreover, O is lower in N_E, so Lemma 3.4 justifies the existence of two other correspondences $\alpha' : N \rightsquigarrow M$ and $\beta' : M_E \rightsquigarrow N_E$ such that $O_2 = (X_E, \beta' \circ \alpha'_E)$ is a direct summand of N_E, and the motive O_2 is outer in N_E since it contains O. The dual motive $O_2^{\dagger} = (X_E, {}^t\alpha'_E \circ {}^t\beta')[-\dim(X)]$ is therefore outer in $N_E^{\dagger}$ by Lemma 3.3 and is a direct summand of the dual motive $M_E^{\dagger}$. Twisting these three motives by $\dim(X)$, we can apply Lemma 3.4 again. The correspondence $\overline{{}^t\alpha'}$ is F-rational, so Lemma 3.4 gives two correspondences $\gamma : N^{\dagger} \rightsquigarrow M^{\dagger}$ and $\delta : M^{\dagger} \rightsquigarrow N^{\dagger}$ such that the motive $(X_E, \delta_E \circ \gamma_E)$ is both an outer direct summand of $N^{\dagger}$ (since it contains the dual motive $O^{\dagger}$) and a direct summand of $M^{\dagger}$. Transposing again, the motive $(X, {}^t\gamma \circ {}^t\delta)$ is an outer direct summand of N and a direct summand of M. □

4. Motivic decompositions for groups of inner type E_6

The purpose of this section is to discuss the complete classification of the motivic decompositions of projective homogeneous varieties of inner type E_6, which is achieved in [4]. Let G be an algebraic group of inner type E_6 and X a projective G-homogeneous variety. We choose the following numbering of the Dynkin diagram G:



The results of Petrov *et al*. [8] show that in the case where X is *generically split*, any indecomposable summand of the $\mathbb{F}_p$-motive of X is isomorphic to a shift of the upper motive $\mathcal{R}_p(G)$ of the variety of Borel subgroups of G. Furthermore, the structure of the motives $\mathcal{R}_p(G)$ is determined in [8] in terms of the so-called J-invariant modulo p of G.

The J-invariant was first introduced by Vishik [11] in the context of quadratic forms. Petrov *et al*. define in [8] the notion of J-invariant modulo p of an arbitrary semisimple algebraic group G, denoted by $J_p(G)$, which is an r-tuple of integers $(j_1, \ldots, j_r)$ given by the rational cycles in $\mathrm{Ch}(\bar{G})$. By Petrov *et al*. [8, Table 4.13], the J-invariant modulo 3 of a semisimple adjoint algebraic group of inner type E_6 is (j_1, j_2), with $0 \leq j_1 \leq 2$ and $0 \leq j_2 \leq 1$.

Another invariant attached to G is the Tits index, which consists of the data of the Dynkin diagram of G with some vertices being circled. The complete classification of the Tits indices of type E_6, provided in [9], is as follows:

Let Θ be a subset of the vertices of the Dynkin diagram of G and X_Θ a projective G-homogeneous variety of type Θ. By Petrov *et al*. [8, Table 3.6] and a case by case analysis of the above Tits indices, the variety X_Θ is either split or generically split if $p \neq 3$, $\Theta \neq \{2\}, \{4\}$ or $\{2, 4\}$ and if $j_1 = 0$. Finally, the results of Chernousov *et al*. [2] and Garibaldi *et al*. [4, Section 8] imply that the understanding of the motivic decompositions of $X_{\{2\}}$, $X_{\{4\}}$ and $X_{\{2,4\}}$ is reduced to the study of the upper motive of X_2 in $\mathrm{CM}(F; \mathbb{F}_3)$, which is denoted by M_{j_1,j_2}.

Using Theorem 1.1, Garibaldi *et al*. provide some restrictions on the Poincaré polynomial of the upper motive of X, if X is an anisotropic projective homogeneous variety satisfying some technical assumptions (see [4, Proposition 7.6]). Assuming that $J_3(G) = (1, 0)$, they observe that although the variety $X_{\{2\}}$ satisfies all those technical assumptions, the Poincaré polynomial of its upper motive does not match with the conclusion of Garibaldi *et al*. [4, Proposition 7.6]. In particular, $X_{\{2\}}$ has a 0-cycle of degree coprime to 3, and $M_{1,0}$ is the Tate motive $\mathbb{F}_3$.

Furthermore, the authors deduce from the fact that $M_{1,0}$ is the Tate motive that the J-invariant modulo 3 of G cannot be $(2, 0)$ (see [4, Corollary 8.10]). Indeed, if $J_3(G) = (2, 0)$ and $\mathrm{SB}(3, A)$ is the Severi–Brauer variety of right ideals of reduced dimension 3 in the Tits algebra of G, then $J_3(G_{F(\mathrm{SB}(3,A))}) = (1, 0)$. In particular, X_2 has a zero-cycle of degree coprime to 3 over the function field of $\mathrm{SB}(3, A)$. It follows that the upper motive of X_2 would be isomorphic to the upper motive of $\mathrm{SB}(3, A)$, and thus the canonical 3-dimensions of X_2 and $\mathrm{SB}(3, A)$ would be equal, which is a contradiction.

The authors use similar techniques to provide isotropy criteria for projective homogeneous varieties. They consider several varieties which satisfy the technical assumptions of Garibaldi *et al*. [4, Proposition 7.6] without fulfilling its conclusion, and thus have a zero cycle of degree 3 (or a rational point). These examples include projective homogeneous varieties for orthogonal group with application to the isotropy of varieties of type E_7 and varieties of type E_8 (see [4, Lemmas 10.15,10.21]). They also produce with these techniques isotropy criteria for projective homogeneous varieties in terms of the Rost invariant (see [4, Proposition 10.18] for type E_7 and [4, Propositions 10.22] for type E_8).

Acknowledgements

I am grateful to N. Karpenko for raising this question and for his suggestions. I also would like to thank N. Semenov for the very useful conversations.

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