Motivic interpretation of the excess intersection formula and ramified case

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Introduction

The starting point of this talk is the localising exact sequence for the Chow group. Recall that if we consider

$$Z \xrightarrow[\text{codim. } n]{i} X \xleftarrow{j} X - Z$$

there is an exact sequence

$$0 \leftarrow \mathrm{CH}^i\left(X - Z\right) \xleftarrow{j^*} \mathrm{CH}^i\left(X\right) \xleftarrow{i_*} \mathrm{CH}^{i-n}\left(Z\right).$$

It is now known that motivic cohomology allows to extend this exact sequence, as we will recall in the course of the talk.

The aim of this lecture is to study the functoriality of this sequence and its extension. For this, we will work directly in the triangulated category of mixed motives of Voevodsky.

The results will then be more general than the corresponding results for higher Chow groups ; the drawback is that we have to supposed that our schemes are smooth.

Table des matières

Notations and recall

Geometry

k a perfect field.
schemes = seperated algebraic k-schemes.
$\mathscr{S}m_k$ = category of smooth schemes

Correspondances and transfers

$\mathscr{S}mCor_k$ = category of smooth schemes equipped with finite correpondances.
$\gamma : \mathscr{S}m_k \to \mathscr{S}mCor_k$ graph functor.
Identity on object, graph on morphism ; it is faithfull
$\mathscr{S}hv(\mathscr{S}mCor_k)$ = category of (Nisnevich) sheaves with transfers. *It is abelian.*

Example 1.– Let X be a smooth scheme. Consider the functor

$$\begin{array}{ccc} \mathscr{S}mCor_k & \xrightarrow{\mathrm{L}[X]} & \mathscr{A}b \\ Y & \mapsto & \mathrm{c}\,(Y, X)\,, \end{array}$$

which maps Y to the abelian group of finite correspondances from Y to X. It is a sheaf with transfers.

Indeed, $\mathrm{L}[X]$ is additive and it's an easy exercice to show that $\mathrm{L}[X] \circ \gamma$ is an étale sheaf.

Motivic complex

The main setting of this talk is the category of motivic complexes $DM_-^{eff}(k)$ such that we have

$$DM_-^{eff}(k) \overset{\text{full}}{\subset} \mathrm{D}^-(\mathscr{S}hv(\mathscr{S}mCor_k)),$$

full subcategory of bounded below complex of sheaves with transfers with homotopy invariants cohomology sheaves.

Example 2.– Following Voevodsky, to any sheaf with transfers F, we associate the motivic complex $\underline{C}^* F$, called the singular simplicial chain complex. Recall that for a smooth scheme X and an integer n, we have

$$\underline{C}^n F(X) = F(\Delta^n \times_k X),$$

$\Delta^\bullet$ meaning the standard cosimplicial scheme. The differential of the complex are given by the alternate sum.

1 General framework

1.1 Closed pairs

Définition 1 *We will call simply* pair *every couple* (X,Z) *of a smooth scheme* X *with a given closed embedding* $i : Z \to X$.

We will say that the pair (X,Z) *is*

1. *smooth if* Z *is smooth,*
2. *of codimension* n *if* Z *is everywhere of codimension* n *in* X.

To such a pair we associate a motivic complex $M(X,Z)$ as follows :

Denote by $\mathrm{L}[X,Z]$ the cokernel of the map $\mathrm{L}[X-Z] \to \mathrm{L}[X]$ induced by the open immersion.

Put $M(X,Z) = \underline{C}^*\mathrm{L}[X,Z]$.

It is thought as the motive of X with support in Z.

The exact sequence of sheaves with transfers

$$0 \to \mathrm{L}[X-Z] \to \mathrm{L}[X] \to \mathrm{L}[X,Z] \to 0$$

induced a distinguished triangle in $DM_-^{eff}(k)$

$$M(X-Z) \to M(X) \to M(X,Z) \to M(X-Z)\,[1].$$

1.2 Morphisms and functoriality

To express the functoriliaty of motives with support, we introduce the followings :

Définition 2 *A morphism of closed pairs* $(f,g) : (Y,T) \to (X,Z)$ *is a commutative diagram*

$$\begin{array}{ccc} T & \rightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \rightarrow & X \end{array}$$

which is cartesian on the underlying topological spaces.

Such a morphism is said to be cartesian is the above square is cartesian in the category of schemes.

For example, if Z is a non reduced closed subscheme in X, and Z_{red} denote its reduction, the canonical morphism $(X,Z_{red}) \to (X,Z)$ is non cartesian.

A morphism of closed pairs $(f,g) : (Y,T) \to (X,Z)$ induces a cartesian diagram of schemes

$$\begin{array}{ccc} Y-T & \rightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ X-Z & \rightarrow & X \end{array}$$

and thus a morphism of motivic complexes

$$M(Y,T) \to M(X,Z)\,.$$

In the example of a non reduced closed subscheme $Z \subset X$, the canonical morphism $M(X, Z_{red}) \to M(X,Z)$ is an isomorphism. However, the use of non cartesian morphism will be usefull in our study of ramification.

1.3 Excision property

(Exc) The following property is relevant for motive with support and underline precisely the role of Nisnevich topology :

A morphism of closed pairs $(f,g) : (Y,T) \to (X,Z)$ will be said excisive if and only if :

1. f is étale,
2. g_{red} is an isomorphism

Then, such a morphism induce an isomorphism on the associated relative motives :
$M(Y,T) \to M(X,Z)$.

Remark 1.– Excisive morphisms are exactly the distinguished squares introduced by Morel and Voevodsky which caracterize Nisnevich topology on $\mathscr{S}m_k$.

2 Purity isomorphism and the Gysin triangle

2.1 Projective bundle theorem

We recall the following result of Voevodsky :
Let X be a smooth scheme, and E be a vector bundle over X of dimension n.
Then, there is a canonical isomorphism

$$M(\mathbb{P}(E)) \to \bigoplus_{i=0}^{n-1} M(X)\,(i)[2i].$$

We shall use this canonical isomorphism to obtain and normalize the purity isomorphism.

2.2 Purity

The following construction is our starting point :

Proposition 1 *Let n be a natural integer.*
There is a unique family of isomorphisms

$$M(X,Z) \xrightarrow{\mathfrak{p}_{(X,Z)}} M(Z)\,(n)[2n]$$

indexed by smooth closed pairs (X,Z) of codimension n such that :

1. *$\mathfrak{p}_{(X,Z)}$ is natural for cartesian morphisms between smooth closed pairs of codimension n.*
2. *for a vector bundle E over a smooth scheme Z of dimension n, the isomorphism $\mathfrak{p}_{(X,Z)}$ is the reciprocla of the composite*

$$M(Z)\,(n)[2n] \xrightarrow{(1)} \bigoplus_{i=0}^{n} M(Z)\,(i)[2i] \xrightarrow{(2)} M(\mathbb{P}(E\oplus 1)) \xrightarrow{(3)} M(\mathbb{P}(E\oplus 1), Z)\,.$$

(1) *is the canonical inclusion,* 2 *is the projective bundle isomorphism,* 3 *is the canonical projection.*

We will just give the tools needed to this proof. Indeed, we use a kind of deformation space to the normal cone.
We will consider $B_Z(\mathbb{A}^1_X)$ the blow-up of $\mathbb{A}^1_X$ with center $Z \times \{0\}$.

$$\begin{array}{c} B_Z(\mathbb{A}^1_X) \\ \big\downarrow{\scriptstyle \text{flat}} \\ \mathbb{A}^1_k \end{array}$$

By definition of the blow-up :
fiber of $B_Z(\mathbb{A}^1_X)$ over $1 =$ X
fiber of $B_Z(\mathbb{A}^1_X)$ over $0 = \mathbb{P}(N_Z X \oplus 1) \cup B_Z X$.

In the last fiber the two subspaces intersects in the infinite projective plan $\mathbb{P}(N_Z X)$.

We set : $D(X,Z) = B_Z(\mathbb{A}^1_X) - (B_Z X - \mathbb{P}(N_Z X))$.

Note that $D(Z,Z) = \mathbb{A}^1_Z$; it is a closed subscheme in $D(X,Z)$.

Then, we can consider the diagram :

$$\begin{array}{ccccc}
M(X,Z) & \longrightarrow & M\big(D,\mathbb{A}^1_Z\big) & \longleftarrow & M(\mathbb{P}(N_Z X \oplus 1), Z) \\
\downarrow{\scriptstyle \mathfrak{p}_{(X,Z)}} & & \downarrow{\scriptstyle \mathfrak{p}_{(D,Z)}} & & \downarrow{\scriptstyle \mathfrak{p}_{(\mathbb{P}(N_Z X\oplus 1),Z)}} \\
M(Z)\,(n)[2n] & \xrightarrow{s_{1*}} & M\big(\mathbb{A}^1_Z\big)\,(n)[2n] & \xleftarrow{s_{0*}} & M(Z)\,(n)[2n]
\end{array}$$

Unicity : From the first condition, the squares in the above diagrams are commutative. Moreover each morphism is an isomorphism. In particular, $\mathfrak{p}_{(X,Z)}$ is determined by the right hand side isomorphism which is forced by the second condition.

Existence : Just indicate

1. By analysing the projective bundle isomorphism, we prove that the morphism considered in condition 2 is an isomorphism (using the homotopy property).
2. We then prove that the morphisms in the upper vertical line of the above diagram are isomorphism by reducing to the case of the closed pair $(\mathbb{A}^n_Z, Z)$ (using excision).

Then $\mathfrak{p}_{(X,Z)}$ is obtained by the curved arrow in the diagram.

2.3 Gysin triangle

As a corollary, we obtain the Gysin triangle which was first considered by Voevodsky :

Let (X,Z) be a smooth pair of codimension n, and $X - Z \xrightarrow{j} X$, $Z \xrightarrow{i} X$ the canonical immersions.

Then the lower triangle in the following commutative diagram is distinguished :

$$\begin{array}{ccccccc}
M(X-Z) & \longrightarrow & M(X) & \longrightarrow & M(X,Z) & \longrightarrow & M(X-Z)\,[1] \\
\| & & \| & & \downarrow{\scriptstyle \mathfrak{p}_{(X,Z)}} & & \| \\
M(X-Z) & \xrightarrow{j_*} & M(X) & \xrightarrow{i^*} & M(Z)\,(n)[2n] & \xrightarrow{\partial_{X,Z}} & M(X-Z)\,[1]
\end{array}$$

Notations i^* and $\partial_{X,Z}$ stands for definitions.

1.– *Link with the Chow groups.*– If we apply the functor $\mathrm{Hom}_{DM_-^{eff}(k)}(?, \mathbb{Z}(i)[2i])$ to the above distinguished triangle, we obtain the following long exact sequence :

$$\begin{array}{ccccccc}
H^{2i}_{\mathcal{M}}(X - Z; \mathbb{Z}(i)) & \xleftarrow{(j_*)^\sharp} & H^{2i}_{\mathcal{M}}(X; \mathbb{Z}(i)) & \xleftarrow{(i^*)^\sharp} & H^{2(i-n)}_{\mathcal{M}}(Z; \mathbb{Z}(i-n)) & \xleftarrow{\partial^\sharp_{X,Z}} & \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \\
\mathrm{CH}^i(X - Z) & \xleftarrow{j^*} & \mathrm{CH}^i(X) & \xleftarrow{i_*} & \mathrm{CH}^{i-n}(Z) & &
\end{array}$$

The commutativity of the squares is obtained by an explicit computation (made in my thesis).

Remark 2.– This is the occasion to underline that Voevodsky choose the homological notation for his motives (countrary to Grothendieck). In particular, the morphism j_* on motives corresponds to the pullback morphism j^* on the Chow groups. Moreover the morphism i^* defined by the Gysin triangle induced the pushout morphism i_* on the Chow group. We should be carefull that i^* is sometimes called the Gysin morphism but it does not induce the Gysin morphism i^* on the Chow group.

3 Functoriality

We fix for all this section a morphism $(f,g):(Y,T)\to(X,Z)$ of smooth closed pairs and consider

$$\begin{array}{ccccc} T & \xrightarrow[\text{codim. } m]{k} & Y & \xleftarrow{l} & Y-T \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ Z & \xrightarrow[\text{codim. } n]{i} & X & \xleftarrow{j} & X-Z. \end{array}$$

It induces a morphism of ditinguished Gysin triangle :

$$\begin{array}{ccccccc} M(Y-T) & \xrightarrow{l_*} & M(Y) & \xrightarrow{k^*} & M(T)\,(m)[2m] & \xrightarrow{\partial_{Y,T}} & M(Y-T)\,[1] \\ {\scriptstyle h_*}\downarrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (f,g)_!} & & \downarrow{\scriptstyle h_*} \\ M(X-Z) & \xrightarrow{j_*} & M(X) & \xrightarrow{i^*} & M(Z)\,(n)[2n] & \xrightarrow{\partial_{X,Z}} & M(X-Z)\,[1]. \end{array}$$

Again, $(f,g)_!$ stands for definition. It is obtained via the purity isomorphisms for (X,Z) and (Y,T). Our study will consist to determine this morphism in certain particular case.

As we saw from the above diagram, this will imply a kind of projection formula from the center square. Indeed, if we apply anew the functor $\mathrm{Hom}_{DM_-^{eff}(k)}(?,\mathbb{Z}(i)[2i])$ to the middle square of the above diagram, we obtain

$$\begin{array}{ccc} \mathrm{CH}^i\,(Y) & \xleftarrow{k_*} & \mathrm{CH}^{i-n}\,(T) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f,g)^!} \\ \mathrm{CH}^i\,(X) & \xleftarrow{i_*} & \mathrm{CH}^{i-n}\,(Z) \end{array}$$

This diagram explain our choice of notations for $(f,g)_!$. Indeed, in case (f,g) is cartesian, Fulton associate to the cartesian square a morphism $\mathrm{CH}^*\,(T)\to\mathrm{CH}^*\,(Z)$ called the refined Gysin morphism (associate to the square) ; he denotes it $f^!$ forgetting in the notation the entire square. The commutativity of the above square can be interpreted as a generalized projection formula.

The morphism we define on motives is the analogue of this refined Gysin morphism. The formulas we obtained allows to recover results of Fulton on this refined Gysin morphism.

3.1 Transversal case

Let $f:Y\to X$ be a morphism and Z be a closed subscheme of codimension n. Recall that we say f is transversal to f if and only if $f^{-1}Z$ is smooth of codimension n.

In the case of a morphism of smooth closed pairs $(Y, T) \to (X, Z)$, we say it is transversal if the codimension of Z in X is equal to the codimension of T in Y.

In this case, the first property of the purity morphism implies :

Proposition 2 *Suppose we are in the transversal case : (f, g) is cartesian and $n = m$.*

Then, $(f, g)_! = g_*(n)[2n]$.

Indeed, as we saw, the purity morphism is functorial againt transversal morphism of closed pairs.

Ther interesting cases follow.

3.2 Excess intersection

To assert the next theorem, we need a piece of notations :

Using the canonical isomorphism $H^{2i}_{\mathcal{M}}(X; \mathbb{Z}(i)) \simeq \mathrm{CH}^i(X)$, we can define Chern classes in motivic cohomology : if X is a smooth scheme and E be a vector bundle over X, we can consider the i-th Chern class of E in motivic cohmology :

$$c_i(E) \in H^{2i}_{\mathcal{M}}(X; \mathbb{Z}(i)).$$

But moreover, by definition, $H^{2i}_{\mathcal{M}}(X; \mathbb{Z}(i)) = \mathrm{Hom}_{DM^{eff}_-(k)}(M(X), \mathbb{Z}(i)[2i])$; we will denote

$$\mathfrak{c}_i(E) : M(X) \to \mathbb{Z}(i)[2i]$$

the morphism corresponding to the i-th Chern class.

The next proposition concern the case where there is a hole in codimension :

Proposition 3 *Suppose there is an excess of intersection : (f, g) is cartesian but $n > m$.*

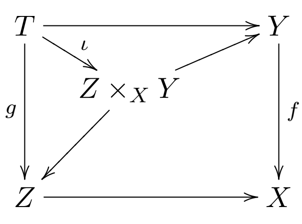
If we set $e = n - m$ and denote $\xi = g^* N_Z X / N_T Y$ the quotient vector bundle on T, then

$$(f, g)_! = (\mathfrak{c}_e(\xi) \otimes g_*) \circ \Delta_{T*}.$$

In the case of motivic comology in bidegree $(2i, i)$, we simply get the excess intersection formula (cf for example [Ful98]). But this formula is also valid for over weight and concerns also the residue morphism.

3.3 Ramification case

The next theorem concerns the case where (f, g) is not necessarily cartesian. In this case, there is a canonical closed immersion ι :



Recall that (f,g) is cartesian on the underlying topological spaces. In particular, ι is a universal homeomorphism. This means that it is radicial.

Proposition 4 *Suppose that $n = m = 1$.*

Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be the ideal of T (resp. $Z \times_X Y$) in Y. Suppose that there exist an integer $r > 0$ such that $\mathcal{I}' = \mathcal{I}^r$.

Then, $(f,g)_! = r.g_*(1)[2]$.

Remark 1.– The condition of the proposition can also be expressed by saying that the ramification index of f at every point of Z is r.

TO obtain this formula, we use the fact that the deformation space is functorial with respect to the morphism (f,g) :

$$\begin{array}{ccccc} Y & \longrightarrow & D_T Y & \longleftarrow & \mathbb{P}(N_T Y \oplus 1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle D_g f} & & \downarrow \\ X & \longrightarrow & D_Z X & \longleftarrow & \mathbb{P}(N_Z X \oplus 1). \end{array}$$

Moreover, the right hand side morphism is elevation to the r-th power on the parameter of the vector bundle which imply the proposition we need by a carefull analysis of the projective bundle isomorphism.

This formula is particularly inspired by the analog result in Milnor K-theory concerning a ramified extension of discrete valuation ring and the residue morphism. In fact, the residue morphism in the Gysin triangle induces the residue morphism in Milnor K-theory, corresponding to motivic cohmology in degree (i,i).

Realisation

Let me just mention another application of these formula.

Consider a cohomological theory $H^* : \mathscr{S}m_k^{op} \to \mathscr{A}b$ which induces a triangulated realisation functor :

$$R_H : DM_{gm}(k)^{op} \to D(\mathscr{A}b).$$

Then, we can define the H-cohomology of a closed pair (X,Z). It satisfies the purity isomorphism and thus we obtain a localisation long exact sequence. Then the funcotriality results we obtained are applicable for this cohomology.

Let mention a few examples, using results of A.Huber (cf [Hub00] and [Hub04]) :

1. the De Rham cohomology in characteristic 0
2. the continuous étale l-adic cohomology with rational coefficients

Also, we can construct such realisations for

1. the continuous étale l-adic cohomology with integral coefficients
2. the rigid cohomology of Berthelot (for characteristic p).

Références

[Dég04] F. Déglise. Interprétation motivique de la formule d'excès d'intersection. *C. R. Math. Acad. Sci. Paris*, 338(1) :41–46, 2004. Presented by J.P. Serre.

[Ful98] W. Fulton. *Intersection theory.* Springer, second edition, 1998.

[Hub00] Annette Huber. Realization of Voevodsky's motives. *J. Algebraic Geom.*, 9(4) :755–799, 2000.

[Hub04] A. Huber. Corrigendum to : "Realization of Voevodsky's motives" [J. Algebraic Geom. **9** (2000), no. 4, 755–799 ; 1775312]. *J. Algebraic Geom.*, 13(1) :195–207, 2004.