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Wiener's test for super-Brownian motion and the Brownian snake

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Summary. We prove a Wiener-type criterion for super-Brownian motion and the Brownian snake. If F is a Borel subset of $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, we provide a necessary and sufficient condition for super-Brownian motion started at δ_x to immediately hit the set F. Equivalently, this condition is necessary and sufficient for the hitting time of F by the Brownian snake with initial point x to be 0. A key ingredient of the proof is an estimate showing that the hitting probability of F is comparable, up to multiplicative constants, to the relevant capacity of F. This estimate, which is of independent interest, refines previous results due to Perkins and Dynkin. An important role is played by additive functionals of the Brownian snake, which are investigated here via the potential theory of symmetric Markov processes. As a direct application of our probabilistic results, we obtain a necessary and sufficient condition for the existence in a domain D of a positive solution of the equation $\Delta u = u^2$ which explodes at a given point of ∂D.

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1 Introduction

This work is concerned with regularity problems for super-Brownian motion and the Brownian snake. These problems are closely related to the characterization of polar sets for super-Brownian motion. A sufficient condition for non-polarity, in terms of a suitable capacity, was first given by Perkins [23], and later Dynkin [8] proved the necessity of this condition, by using analytic results of Baras and Pierre [2] concerning removable singularities of semilinear partial differential equations. In the present work, we refine the Dynkin–Perkins result by showing that the hitting probability of a set is comparable, up to multiplicative constants, to its capacity. We then apply these estimates to an analogue of the classical Wiener criterion for super-Brownian motion. In the same way as for usual Brownian motion, this Wiener criterion has an interesting analytic counterpart. Namely, if D is a domain in $\mathbb{R}^d$ and $x \in \partial D$, we obtain a necessary and sufficient condition for the existence of a positive solution of the partial differential equation $\Delta u = u^2$ in D that tends to ∞ at x. The existence of such solutions has been investigated in a more general setting by Marcus and Véron [18, 19]. However, in the special case of equation $\Delta u = u^2$, our results give much more precise information than the analytic methods.

We use the path-valued process called the Brownian snake as a basic tool. Let us briefly describe this process, referring to [14, 15] for more detailed information. The Brownian snake is a Markov process taking values in the set of all stopped paths in $\mathbb{R}^d$. By definition, a stopped path in $\mathbb{R}^d$ is a continuous mapping $w : [0, \zeta] \to \mathbb{R}^d$. The number $\zeta = \zeta_w \geqq 0$ is called the lifetime of the path. We sometimes write $\hat{w} = w(\zeta_w)$ for the tip of the path w. We denote by $\mathscr{W}$ the set of all stopped paths in $\mathbb{R}^d$. This set is equipped with the distance $d(w, w') = |\zeta - \zeta'| + \sup_{t \geqq 0} |w(t \wedge \zeta) - w'(t \wedge \zeta')|$. Let us fix $x \in \mathbb{R}^d$ and denote by $\mathscr{W}_x$ the set of all stopped paths with initial point $w(0) = x$. The Brownian snake with initial point x is the continuous strong Markov process $W = (W_s, s \geqq 0)$ in $\mathscr{W}_x$ whose law is characterized as follows.

1. If ζ_s denotes the lifetime of W_s, the process $(\zeta_s, s \geqq 0)$ is a reflecting Brownian motion in $\mathbb{R}_+$.
2. Conditionally on $(\zeta_s, s \geqq 0)$, the process W remains a (time-inhomogeneous) Markov process. Its conditional transition kernels are described by the following properties: For $s < s'$,

– $W_{s'}(t) = W_s(t)$ for every $t \leqq m(s, s') := \inf_{[s,s']} \zeta_r$;
– $(W_{s'}(m(s, s') + t) - W_{s'}(m(s, s')), 0 \leqq t \leqq \zeta_{s'} - m(s, s'))$ is a standard Brownian motion in $\mathbb{R}^d$ independent of W_s.

Informally, one should think of W_s as a Brownian path in $\mathbb{R}^d$ with a random lifetime ζ_s evolving like (reflecting) linear Brownian motion. When ζ_s decreases, the path W_s is "erased" from its tip. When ζ_s increases, the path W_s is extended (independently of the past) by adding "small pieces" of Brownian motion at its tip.

We may and will assume that the process $(W_s, s \geqq 0)$ is the canonical process on the space $\Omega := C(\mathbb{R}_+, \mathscr{W})$ of all continuous functions from $\mathbb{R}_+$ into $\mathscr{W}$. The canonical filtration on Ω is denoted by $(\mathscr{F}_s^\circ)_{s \geqq 0}$. For $w \in \mathscr{W}$, we denote by $\mathbb{P}_w$ the law of the Brownian snake started at w. We denote by $\underline{x}$ the trivial path in $\mathscr{W}_x$ with lifetime 0 and for simplicity we write $\mathbb{P}_x$ instead of $\mathbb{P}_{\underline{x}}$.

It is immediate that $\underline{x}$ is a regular recurrent point for the Markov process W. We denote by $\mathbb{N}_x$ the associated excursion measure. Then $\mathbb{N}_x$ is a σ-finite measure on the set Ω_0 of all $\omega \in \Omega$ for which there exists a number $\sigma = \sigma(\omega) \geqq 0$ such that $\zeta_s(\omega) \neq 0$ if and only if $0 < s < \sigma$. The law of W under $\mathbb{N}_x$ is described by properties analogous to 1 and 2, with the only difference that the law of reflecting Brownian motion in 1 is replaced by the Itô measure of positive excursions of linear Brownian motion (see [15]). We can normalize $\mathbb{N}_x$ so that, for every $\varepsilon > 0$, $\mathbb{N}_x(\sup_{s \geqq 0} \zeta_s > \varepsilon) = (2\varepsilon)^{-1}$. The following scaling property of $\mathbb{N}_x$ is often useful. For $\lambda > 0$, define $W^{(\lambda)} = (W_s^{(\lambda)}, s \geqq 0)$ by

$$W_s^{(\lambda)}(t) = x + \lambda(W_{\lambda^{-4}s}(\lambda^{-2}t) - x), \quad 0 \leqq t \leqq \zeta_{W_s^{(\lambda)}} = \lambda^2 \zeta_{\lambda^{-4}s}.$$

Then the law of $W^{(\lambda)}$ under $\mathbb{N}_x$ is $\lambda^2 \mathbb{N}_x$. As a consequence, one can check that, for every $\delta > 0$,

$$\mathbb{N}_x\left(\sup_{s \geqq 0} |\hat{W}_s - x| \geqq \delta\right) = c(d)\,\delta^{-2},$$

with a constant $c(d) < \infty$ depending only on d. We also introduce the random sets

$$\mathscr{R} = \{\hat{W}_s,\ s \geqq 0\}, \qquad \mathscr{R}^* = \{\hat{W}_s,\ s \geqq 0,\ \zeta_s > 0\},$$

that will be of interest under the excursion measure $\mathbb{N}_x$. Notice that $\mathscr{R} = \mathscr{R}^* \cup \{x\}$, $\mathbb{N}_x$ a.e.

Let us now explain the connection between the Brownian snake and super-Brownian motion. Denote by $(L_s^t(\zeta), s \geqq 0, t \geqq 0)$ the jointly continuous collection of local times of the process $(\zeta_s, s \geqq 0)$. Notice that $(L_s^0(\zeta), s \geqq 0)$ is also the local time of W at $\underline{x}$. We set $\tau_1 := \inf\{s \geqq 0, L_s^0(\zeta) > 1\}$ and, for every $t \geqq 0$, we let X_t be the random measure on $\mathbb{R}^d$ defined by

$$\langle X_t, \varphi \rangle = \int_0^{\tau_1} d_s L_s^t(\zeta)\, \varphi(\hat{W}_s).$$

Then $(X_t, t \geqq 0)$ is under $\mathbb{P}_x$ a super-Brownian motion in $\mathbb{R}^d$ started at δ_x (see [14]). The associated historical process is the process $(\mathscr{X}_t, t \geqq 0)$ defined by

$$\langle \mathscr{X}_t, \Phi \rangle = \int_0^{\tau_1} d_s L_s^t(\zeta)\, \Phi(W_s).$$

Let $\mathscr{R}^X$ denote the range of X, defined as in [4]:

$$\mathscr{R}^X = \bigcup_{\varepsilon > 0} \left(\overline{\bigcup_{t \geqq \varepsilon} \operatorname{supp} X_t} \right),$$

where $\operatorname{supp} X_t$ is the topological support of X_t, and $\bar{A}$ denotes the closure of a subset A of $\mathbb{R}^d$. Then, it is easily checked (cf. [14], proof of Proposition 2.2) that

$$\mathscr{R}^X = \{\hat{W}_s,\ 0 \leqq s \leqq \tau_1,\ \zeta_s > 0\}. \tag{1}$$

A Borel subset F of $\mathbb{R}^d$ is called $\mathscr{R}$-polar if $\mathbb{P}_x(\mathscr{R}^X \cap F \neq \emptyset) = 0$ for some (or equivalently for every) $x \in \mathbb{R}^d$. From (1) and excursion theory, we have

$$\mathbb{P}_x(\mathscr{R}^X \cap F \neq \emptyset) = 1 - \exp -\mathbb{N}_x(\mathscr{R}^* \cap F \neq \emptyset), \tag{2}$$

and therefore F is $\mathscr{R}$-polar if and only if $\mathbb{N}_x(\mathscr{R}^* \cap F \neq \emptyset) = 0$ for $x \in \mathbb{R}^d$. For every $\beta \geqq 0$, let $\operatorname{cap}_\beta(F)$ be the capacity defined by

$$\operatorname{cap}_\beta(F) = (\inf(\textstyle\iint \nu(dy)\nu(dy')\, \psi_\beta(y, y')))^{-1},$$

where the infimum is taken over all probability measures ν supported on F, and

$$\psi_\beta(y, y') := \begin{cases} 1 + \log_+ \dfrac{1}{|y - y'|} & \text{if } \beta = 0, \\ |y - y'|^{-\beta} & \text{if } \beta > 0. \end{cases}$$

Perkins and Dynkin proved that there are no nonempty $\mathscr{R}$-polar sets when $d \leqq 3$ and, when $d \geqq 4$, a set F is $\mathscr{R}$-polar if and only if $\mathrm{cap}_{d-4}(F) = 0$.

We denote by $B(y, r)$ the open ball of radius r centered at y, and by $\bar{B}(y, r)$ the closed ball.

Theorem 1. *Suppose that $d \geqq 4$. There exist two positive constants α, β such that, if F is a Borel set contained in the ball $\bar{B}(0, 1/2)$, and if $|x| \geqq 1$, then*

$$\frac{\alpha |x|^{-2}\, \mathrm{cap}_0(F)}{1 + (\log |x|)\, \mathrm{cap}_0(F)} \leqq \mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset) \leqq \frac{\beta |x|^{-2}\, \mathrm{cap}_0(F)}{1 + (\log |x|)\, \mathrm{cap}_0(F)} \quad \text{if } d = 4,$$

$$\alpha |x|^{2-d}\, \mathrm{cap}_{d-4}(F) \leqq \mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset) \leqq \beta |x|^{2-d}\, \mathrm{cap}_{d-4}(F), \qquad \text{if } d \geqq 5.$$

The same bounds hold with $\mathbb{P}_x(\mathscr{R}^X \cap F \neq \emptyset)$ instead of $\mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset)$.

Remarks. (i) The last assertion is an immediate consequence of the first part of the theorem and (2).

(ii) When $d \leqq 3$, since points are not $\mathscr{R}$-polar, a simple scaling argument shows that, under the assumptions of Theorem 1, if $F \neq \emptyset$,

$$\alpha |x|^{-2} \leqq \mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset) \leqq \beta |x|^{-2}.$$

(iii) Bounds analogous to Theorem 1 can also be given for the probability that the ranges of p independent super-Brownian motions have a common point in the set F. For simplicity, consider the case $d \geqq 5$. Let $k \geqq 1$ and let $Y^1, \ldots, Y^k$ be k independent super-Brownian motions started respectively at $\delta_{x_1}, \ldots, \delta_{x_k}$, where $1 \leqq |x_j| \leqq 2$. There exist two positive constants α_k and β_k such that, if F is a Borel subset of $B(0, 1/2)$, then

$$\alpha_k\, \mathrm{cap}_{k(d-4)}(F) \leqq P(\mathscr{R}^{Y^1} \cap \cdots \cap \mathscr{R}^{Y^k} \cap F \neq \emptyset) \leqq \beta_k\, \mathrm{cap}_{k(d-4)}(F), \qquad (3)$$

where $\mathscr{R}^{Y^j}$ denotes the range of Y^j. These bounds follow from Theorem 1 by using a technique of Peres [22]: By Theorem 1 and Corollary 4.3(i) of [22], each set $\mathscr{R}^{Y^j}$ is intersection-equivalent in $B(0, 1/2)$ (in the sense of [22, Definition 1]) to a percolation set $Q_d(2^{-(d-4)})$ constructed as in [22], with the minor modification that one starts from the cube $[-1/2, 1/2]^d$ rather than from $[0, 1]^d$. From Lemmas 2.2 and 2.3 of [22], one gets that $\mathscr{R}^{Y^1} \cap \cdots \cap \mathscr{R}^{Y^k}$ is intersection equivalent in $B(0, 1/2)$ to $Q_d(2^{-k(d-4)})$, and the bounds (3) then follow from Corollary 4.3(i) of [22]. As was observed by Yuval Peres (personal communication), a qualitative version of the previous argument, using only the Dynkin–Perkins characterization of polar sets, already gives the weaker fact that the probability considered in (3) is strictly positive if and only if $\mathrm{cap}_{k(d-4)}(F) > 0$.

From (3), or from its weak version, one can derive the converse of a theorem of Perkins ([23, Theorem 5.9]). According to this theorem, the condition $\mathrm{cap}_{k(d-4)}(F) > 0$ is sufficient for the set F to contain k-multiple points of super-Brownian motion with positive probability. The necessity of the condition follows from (3), using also Lemma 16 of Serlet [25] which allows us to derive information on k-multiple points of a single super-Brownian motion from intersections of k independent superprocesses. As a special case, we recover the known fact that there are no k-multiple points when $k(d-4) \geqq d$.

The lower and upper bounds of Theorem 1 are proved in a very different way. The proof of the lower bound uses the familiar idea of constructing an additive functional of the Brownian snake that will be nonzero only if the process hits F. In Sect. 2, we develop a general theory of additive functionals of the Brownian snake. In view of forthcoming applications, we have presented this theory in more generality than was really needed here. Additive functionals of the Brownian snake play an important role in the probabilistic representation of solutions of $\Delta u = u^2$ in a domain (see [17]). Our construction of additive functionals should also be compared with the results of Dynkin and Kuznetsov [10], who introduce more general functionals of Markov snakes. When the relevant properties of additive functionals have been established, the proof of the lower bound of Theorem 1 becomes easy, and is presented in Sect. 3.

The proof of the upper bound of Theorem 1 is given in Sect. 4. It relies on analytic estimates inspired from Baras and Pierre [2], along the lines of [16], Sect. 3. These analytic ingredients are here supplemented by a suitable application of Itô's formula.

Let us now turn to the Wiener criterion. We set

$$\tau_F := \inf\{t > 0, \operatorname{supp} X_t \cap F \neq \emptyset\}.$$

Let $(\mathscr{G}_t^\circ)_{t \geqq 0}$ denote the natural filtration of X, and let $(\mathscr{G}_t)_{t \geqq 0}$ be its augmentation with the class of all $\mathbb{P}_x$-negligible sets of $(\Omega, \mathscr{F}_\infty^\circ)$. As X is Markov with respect to $(\mathscr{G}_{t+}^\circ)$, standard arguments show that the filtration $(\mathscr{G}_t)_{t \geqq 0}$ is right-continuous. In particular $\mathscr{G}_{0+} = \mathscr{G}_0$ is $\mathbb{P}_x$-trivial. We show in Sect. 5 that τ_F is a stopping time of the filtration $(\mathscr{G}_t)$. Thus, $\mathbb{P}_x(\tau_F = 0) = 0$ or 1.

We say that x is *super-regular* for F if $\mathbb{P}_x(\tau_F = 0) = 1$. Let F^{sr} denote the set of all points that are super-regular for F. From the known path properties of super-Brownian motion (see [4, 23]) it is obvious that $\mathrm{int}(F) \subset F^{sr} \subset \overline{F}$, where $\mathrm{int}(F)$ denotes the interior of F.

We also set $T_F = \inf\{s > 0, \hat{W}_s \in F, \zeta_s > 0\}$. The following proposition shows that super-regularity can be characterized from the behavior of the Brownian snake in different ways.

Proposition 2. *Let F be a Borel subset of $\mathbb{R}^d$. The following properties are equivalent*:

(i) *x is super-regular for F*;
(ii) $\mathbb{P}_x(T_F = 0) = 1$;
(iii) $\mathbb{N}_x(\mathscr{R}^* \cap F \neq \emptyset) = \infty$.

The next theorem, which is analogous to the classical Wiener criterion (see e.g. [24]), provides a characterization of the set of super-regular points.

Theorem 3. *Let F be a Borel subset of $\mathbb{R}^d$ and, for every $x \in \mathbb{R}^d$, $n \geqq 1$, let*

$$F_n(x) = \{y \in F,\ 2^{-n} \leqq |y - x| < 2^{-n+1}\}.$$

If $d \leqq 3$, $F^{sr} = \overline{F}$. If $d \geqq 4$, $x \in F^{sr}$ if and only if

$$\sum_{n=1}^{\infty} 2^{n(d-2)} \mathrm{cap}_{d-4}(F_n(x)) = \infty. \tag{4}$$

The case $d \leqq 3$ of Theorem 3 is very easy: If $x \in \overline{F}$, then we may find a sequence (x_n) in F converging to x. By the scaling properties of $\mathbb{N}_x$, we have $\mathbb{N}_x(\mathscr{R}^* \cap F \neq \emptyset) \geqq \mathbb{N}_x(x_n \in \mathscr{R}^*) = c|x - x_n|^{-2}$, where $c > 0$ because points are not $\mathscr{R}$-polar in dimension $d \leqq 3$. The desired result then follows from Proposition 2.

The proofs of Proposition 2 and Theorem 3 (when $d \geqq 4$) are given in Sect. 5. The necessity of the condition (4) is easy from the upper bound in Theorem 1 and the characterization of super-regularity given by Proposition 2(iii). The key idea of the sufficiency part is to introduce $M = \sup\{\zeta_s, s \geqq 0\}$ and to observe that

$$\mathbb{N}_x(\mathscr{R}^* \cap F \neq \emptyset) = \mathbb{N}_x(T_F < \infty) \geqq \sum_{n=1}^{\infty} \mathbb{N}_x(T_F < \infty, 2^{-2n} \leqq M < 2^{-2n+2}).$$

Using a refinement of the lower bound of Theorem 1, one can check that each term $\mathbb{N}_x(T_F < \infty, 2^{-2n} \leqq M < 2^{-2n+2})$ is bounded below by a constant times the corresponding term in the series (4).

The classical Wiener criterion was originally formulated in analytic terms [27]. Similarly, one can give an analytic formulation of Theorem 3. The proof of the next theorem, which is also given in Sect. 5, is easy from the connections between super-Brownian motion or the Brownian snake and the partial differential equation $\Delta u = u^2$ (see [8] or [15]).

Theorem 4. *Let D be a domain in $\mathbb{R}^d$ such that $x \in \partial D$. Set $F = \mathbb{R}^d \backslash D$, and let $F_n(x)$ be as in the previous theorem. The following two conditions are equivalent.*

(i) *There exists a nonnegative solution of $\Delta u = u^2$ in D such that*

$$\lim_{D \ni y \to x} u(y) = \infty.$$

(ii) *Either $d \leqq 3$, or $d \geqq 4$ and (4) holds.*

Notice that every nonnegative solution of $\Delta u = u^2$ is bounded above by the maximal solution, which has a simple probabilistic expression (see e.g. [15, Sect. 5]). Therefore, if $d \leqq 3$, or if $d \geqq 4$ and (4) holds at every $x \in \partial D$, the maximal solution tends to ∞ at every $x \in \partial D$.

The existence of positive solutions to the more general equation $\Delta u = u^p$ with infinite boundary conditions has been recently investigated by Marcus and Véron ([18, Theorem 2.3] see also [26, Sect. 4.2]). Marcus and Véron show that such solutions exist under an exterior cone condition on D, which can be weakened to an exterior segment condition when $1 < p < (d-1)/(d-3)$. Clearly, Theorem 4 gives a much more precise result in the special case $p = 2$.

In Sect. 6, as an application of Theorem 3, we treat the special case when D is the complement of a thorn with vertex at x. The super-regularity of x is then characterized by an integral test analogous to the classical result for usual Brownian motion.

2 Additive functionals of the Brownian snake

Our goal in this section is to construct a general class of additive functionals of the Brownian snake and to investigate their properties. Clearly our results are related to the theory of additive functionals for general Markov processes that has been developed by different authors under various assumptions (see, in particular, [3, 7, 12]). However, as we were unable to derive our results from a known theory, and also because we will need several specific facts, we will present a short self-contained construction in the special case of the Brownian snake. We follow ideas from Dynkin [7] and rely on some results of Fitzsimmons and Getoor [11].

Throughout this section, x is a fixed point of $\mathbb{R}^d$. For $w \in \mathscr{W}_x$, the law of the Brownian snake started at w and killed when it hits the trivial path $\underline{x}$ is denoted by $\mathbb{P}_w^*$. We may consider $(W_s, \mathbb{P}_w^*)$ as a Markov process in $\mathscr{W}_x$, with transition kernels Q_s^*, and the trivial path $\underline{x}$ plays the role of a cemetery point for this Markov process. The set of nontrivial paths in $\mathscr{W}_x$ is denoted as $\mathscr{W}_x^*$. Let $U(w, dw')$ denote the potential kernel of the killed process. According to [16], we have for $w \in \mathscr{W}_x^*$

$$U(w, dw') = 2 \int_0^{\zeta_w} da \int_a^{\infty} db \, R_{a,b}(w, dw'),$$

where the kernel $R_{a,b}(w, dw')$ can be defined as follows. Under $R_{a,b}(w, dw')$, the path w' coincides with w on $[0, a]$, and is then distributed as a Brownian path started from $w(a)$ at time a, and stopped at time b. Let $P_x^b(dw) = R_{0,b}(\underline{x}, dw)$ be the law of a Brownian path started at x and stopped at time b. Then, the process $(W_s, \mathbb{P}_w^*)$ is symmetric with respect to the σ-finite measure

$$M_x(dw) = \int_0^{\infty} db \, P_x^b(dw).$$

The measure M_x and the excursion measure $\mathbb{N}_x$ are linked by the formula

$$\mathbb{N}_x \left(\int_0^{\sigma} \varphi(W_s) \, ds \right) = M_x(\varphi),$$

for any nonnegative measurable function φ on $\mathscr{W}_x$. It is easy to verify that the process $(W_s, \mathbb{P}_w^*)$ satisfies all assumptions of Fitzsimmons and Getoor [11]. Let F be a Borel subset of $\mathscr{W}_x^*$ and let $H_F = \inf\{s \geqq 0, W_s \in F\}$. Then F is called M_x-polar if

$$M_x(dw) \text{ a.e.} \quad \mathbb{P}_w^*(H_F < \infty) = 0.$$

A property is said to hold quasi-everywhere (q.e.) if it holds outside an M_x-polar set.

Let μ be a finite measure on $\mathscr{W}_x^*$. It is straightforward to check that μU is then σ-finite (in fact $\mu U(\{w, \zeta_w \leqq b\}) < \infty$ for every $b > 0$). When μU is absolutely continuous with respect to M_x, we may choose (see [11] and the references therein) a Borel measurable excessive version $U(\mu)$ of the Radon–Nikodym derivative of μU with respect to M_x. We say that μ has finite energy if in addition

$$\mathscr{E}(\mu) := \langle \mu, U(\mu) \rangle < \infty.$$

This condition and the quantity $\mathscr{E}(\mu)$ do not depend on the choice of $U(\mu)$ (see [11]).

Recall that $(\mathscr{F}_s^\circ, s \geqq 0)$ is the canonical filtration on Ω. For every $s \geqq 0$, the shift operator θ_s on Ω is defined by $W_r(\theta_s \omega) = W_{s+r}(\omega)$. Finally, we extend the definition given in the introduction by setting $\sigma(\omega) = \inf\{s > 0, \zeta_s(\omega) = 0\}$, for every $\omega \in \Omega$.

Definition. *An $(\mathscr{F}_t^\circ)$-adapted increasing process A on Ω, such that $A_0 = 0$, is called an additive functional of the killed Brownian snake if the following properties hold. There exist a Borel subset N of $\mathscr{W}_x^*$ and a Borel subset Λ of Ω such that*:

(a) *N is M_x-polar, and for every $w \in \mathscr{W}_x^* \backslash N$, $\mathbb{P}_w^*(H_N < \infty) = 0$;*
(b) *for every $s \geqq 0$, $\theta_s(\Lambda) \subset \Lambda$;*
(c) *$\mathbb{N}_x(\Omega \backslash \Lambda) = 0$ and, for every $w \in \mathscr{W}_x^* \backslash N$, $\mathbb{P}_w^*(\Lambda) = 1$;*
(d) *for every $\omega \in \Lambda$, the function $s \to A_s(\omega)$ is continuous, constant over $[\sigma, \infty)$ and such that, for every $r, s \geqq 0$, $A_{r+s}(\omega) = A_r(\omega) + A_s(\theta_r \omega)$.*

Theorem 5. *Let μ be a finite measure on $\mathscr{W}_x^*$ with finite energy. There exists an additive functional A of the killed Brownian snake such that the following properties hold*:

(i) *$\mathbb{E}_w^*(A_\infty) = U(\mu)(w)$, q.e.;*
(ii) *for every nonnegative Borel function f on $\mathscr{W}_x^*$,*

$$\mathbb{N}_x\left(\int_0^\infty f(W_s)\, dA_s\right) = \langle \mu, f \rangle \;;$$

(iii) *$\mathbb{N}_x(A_\infty^2) = 2\,\mathscr{E}(\mu)$.*

Moreover, if ρ denotes the time-reversal operator on Ω_0 (i.e. $W_s(\rho\omega) = W_{(\sigma-s)_+}(\omega)$), then

$$A_s \circ \rho = A_\sigma - A_{(\sigma-s)_+} \quad \text{for every } s \geqq 0,\ \mathbb{N}_x \text{ a.e.}$$

Proof. Denote by $\mathscr{B}(E)$ the Borel σ-field on a metric space E. Because $\mathscr{E}(\mu) < \infty$, we may apply Proposition 3.7 of [11] to construct an M_x-exit law ϕ associated with $U(\mu)$. Precisely, there exists a family $\phi = (\phi_s)_{s>0}$ of nonnegative functions on $\mathscr{W}_x^*$ such that the mapping $(s, x) \to \phi_s(x)$ is

$\mathscr{B}(]0,\infty[) \otimes \mathscr{B}(\mathscr{W}_x^*)$-measurable, for every $s, s' > 0$, $Q_s^* \phi_{s'} = \phi_{s+s'}$, M_x a.e., and

$$U(\mu) = \int_0^\infty \phi_s \, ds, \, M_x \text{ a.e.} \tag{5}$$

Furthermore, $\phi_s \in L^2(M_x)$ for every $s > 0$ and

$$M_x \left(\int_0^\infty \phi_s^2 \, ds \right) = \frac{1}{2} \mathscr{E}(\mu) < \infty. \tag{6}$$

Finally, we have also $\mu Q_s^* = \phi_s \cdot M_x$ for every $s > 0$ ([11, Theorem (3.16)]). Notice that, in [11], the mapping $(s,x) \to \phi_s(x)$ is $\mathscr{B}(]0,\infty[) \otimes \mathscr{E}^e$-measurable, where $\mathscr{E}^e$ denotes the σ-field generated by excessive functions. Here however, as the excessive function $U(\mu)$ is Borel measurable, the same argument as in [11] shows that this mapping can be taken to be $\mathscr{B}(]0,\infty[) \otimes \mathscr{B}(\mathscr{W}_x^*)$-measurable.

For every $\varepsilon > 0$, we set

$$A_s^\varepsilon = \int_0^s \phi_\varepsilon(W_r) \, dr.$$

We will construct the additive functional A as a suitable limit of A^ε when ε goes to 0. It will be convenient to use the Kuznetsov measure $\mathbb{K}_x$ associated with M_x. Let us briefly describe the construction of $\mathbb{K}_x$ (this construction is a trivial extension of the well-known case of linear Brownian motion killed at 0). Let $\tilde{\Omega} := C(\mathbb{R}, \mathscr{W})$ be the canonical space of all continuous functions from $\mathbb{R}$ into $\mathscr{W}$, and denote again by W the canonical process on $\tilde{\Omega}$. For $\omega \in \tilde{\Omega}$ and $s \in \mathbb{R}$, let $\theta_s \omega \in \Omega$ be defined by $W_r(\theta_s \omega) = W_{s+r}(\omega)$, for every $r \geqq 0$. The measure $\mathbb{K}_x$ is characterized by the following two properties:

(a) $\mathbb{K}_x(d\omega)$ a.e., there exist $\alpha(\omega) < \beta(\omega)$ such that $W_s(\omega) \neq \underline{x}$ if and only if $\alpha(\omega) < s < \beta(\omega)$;

(b) the law under $\mathbb{K}_x(d\omega)$ of the pair $(\alpha(\omega), \theta_{\alpha(\omega)}\omega)$ is $\lambda \otimes \mathbb{N}_x$, where λ denotes Lebesgue measure on $\mathbb{R}$.

For every fixed $s \in \mathbb{R}$, the law of W_s under $\mathbb{K}_x(\cdot \cap \{W_s \neq \underline{x}\})$ is M_x. Furthermore, on the set $\{W_s \neq \underline{x}\}$, the conditional distribution of $(W_{s+r}, r \geqq 0)$ knowing $(W_u, u \leqq s)$ is $\mathbb{P}_{W_s}^*$.

For $\omega \in \tilde{\Omega}$ and $-\infty < r < s < \infty$, we set

$$A_{[r,s]}^\varepsilon(\omega) = \int_r^s \phi_\varepsilon(W_u(\omega)) \, du.$$

Lemma 6. *For $-\infty < r < s < \infty$, the limit*

$$A_{[r,s]} = \lim_{\varepsilon \to 0} A_{[r,s]}^\varepsilon$$

exists in $L^2(\mathbb{K}_x)$, and

$$\mathbb{K}_x((A_{[r,s]})^2) = 2 M_x \left(\int_0^{s-r} du \, (s - r - u) \, \phi_{u/2}^2 \right) < \infty. \tag{7}$$

Proof. We evaluate

$$\begin{aligned}\mathbb{K}_x(A^{\varepsilon}_{[r,s]}A^{\varepsilon'}_{[r,s]}) &= \mathbb{K}_x\left(\int_r^s du \int_r^s dv\, \phi_\varepsilon(W_u)\phi_{\varepsilon'}(W_v)\right)\\ &= \int_r^s du \int_u^s dv\, M_x(\phi_\varepsilon Q^*_{v-u}\phi_{\varepsilon'}) + \int_r^s dv \int_v^s du\, M_x(\phi_{\varepsilon'} Q^*_{u-v}\phi_\varepsilon)\\ &= 2\int_r^s du \int_u^s dv\, M_x(\phi^2_{(v-u+\varepsilon+\varepsilon')/2})\\ &= 2\int_0^{s-r} du\,(s-r-u)\, M_x(\phi^2_{(u+\varepsilon+\varepsilon')/2}).\end{aligned}$$

We used the symmetry of the operators Q^*_s on $L^2(M_x)$ together with the equality $Q^*_s\phi_r = \phi_{r+s}$. From (6), it follows that

$$\lim_{\varepsilon,\varepsilon'\downarrow 0} \mathbb{K}_x(A^{\varepsilon}_{[r,s]}A^{\varepsilon'}_{[r,s]}) = 2\,M_x\left(\int_0^{s-r} du\,(s-r-u)\,\phi^2_{u/2}\right) < \infty,$$

which completes the proof. □

Lemma 7. *There exists* $\mathbb{K}_x$ *a.e. a finite measure* $\tilde{A}(ds)$ *on* $\mathbb{R}$, *supported on* $]\alpha(\omega),\beta(\omega)[$ *and without atoms, such that, for every* $r<s$, $A_{[r,s]} = \int_r^s \tilde{A}(du)$, $\mathbb{K}_x$ *a.e.*

Proof. Let m, n be two positive integers. By (7), we have

$$\mathbb{K}_x\left(\sum_{i=-2^n m+1}^{2^n m} (A_{[(i-1)2^{-n},i2^{-n}]})^2\right) = 2\,m\int_0^{2^{-n}} dv\, M_x(\phi^2_{v/2})\,2^n(2^{-n}-v),$$

and by (6), this quantity converges to 0 as n tends to ∞, for any fixed m. We conclude that, for every $m \geqq 1$,

$$\sup_{-2^{-n}m<i\leqq 2^{-n}m} A_{[(i-1)2^{-n},i2^{-n}]}$$

decreases to 0 as $n\to\infty$, $\mathbb{K}_x$ a.e. For any real number s of the type $s=i2^{-n}$, set $\tilde{A}_s = A_{[0,s]}$ if $s\geqq 0$, $\tilde{A}_s = -A_{[s,0]}$ if $s<0$. The previous observation implies that $\mathbb{K}_x$ a.e. the function $s\to\tilde{A}_s$ can be extended to a continuous nondecreasing function from $\mathbb{R}$ into $\mathbb{R}$. By construction, the function $\tilde{A}$ is constant over $]-\infty,\alpha]$ and over $[\beta,\infty[$. It is then easy to check that the measure $\tilde{A}(ds)$ has the desired properties. □

We now choose a sequence (ε_p) decreasing to 0 in such a way that the convergence of Lemma 6 holds $\mathbb{K}_x$ a.e. for all rational $r<s$ along the subsequence (ε_p). We set for every $\omega\in\Omega$ and every $s\geqq 0$,

$$A_s(\omega) = \liminf_{p\to\infty} A^{\varepsilon_p}_s(\omega),$$

so that the process $(A_s, s \geqq 0)$ is clearly adapted with respect to the filtration $(\mathscr{F}_s^\circ)$. We let Λ be the Borel subset of all $\omega \in \Omega$ such that the functions $A_s^{\varepsilon_p}(\omega)$ are finite and converge to $A_s(\omega)$ uniformly on $\mathbb{R}_+$. For $\omega \in \Lambda$, the function $s \to A_s(\omega)$ is continuous, nondecreasing and constant over $[\sigma, \infty[$. It is also clear that $\theta_s(\Lambda) \subset \Lambda$, and the additivity property $A_{r+s}(\omega) = A_r(\omega) + A_s(\theta_r \omega)$ holds for every $\omega \in \Lambda$.

From Lemma 7 and the choice of the sequence (ε_p), we see that $\mathbb{K}_x$ a.e. $A_{[r,s]}^{\varepsilon_p}$ converges to $A_{[r,s]}$ uniformly on the set $\{(r,s) \in \mathbb{R}^2,\ r \leqq s\}$. It follows that $\mathbb{N}_x(\Lambda^c) = 0$ and $\mathbb{P}_{M_x}^*(\Lambda^c) = 0$. We set

$$N := \{w \in \mathscr{W}_x^*,\ \mathbb{P}_w^*(\Lambda) < 1\}$$

Then $M_x(N) = 0$. Let $w \in \mathscr{W}_x \backslash N$, and assume that $\mathbb{P}_w^*(H_N < \infty) > 0$. By the section theorem (see [5, p. 219]), we may find a stopping time T of the filtration $(\mathscr{F}_s^\circ)$ such that $\mathbb{P}_w^*(T < \infty) > 0$ and $W_T \in N$ on the set $\{T < \infty\}$. By the strong Markov property, the conditional distribution of $\theta_T \omega$ knowing $\mathscr{F}_T^\circ$ is $\mathbb{P}_{W_T}^*$. By the definition of N, we get that $\mathbb{P}_w^*(T < \infty, \theta_T \omega \notin \Lambda) > 0$. On the other hand, since $w \notin N$, we know that, on $\{T < \infty\}$,

$$A_s^{\varepsilon_p}(\theta_T \omega) = A_{T+s}^{\varepsilon_p}(\omega) - A_T^{\varepsilon_p}(\omega)$$

converges $\mathbb{P}_w^*$ a.s. uniformly on $\mathbb{R}_+$. We thus get a contradiction, which proves that $\mathbb{P}_w^*(H_N < \infty) = 0$ when $w \notin N$. In particular, N is M_x-polar.

At this point, we have checked that A satisfies all properties of the definition of an additive functional. Let us turn to the proof of (i). As a consequence of Lemma 6, we have

$$A_s = \lim_{p \to \infty} A_s^{\varepsilon_p},$$

in $L^2(\mathbb{P}_{M_x}^*)$. Hence, $M_x(dw)$ a.e.,

$$\mathbb{E}_w^*(A_s) = \lim_{p \to \infty} \mathbb{E}_w^*(A_s^{\varepsilon_p}) = \lim_{p \to \infty} \int_0^s Q_r^* \phi_{\varepsilon_p}(w)\, dr$$

$$= \lim_{p \to \infty} \int_{\varepsilon_p}^{\varepsilon_p + s} \phi_u(w)\, du = \int_0^s \phi_u(w)\, du.$$

It follows that, M_x a.e.,

$$\mathbb{E}_w^*(A_\infty) = \int_0^\infty \phi_u(w)\, du = U(\mu)(w),$$

using (5). We can then replace M_x a.e. by q.e. To this end, note that by the property (a) of the definition of an additive functional, we may restrict the state space of the killed Brownian snake to $\mathscr{W}_x \backslash N$. The additivity property (d) then shows that the function $w \to \mathbb{E}_w^*(A_\infty)$ is excessive. We have just checked that this function coincides with the excessive function $U(\mu)$, M_x a.e. However two excessive functions that are equal M_x a.e. must coincide q.e.

Let us now check (iii). We observe that the convergence of $A_\infty^{\varepsilon_p}$ towards A_∞ holds in $L^2(\mathbb{N}_x)$. Indeed

$$\begin{aligned}
\mathbb{N}_x(A_\infty^{\varepsilon} A_\infty^{\varepsilon'}) &= 2\,\mathbb{N}_x\left(\int_0^\infty dr\,\phi_\varepsilon(W_r)\int_r^\infty ds\,\phi_{\varepsilon'}(W_s)\right)\\
&= 2\,\mathbb{N}_x\left(\int_0^\infty dr\,\phi_\varepsilon(W_r)\,U\phi_{\varepsilon'}(W_r)\right)\\
&= 2\,M_x(\phi_\varepsilon U\phi_{\varepsilon'})\\
&= 2\int_0^\infty ds\int M_x(dw)\,\phi_\varepsilon(w)\,Q_s^*\phi_{\varepsilon'}(w)\\
&= 2\int_0^\infty ds\,M_x(\phi^2_{(s+\varepsilon+\varepsilon')/2}),
\end{aligned}$$

which increases as $\varepsilon, \varepsilon' \to 0$ to $2\int_0^\infty ds\,M_x(\phi^2_{s/2}) = 2\,\mathscr{E}(\mu)$. Formula (iii) follows.

It remains to prove (ii). We may assume that f is bounded and continuous and that $f(w)=0$ if $\zeta_w \leqq \delta$, for some fixed $\delta>0$. Clearly, the event $\{\int_0^\infty f(W_s)\,dA_s>0\}$ has finite $\mathbb{N}_x$-measure. Since A_∞^ε is bounded in $L^2(\mathbb{N}_x)$, we have

$$\begin{aligned}
\mathbb{N}_x\left(\int_0^\infty f(W_s)\,dA_s\right) &= \lim_{p\to\infty}\mathbb{N}_x\left(\int_0^\infty f(W_s)\,dA_s^{\varepsilon_p}\right)\\
&= \lim_{p\to\infty}\mathbb{N}_x\left(\int_0^\infty f(W_s)\,\phi_{\varepsilon_p}(W_s)\,ds\right)\\
&= \lim_{p\to\infty}\int M_x(dw)\,f(w)\,\phi_{\varepsilon_p}(w)\\
&= \lim_{p\to\infty}\langle \mu Q^*_{\varepsilon_p}, f\rangle\\
&= \lim_{p\to\infty}\langle \mu, Q^*_{\varepsilon_p} f\rangle\\
&= \langle \mu, f\rangle.
\end{aligned}$$

In the fourth equality, we used the identity $\mu Q_s^* = \phi_s\cdot M_x$.

The last assertion in Theorem 5 is immediate from our construction since it is obvious that $A_s^\varepsilon\circ\rho = A_\sigma^\varepsilon - A^\varepsilon_{(\sigma-s)_+}$, for every $s\geqq 0$, $\mathbb{N}_x$ a.e. □

For $\omega\in\Omega_0$ and $s\geqq 0$, we define $\hat\theta_s\omega\in\Omega$ by $W_r(\hat\theta_s\omega) = W_{(s-r)_+}(\omega)$. The following technical lemma will be useful in Sect. 3.

Lemma 8. *Let μ and A be as in Theorem 5, and let φ, ψ be two nonnegative Borel functions on Ω. Then,*

$$\mathbb{N}_x\left(\int_0^\sigma \varphi\circ\theta_s\,\psi\circ\hat\theta_s\,dA_s\right) = \int \mu(dw)\,\mathbb{E}_w^*(\varphi)\,\mathbb{E}_w^*(\psi).$$

Proof. As a consequence of the strong Markov property under $\mathbb{N}_x$ (see [15]), we have for any progressively measurable nonnegative function $g_s(\omega)$ on $\mathbb{R}_+ \times \Omega$,

$$\mathbb{N}_x\left(\int_0^\sigma g_s\, \varphi \circ \theta_s\, dA_s\right) = \mathbb{N}_x\left(\int_0^\sigma g_s\, \mathbb{E}^*_{W_s}(\varphi)\, dA_s\right).$$

On the other hand, the time-reversal invariance of $\mathbb{N}_x$, and the time-reversal property stated in Theorem 5 give for any Borel function f on $\mathscr{W}_x$

$$\mathbb{N}_x\left(\int_0^\sigma f(W_s)\, \psi \circ \hat{\theta}_s\, dA_s\right) = \mathbb{N}_x\left(\int_0^\sigma f(W_s)\, \psi \circ \theta_s\, dA_s\right).$$

We use both facts in the following calculation

$$\begin{aligned}
\mathbb{N}_x\left(\int_0^\sigma \varphi \circ \theta_s\, \psi \circ \hat{\theta}_s\, dA_s\right) &= \mathbb{N}_x\left(\int_0^\sigma \mathbb{E}^*_{W_s}(\varphi)\, \psi \circ \hat{\theta}_s\, dA_s\right) \\
&= \mathbb{N}_x\left(\int_0^\sigma \mathbb{E}^*_{W_s}(\varphi)\, \psi \circ \theta_s\, dA_s\right) \\
&= \mathbb{N}_x\left(\int_0^\sigma \mathbb{E}^*_{W_s}(\varphi)\, \mathbb{E}^*_{W_s}(\psi)\, dA_s\right) \\
&= \int \mu(dw)\, \mathbb{E}^*_w(\varphi)\, \mathbb{E}^*_w(\psi),
\end{aligned}$$

by Theorem 5(ii). □

3 Lower bounds for hitting probabilities

Our main goal in this section is to prove the lower bound of Theorem 1. In this section and the next one, we assume that $d \geqq 4$. We denote by M the maximum of the lifetime process:

$$M := \sup\{\zeta_s, 0 \leqq s \leqq \sigma\}\,.$$

Lemma 9. *There exists a positive constant $\gamma = \gamma(d)$ such that, if F is a Borel subset of $\overline{B}(0,1)\backslash B(0,1/3)$ and $x \in \overline{B}(0,1/4)$, then*

$$\mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset,\ \tfrac{1}{4} < M \leqq 1) \geqq \gamma\, \mathrm{cap}_{d-4}(F)\,.$$

Proof. We may suppose that $\mathrm{cap}_{d-4}(F) > 0$, and then find a probability measure ν on F such that

$$\int\int \nu(dy)\nu(dy')\, \psi_{d-4}(y'-y) \leqq 2\,(\mathrm{cap}_{d-4}(F))^{-1}\,. \tag{8}$$

We denote by G the Green function of Brownian motion in $\mathbb{R}^d$, $G(x,y) = c_d|y-x|^{2-d}$ for a positive constant c_d. For $y \neq x$, let P_{xy} denote the law of Brownian motion started at x and conditioned to die at y. In other words, P_{xy} is the law of the h-transform of Brownian motion started at x, with $h = G(\cdot, y)$

(see [6]). We may and will consider P_{xy} as a probability measure on the set $\{w \in \mathscr{W}_x, \hat{w} = y\}$. Fix $x \in \overline{B}(0, 1/4)$ and let μ be the finite measure on $\mathscr{W}_x$ defined by

$$\mu(dw) = \int \nu(dy)\, G(x, y)\, P_{xy}(dw)\,.$$

Notice that μ is supported on $\{w \in \mathscr{W}_x, \hat{w} \in F\}$.

According to [16], Sect. 3, the energy of μ is

$$\begin{aligned}\mathscr{E}(\mu) &= 2 \int dz\, G(x,z) (\int \nu(dy)\, G(z,y))^2 \\ &= 2 \int\int \nu(dy) \nu(dy') (\int dz\, G(x,z) G(z,y) G(z,y')).\end{aligned}$$

On the other hand, an elementary calculation shows that there exists a constant C_d (independent of the choice of $x \in \overline{B}(0,1/4)$) such that for $y, y' \in \overline{B}(0,1) \backslash B(0, \frac{1}{3})$,

$$\int dz\, G(x,z) G(z,y) G(z,y') \leqq C_d\, \psi_{d-4}(y' - y)\,.$$

Using (8), we now obtain

$$\mathscr{E}(\mu) \leqq 4\, C_d\, (\mathrm{cap}_{d-4}(F))^{-1} < \infty\,. \tag{9}$$

Therefore we can introduce the additive functional A^μ associated with μ by Theorem 5. From Theorem 5 (ii) applied with $f(w) = 1_{F^c}(\hat{w})$, it is obvious that the event $\{A^\mu_\infty > 0\}$ is $\mathbb{N}_x$ a.e. contained in $\{\mathscr{R} \cap F \neq \emptyset\}$. Therefore,

$$\begin{aligned}\mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset,\, \tfrac{1}{4} < M \leqq 1) &\geqq \mathbb{N}_x(A^\mu_\infty > 0,\, \tfrac{1}{4} < M \leqq 1) \\ &\geqq \frac{\mathbb{N}_x(A^\mu_\infty,\, \frac{1}{4} < M \leqq 1)^2}{\mathbb{N}_x((A^\mu_\infty)^2)}\,,\end{aligned} \tag{10}$$

by the Cauchy–Schwarz inequality.

From Theorem 5(iii) and (9) we have

$$\mathbb{N}_x((A^\mu_\infty)^2)) \leqq 8\, C_d\, (\mathrm{cap}_{d-4}(F))^{-1}\,. \tag{11}$$

It remains to get a lower bound on $\mathbb{N}_x(A^\mu_\infty,\, 1/4 < M \leqq 1)$. To this end, we apply Lemma 8 with $\varphi = \psi = 1_{\{1/4 < M \leqq 1\}}$, observing that $\varphi \geqq \varphi \circ \theta_s \cdot \varphi \circ \hat{\theta}_s$, for every $s \in (0, \sigma)$, $\mathbb{N}_x$ a.e. We obtain

$$\begin{aligned}\mathbb{N}_x(A^\mu_\infty,\, \tfrac{1}{4} < M \leqq 1) &\geqq \int \mu(dw) (\mathbb{P}^*_w(1/4 < M \leqq 1))^2 \\ &\geqq \int \mu(dw)\, 1_{\{1/4 < \zeta_w \leqq 1/2\}}\, (\mathbb{P}^*_w(M \leqq 1))^2 \\ &= \int \mu(dw)\, 1_{\{1/4 < \zeta_w \leqq 1/2\}} (1 - \zeta_w)^2 \\ &\geqq \tfrac{1}{4}\, \mu(\tfrac{1}{4} < \zeta_w \leqq \tfrac{1}{2})\,.\end{aligned}$$

We have used the fact that $(\zeta_s, s \geqq 0)$ is distributed under $\mathbb{P}^*_w$ as a linear Brownian motion started at ζ_w and stopped when it hits 0.

From the definition of μ, we have

$$\mu(\tfrac{1}{4} < \zeta_w \leqq \tfrac{1}{2}) = \int \nu(dy)\, G(x,y)\, P_{xy}(\tfrac{1}{4} < \zeta \leqq \tfrac{1}{2}),$$

and, if p_t stands for the Brownian transition kernel,

$$P_{xy}(\tfrac{1}{4} < \zeta \leqq \tfrac{1}{2}) = \frac{\int_{1/4}^{1/2} dt\ p_t(x,y)}{\int_0^\infty dt\ p_t(x,y)} \geqq C'_d\,,$$

for some positive constant C'_d independent of $x \in \overline{B}(0,1/4)$, $y \in \overline{B}(0,1)\backslash B(0,\frac{1}{3})$. We thus obtain

$$\mathbb{N}_x(A^\mu_\infty,\ \tfrac{1}{4} < M \leqq 1) \geqq \tfrac{1}{4}(c_d 2^{2-d})C'_d\,.$$

Lemma 9 follows by combining this bound with (10) and (11). □

Proof of the lower bound of Theorem 1 (*case* $d \geqq 4$). We use a simple scaling lemma whose proof follows immediately from the definition of $\mathrm{cap}_{d-4}(F)$.

Lemma 10. *If F is a Borel subset of $\overline{B}(0,1/2)$ and $\lambda \in]0,1]$,*

(i) *if* $d = 4$, $(\mathrm{cap}_0(\lambda F))^{-1} = (\mathrm{cap}_0(F))^{-1} + \log \frac{1}{\lambda}$,
(ii) *if* $d \geqq 5$, $\mathrm{cap}_{d-4}(\lambda F) = \lambda^{4-d}\,\mathrm{cap}_{d-4}(F)$.

Then, let F be a Borel subset of $\overline{B}(0,1/2)$ and $|x| \geqq 1$. We set

$$\tilde{F} = \left\{ z = x + \frac{1}{|x| + 1/2}(y - x), y \in F \right\},$$

in such a way that $\tilde{F} \subset \overline{B}(x,1)\backslash B(x,1/3)$. By Lemma 9 and a trivial translation argument, we have

$$\mathbb{N}_x(\mathscr{R} \cap \tilde{F} \neq \emptyset) \geqq \gamma\, \mathrm{cap}_{d-4}(\tilde{F})\,. \tag{12}$$

On one hand, Lemma 10 gives

$$\mathrm{cap}_{d-4}(\tilde{F}) = \begin{cases} \dfrac{\mathrm{cap}_0(F)}{1 + \log(|x| + 1/2)\mathrm{cap}_0(F)} & \text{if } d = 4\,, \\[2ex] (|x| + 1/2)^{4-d}\mathrm{cap}_{d-4}(F) & \text{if } d \geqq 5\,. \end{cases} \tag{13}$$

On the other hand, the scaling properties of $\mathbb{N}_x$ give

$$\mathbb{N}_x(\mathscr{R} \cap \tilde{F} \neq \emptyset) = (|x| + \tfrac{1}{2})^2\, \mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset)\,. \tag{14}$$

The lower bound of Theorem 1 follows immediately from (12)–(14). □

Remark. In the previous proof, we did not use the exact statement of Lemma 9 but only the lower bound $\mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset) \geqq \gamma\, \mathrm{cap}_{d-4}(F)$, which can be obtained more easily. The precise form of Lemma 9 will be needed in Sect. 5.

4 Upper bounds for hitting probabilities

In this section, we prove the upper bound of Theorem 1. We use analytic techniques inspired from Baras and Pierre [2]. First recall that if K is a compact subset of $\mathbb{R}^d$, the capacity $c_{2,2}(K)$ is defined by

$$c_{2,2}(K) = \inf\{\|\varphi\|_{2,2}^2, \varphi \in C_0^\infty(\mathbb{R}^d),\ \varphi = 1 \text{ on a neighborhood of } K\},$$

where $C_0^\infty(\mathbb{R}^d)$ denotes the set of all C^∞ functions on $\mathbb{R}^d$ with compact support, and $\|\ \|_{2,2}$ stands for the Sobolev norm:

$$\|\varphi\|_{2,2} = \|\varphi\|_2 + \|\nabla\varphi\|_2 + \|\nabla^2\varphi\|_2 .$$

For the sake of completeness, we recall the relations between $\mathrm{cap}_{d-4}(K)$ and $c_{2,2}(K)$.

Lemma 11. *There exist two positive constants c_1, c_2 such that, for every compact subset K of $\overline{B}(0,1)$,*

$$c_1\,\mathrm{cap}_{d-4}(K) \leqq c_{2,2}(K) \leqq c_2\,\mathrm{cap}_{d-4}(K) .$$

Proof. We use some results about equivalence of capacities due to Adams and Polking [1] and Meyers [21]. Denote by $g_2(x,y)$ the usual Bessel kernel

$$g_2(x,y) = \int_0^\infty dt\, e^{-t}\,(4\pi t)^{-d/2}\,\exp -\frac{|y-x|^2}{4t} .$$

The Bessel capacity $B_2(K)$ of a compact subset K of $\mathbb{R}^d$ is then defined by

$$B_2(K) = \inf\{\|f\|_2^2,\ f \in C_0^\infty(\mathbb{R}^d),\ \int dy\, f(y)\, g_2(x,y) \geqq 1,\ \forall x \in K\} .$$

According to Theorem A of [1], the capacities $c_{2,2}(K)$ and $B_2(K)$ are equivalent, in the sense that $\bar{c}_1\,B_2(K) \leqq c_{2,2}(K) \leqq \bar{c}_2\,B_2(K)$ for some positive constants $\bar{c}_1, \bar{c}_2$ independent of K. On the other hand, Theorem 14 of [21] shows that the Bessel capacity $B_2(K)$ is equivalent to $C_2(K)^2$, where

$$C_2(K) = \sup\{\langle \mu, 1\rangle,\ \mu \text{ is a finite measure supported on } K \text{ and } \|\mu g_2\|_2 \leqq 1\} .$$

Clearly, we have also

$$C_2(K)^2 = (\inf\{\|\mu g_2\|_2^2,\ \mu \text{ is a probability measure supported on } K\})^{-1} .$$

However, a few lines of calculations show that, if μ is supported on $\overline{B}(0,1)$,

$$\begin{aligned}\alpha_1 \iint \mu(dy)\mu(dy')\,\psi_4(y'-y) \leqq \|\mu g_2\|_2^2 &= \int dz\Big(\int \mu(dy)\, g_2(y,z)\Big)^2 \\ &\leqq \alpha_2 \iint \mu(dy)\mu(dy')\,\psi_4(y'-y) ,\end{aligned}$$

where α_1, α_2 are positive constants depending only on d. Thus, $\mathrm{cap}_{d-4}(K)$ and $C_2(K)^2$ are equivalent in the previous sense, provided we consider only compact subsets of $\overline{B}(0,1)$. The desired result follows. □

Lemma 12. *There exists a positive constant $\beta' = \beta'(d)$ such that, for every compact subset K of $\overline{B}(0,1/2)$, if $c_{2,2}(K) > 0$, one can find a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that*

(i) $0 \leqq \varphi \leqq 1$,
(ii) $\varphi = 1$ *on a neighborhood of* K,
(iii) $\varphi = 0$ *on* $B(0,3/4)^c$,
(iv) $\|\varphi\|_{2,2}^2 \leqq \beta' \, c_{2,2}(K)$.

Proof. According to [1, p. 530], there exist two constants Q, β_1 depending only on d, such that, for every compact subset K of $\overline{B}(0,\frac{1}{2})$ with $c_{2,2}(K) > 0$, we can find a function $\phi \in C^\infty(\mathbb{R}^d)$ which satisfies

(a) $0 \leqq \phi \leqq Q$,
(b) $\phi \geqq 1$ on a neighborhood of K,
(c) $\|\phi\|_{2,2}^2 \leqq \beta_1 \, c_{2,2}(K)$.

We let H be a C^∞ function from $\mathbb{R}_+$ into $[0,1]$ such that $H(t) = t$ for $0 \leqq t \leqq \frac{1}{2}$ and $H(t) = 1$ for $t \geqq 1$. We also choose $\eta \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leqq \eta(x) \leqq 1$, $\eta(x) = 1$ if $|x| \leqq \frac{2}{3}$, $\eta(x) = 0$ if $|x| \geqq \frac{3}{4}$. We claim that the function

$$\varphi(x) = \eta(x) \, H(\phi(x))$$

satisfies the desired properties. Properties (i)–(iii) are immediate. We then observe that

$$\|\varphi\|_2 \leqq C \, \|\phi\|_2, \qquad \|\nabla\varphi\|_2 \leqq C \left(\|\phi\|_2 + \|\nabla\phi\|_2 \right),$$

where the constant C only depends on the choice of η and H. Furthermore,

$$\|\nabla^2\varphi\|_2 \leqq C' (\|\phi\|_2 + \|\nabla\phi\|_2 + \|\nabla^2\phi\|_2 + \||\nabla\phi|^2\|_2) .$$

To handle the term $\||\nabla\phi|^2\|_2$ we use the easy inequality

$$\left(\int |\nabla\phi(y)|^4 dy \right)^{1/2} \leqq C'' \|\phi\|_\infty \|\nabla^2\phi\|_2 \leqq C'' Q \|\nabla^2\phi\|_2 , \tag{15}$$

by (a). Property (iv) then follows from (c) and the previous observations. □

Proof of the upper bound of Theorem 1. It is enough to consider the case when $\mathrm{cap}_{d-4}(F) > 0$. We first assume that $F = K$ is a compact subset of $\overline{B}(0,1/2)$ such that $c_{2,2}(K) > 0$. We know from [15], Proposition 5.3 (see also [8]) that

$$u(x) = \mathbb{N}_x(\mathscr{R} \cap K \neq \emptyset)$$

solves the equation $\Delta u = 4u^2$ in K^c. Then, let φ be as in Lemma 12 and set $\psi = 1 - \varphi$, so that ψ is a C^∞ function which vanishes on a neighborhood of K and is equal to 1 on $B(0,3/4)^c$.

First step. We prove that there exists a constant $A = A(d)$ such that

$$\int_{\mathbb{R}^d} \psi(y)^4 \, u(y)^2 \, dy \leqq A \, \|\varphi\|_{2,2}^2 . \tag{16}$$

It will be convenient to replace u by a periodic function. To this end, for every $R > 2$, we set

$$K_R = K + R\mathbb{Z}^d = \{z = y + (k_1 R, \dots, k_d R),\ y \in K,\ k_1, \dots, k_d \in \mathbb{Z}\}\,,$$

and

$$u_R(x) = \mathbb{N}_x(\mathscr{R} \cap K_R \neq \emptyset)\,.$$

Then u_R solves $\Delta u_R = 4\,u_R^2$ on K_R^c, $u \leqq u_R$ and u_R has period R in every direction. If $U_R = [-R/2, R/2]^d$, we have thus

$$4 \int_{U_R} \psi(y)^4\, u_R(y)^2\, dy = \int_{U_R} \psi(y)^4\, \Delta u_R(y)\, dy = \int_{U_R} \Delta(\psi^4)(y)\, u_R(y)\, dy\,,$$

where the last equality follows from integration by parts using the periodicity of u_R and the fact that $\psi = 1$ on a neighborhood of ∂U_R. Then,

$$\begin{aligned}
\frac{1}{4} \int_{U_R} |\Delta(\psi^4)|\, u_R\, dy &\leqq 3 \int_{U_R} \psi^2 |\nabla\psi|^2 u_R\, dy + \int_{U_R} \psi^3 |\Delta\psi|\, u_R\, dy \\
&\leqq 3\Big(\int_{U_R} \psi^4 u_R^2\, dy\Big)^{1/2} \Big(\int |\nabla\psi|^4 dy\Big)^{1/2} \\
&\quad + \Big(\int_{U_R} \psi^6 u_R^2\, dy\Big)^{1/2} \Big(\int |\Delta\psi|^2 dy\Big)^{1/2} \\
&\leqq \Big(\int_{U_R} \psi^4 u_R^2\, dy\Big)^{1/2} \Big(3\Big(\int |\nabla\psi|^4 dy\Big)^{1/2} + \Big(\int |\Delta\psi|^2 dy\Big)^{1/2}\Big).
\end{aligned} \tag{17}$$

Since $0 \leqq \psi \leqq 1$, the same argument as in (15) gives $\int |\nabla\psi|^4 dy \leqq A'\, \|\nabla^2\psi\|_2^2 \leqq A'\, \|\varphi\|_{2,2}^2$, where the constant A' only depends on d. The previous bound then gives

$$\int_{U_R} \psi^4\, u_R^2\, dy \leqq A \|\varphi\|_{2,2}^2$$

again with a constant A that depends only on d. Since $u \leqq u_R$, the same bound holds with u_R replaced by u, and (16) follows by letting R tend to infinity.

From (17), we have also, with a constant $A'' = A''(d)$,

$$\int_{U_R} |\Delta(\psi^4)|\, u_R\, dy \leqq A'' \|\varphi\|_{2,2}^2\,,$$

and the same argument gives

$$\int_{\mathbb{R}^d} |\Delta(\psi^4)|\, u\, dy \leqq A'' \|\varphi\|_{2,2}^2\,. \tag{18}$$

Second step. Let $x \in B(0,1)^c$ and let $(B_t, t \geqq 0)$ denote a d-dimensional Brownian motion that starts at x under the probability P_x. For $a > |x|$, we set $T_a = \inf\{t \geqq 0, |B_t| \geqq a\}$. Then from Itô's formula, P_x a.s.,

$$(\psi^4 u)(B_t) = u(x) + \int_0^t \nabla(\psi^4 u)(B_s) \cdot dB_s + \frac{1}{2} \int_0^t \Delta(\psi^4 u)(B_s)\, ds\,.$$

By applying the optional stopping theorem at $T_a \wedge t$, we get

$$E_x((\psi^4 u)(B_{T_a \wedge t})) = u(x) + \frac{1}{2} E_x \left(\int_0^{T_a \wedge t} \Delta(\psi^4 u)(B_s)\, ds \right)$$

$$= u(x) + \frac{1}{2} E_x \left(\int_0^{T_a \wedge t} (\psi^4 \Delta u + 2\nabla(\psi^4) \cdot \nabla u + u\Delta(\psi^4))(B_s)\, ds \right).$$

Since $\Delta u = 4u^2 \geqq 0$ on K^c, we thus get

$$u(x) \leqq E_x((\psi^4 u)(B_{T_a \wedge t})) - \frac{1}{2} E_x \left(\int_0^{T_a \wedge t} (2\nabla(\psi^4) \cdot \nabla u + u\Delta(\psi^4))(B_s)\, ds \right).$$

Note that both functions $\nabla(\psi^4)$ and $\Delta(\psi^4)$ vanish outside the ball $B(0, 3/4)$, and that $u(y)$ tends to 0 as $|y| \to \infty$. By letting t, and then a tend to infinity, we easily obtain

$$u(x) \leqq -\frac{1}{2} E_x \left(\int_0^{\infty} (2\nabla(\psi^4) \cdot \nabla u + u\Delta(\psi^4))(B_s)\, ds \right)$$

$$= -c_d \int_{\mathbb{R}^d} (\nabla(\psi^4) \cdot \nabla u + \frac{1}{2} u\Delta(\psi^4))(y)\, |y - x|^{2-d}\, dy\,. \tag{19}$$

We will now bound the right-hand side of (19). Since $|x| \geqq 1$ and $\psi = 1$ outside $B(0, \frac{3}{4})$, we can find a constant C depending only on d such that

$$\int_{\mathbb{R}^d} |(u\Delta(\psi^4))(y)|\, |y - x|^{2-d}\, dy \leqq C\, |x|^{2-d} \int_{\mathbb{R}^d} |u\Delta(\psi^4))(y)|\, dy$$

$$\leqq C\, A''\, |x|^{2-d}\, \|\varphi\|_{2,2}^2\,, \tag{20}$$

by (18).

We then consider the other term in the right-hand side of (19). We observe that if $h_d(y) = |y - x|^{2-d}$, we can find a constant C' (independent of $x \in B(0,1)^c$) such that $|\nabla h_d(y)| \leqq C'|x|^{1-d}$, for every $y \in B(0, \frac{3}{4})$. Thus an integration by parts gives

$$\left| \int_{\mathbb{R}^d} (\nabla(\psi^4) \cdot \nabla u)(y)\, |y - x|^{2-d}\, dy \right|$$

$$= \left| \int_{\mathbb{R}^d} (u\Delta(\psi^4))(y)\, |y - x|^{2-d}\, dy + \int_{\mathbb{R}^d} (u\, \nabla \psi^4 \cdot \nabla h_d)(y)\, dy \right|$$

$$\leqq C\, A''\, |x|^{2-d}\, \|\varphi\|_{2,2}^2 + 4\, C'|x|^{1-d} \int_{\mathbb{R}^d} u\, \psi^3 |\nabla \psi|\, dy$$

$$\leqq C\, A''\, |x|^{2-d}\, \|\varphi\|_{2,2}^2 + 4\, C'|x|^{1-d} \left(\int_{\mathbb{R}^d} u^2\, \psi^4\, dy \right)^{1/2} \left(\int_{\mathbb{R}^d} |\nabla \psi|^2 dy \right)^{1/2}$$

$$\leqq (C\, A'' + 4\, C'\, A^{1/2})\, |x|^{2-d}\, \|\varphi\|_{2,2}^2\,, \tag{21}$$

using (16) once again. Then, by combining (19)–(21), Lemma 12(iv) and Lemma 11, we get

$$\mathbb{N}_x(\mathscr{R} \cap K \neq \emptyset) = u(x) \leqq \beta\, |x|^{2-d} \operatorname{cap}_{d-4}(K)\,,$$

with a constant β depending only on d. This is the desired upper bound when $d \geqq 5$. When $d = 4$, we use a scaling argument and apply the previous bound to $x' = x/|x|$, $K' = |x|^{-1}K$. It follows that

$$\begin{aligned}\mathbb{N}_x(\mathscr{R} \cap K \neq \emptyset) = |x|^{-2}\mathbb{N}_{x'}(\mathscr{R} \cap K' \neq \emptyset) \ &\leqq \beta|x|^{-2} \operatorname{cap}_0(K') \\ &= \frac{\beta|x|^{-2} \operatorname{cap}_0(K)}{1 + (\log|x|)\operatorname{cap}_0(K)}\,,\end{aligned}$$

by Lemma 10. This gives the upper bound of Theorem 1, when F is compact.

When F is not compact, we observe that the mapping $F \to \mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset)$ is the restriction to Borel sets of a Choquet capacity (see e.g. [20, Ch. 2]). Therefore,

$$\mathbb{N}_x(\mathscr{R} \cap F \neq \emptyset) = \sup\{\mathbb{N}_x(\mathscr{R} \cap K \neq \emptyset), K \subset F,\ K \text{ compact}\}\,.$$

This completes the proof of Theorem 1. □

5 Regularity and Wiener's test

5.1 Regularity for super-Brownian motion and for the Brownian snake

In this subsection, we prove Proposition 2. We first state a technical lemma.

Lemma 13. *For every $t > 0$, set*

$$\mathscr{R}_t^X = \bigcup_{\varepsilon > 0} \left(\overline{\bigcup_{\varepsilon \leqq r \leqq t} \operatorname{supp} X_t} \right),$$

and, for every Borel subset F of $\mathbb{R}^d$, set

$$\tau'_F = \inf\{t > 0,\ \mathscr{R}_t^X \cap F \neq \emptyset\}\,.$$

Then $\tau_F = \tau'_F$, $\mathbb{P}_x$ a.s., and τ_F is a stopping time of the filtration $(\mathscr{G}_t)$.

Proof. From the definition of the process X in terms of W, it is easy to verify that

$$\mathscr{R}_t^X = \{\hat{W}_s; 0 \leqq s \leqq \tau_1,\ 0 < \zeta_s \leqq t\}, \quad \mathbb{P}_x \text{ a.s.} \tag{22}$$

We then check that, if K is compact, then

$$\{\mathscr{R}_t^X \cap K \neq \emptyset\} = \{\tau_K \leqq t\} \quad \mathbb{P}_x \text{ a.s.} \tag{23}$$

The inclusion $\{\tau_K \leqq t\} \subset \{\mathscr{R}_t^X \cap K \neq \emptyset\}$ a.s. is easy, as we know from Perkins [23] that the mapping $t \to \operatorname{supp} X_t$ is a.s. right-continuous with respect to the

Hausdorff metric, which implies that

$$\operatorname{supp} X_{\tau_K} \cap K \neq \emptyset \quad \text{a.s. on } \{\tau_K < \infty\}.$$

To get the reverse inclusion, set for every $\varepsilon \in (0, t)$

$$T_K^{\varepsilon,t} = \inf\{s \geqq 0;\, \varepsilon \leqq \zeta_s \leqq t,\, \hat{W}_s \in K\},$$

in such a way that

$$\{\mathscr{R}_t^X \cap K \neq \emptyset\} = \bigcup_{0<\varepsilon<t} \{T_K^{\varepsilon,t} \leqq \tau_1\}, \quad \mathbb{P}_x \text{ a.s.}$$

by (22). Notice that $T_K^{\varepsilon,t}$ is a stopping time of the filtration $(\mathscr{F}_t^\circ)$. Set $S = \zeta_{T_K^{\varepsilon,t}}$ on $\{T_K^{\varepsilon,t} < \infty\}$. The strong Markov property of reflected Brownian motion ensures that

$$L^S_{T_K^{\varepsilon,t}+\delta}(\zeta) > L^S_{T_K^{\varepsilon,t}}(\zeta) \quad \forall \delta > 0, \quad \mathbb{P}_x \text{ a.s. on } \{T_K^{\varepsilon,t} < \infty\}.$$

From the definition of the random measure X_S, it follows that $\hat{W}_{T_K^{\varepsilon,t}} \in \operatorname{supp} X_S$, a.s. on $\{T_K^{\varepsilon,t} < \tau_1\}$. Hence $\operatorname{supp} X_S \cap K \neq \emptyset$ a.s. on $\{T_K^{\varepsilon,t} < \tau_1\}$. This completes the proof of (23).

We also observe that $\{(t,\omega), \operatorname{supp} X_t \cap K \neq \emptyset\}$ is progressive with respect to the filtration $(\mathscr{G}_t)$, and therefore τ_K is a $(\mathscr{G}_t)$-stopping time. In particular, $\{\mathscr{R}_t^X \cap K \neq \emptyset\} \in \mathscr{G}_t$.

Then, let F be a Borel subset of $\mathbb{R}^d$. Because the mapping $F \to \mathbb{P}_x(\mathscr{R}_t^X \cap F \neq \emptyset)$ is a Choquet capacity, we may find an increasing sequence (K_n) of compact subsets of F such that

$$\{\mathscr{R}_t^X \cap F \neq \emptyset\} = \bigcup_{n=1}^{\infty} \{\mathscr{R}_t^X \cap K_n \neq \emptyset\} \quad \mathbb{P}_x \text{ a.s.}$$

In particular, $\{\mathscr{R}_t^X \cap F \neq \emptyset\} \in \mathscr{G}_t$. By taking a diagonal subsequence, we may assume that the previous equality holds for every rational t with the same sequence (K_n). Then, for every rational $t > 0$,

$$\begin{aligned}\{\tau'_F < t\} &= \bigcup_{p=1}^{\infty} \{\mathscr{R}^X_{t-p^{-1}} \cap F \neq \emptyset\} \\ &= \bigcup_{p=1}^{\infty}\bigcup_{n=1}^{\infty} \{\mathscr{R}^X_{t-p^{-1}} \cap K_n \neq \emptyset\} \subset \{\tau_F < t\}, \quad \text{a.s.}\end{aligned} \tag{24}$$

by (23). Since the inequality $\tau'_F \leqq \tau_F$ also holds trivially, we conclude from (24) that $\tau_F = \tau'_F$ a.s. Moreover, (24) shows that $\{\tau_F < t\} \in \mathscr{G}_t$. Since the filtration $(\mathscr{G}_t)$ is right-continuous, we conclude that τ_F is a stopping time. □

Proof of Proposition 2. The implication (iii) ⇒ (ii) is clear from excursion theory. If (ii) holds, then (22) and the continuity of the mapping $s \to \zeta_s$ ensure that $\mathbb{P}_x(\mathscr{R}_t^X \cap F \neq \emptyset) = 1$ for every $t > 0$. With the notation of Lemma 13, we have thus $\mathbb{P}_x(\tau'_F \leqq t) = 1$ for every $t > 0$. Since $\tau_F = \tau'_F$ a.s. we get (i). Finally, if (i) holds, it is plain that $\mathbb{P}_x(\mathscr{R}^X \cap F \neq \emptyset) = 1$, and (iii) follows from (2). □

5.2. Wiener's test: the necessity part

We consider a Borel subset F of $\mathbb{R}^d$ $(d \geqq 4)$ and $x \in F$. By a translation argument, we may assume that $x = 0$. We then set for every $n \geqq 1$

$$F_n := \{ y \in F,\, 2^{-n} \leqq |y| < 2^{-n+1} \}.$$

We aim to prove that the condition

$$\sum_{n=0}^{\infty} 2^{n(d-2)} \operatorname{cap}_{d-4}(F_n) < \infty \tag{25}$$

implies that 0 is not super-regular for F, or equivalently that $\mathbb{N}_0(\mathscr{R}^* \cap F \neq \emptyset) < \infty$. Notice that since points are $\mathscr{R}$-polar in dimension $d \geqq 4$, we have $\mathscr{R}^* = \mathscr{R} \backslash \{0\}$, $\mathbb{N}_0$ a.s.

We set $\tilde{F}_n := 2^n F_n$ in such a way that $\tilde{F}_n$ is a Borel subset of $B(0,2) \backslash B(0,1)$. By a scaling argument,

$$\begin{aligned} \mathbb{N}_0(\mathscr{R}^* \cap F \neq \emptyset) &\leqq \sum_{n=0}^{\infty} \mathbb{N}_0(\mathscr{R} \cap F_n \neq \emptyset) + \mathbb{N}_0(\mathscr{R} \cap B(0,1)^c \neq \emptyset) \\ &= \sum_{n=0}^{\infty} 2^{2n}\, \mathbb{N}_0(\mathscr{R} \cap \tilde{F}_n \neq \emptyset) + \mathbb{N}_0(\mathscr{R} \cap B(0,1)^c \neq \emptyset). \end{aligned} \tag{26}$$

Then, let $(B_i,\, 1 \leqq i \leqq p)$ be a finite collection of balls of radius $\frac{1}{2}$ whose centers belong to $B(0,2) \backslash B(0,1)$ and which cover $B(0,2) \backslash B(0,1)$. Using Theorem 1, we have

$$\mathbb{N}_0(\mathscr{R} \cap \tilde{F}_n \neq \emptyset) \leqq \sum_{i=1}^{p} \mathbb{N}_0(\mathscr{R} \cap (\tilde{F}_n \cap B_i) \neq \emptyset) \leqq \beta \sum_{i=1}^{p} \operatorname{cap}_{d-4}(\tilde{F}_n \cap B_i). \tag{27}$$

From Lemma 10, we know that

$$\operatorname{cap}_{d-4}(\tilde{F}_n \cap B_i) = \begin{cases} \dfrac{\operatorname{cap}_0(F_n \cap 2^{-n} B_i)}{1 - \log(2^n) \operatorname{cap}_0(F_n \cap 2^{-n} B_i)} & \text{if } d = 4, \\ 2^{n(d-4)} \operatorname{cap}_{d-4}(F_n \cap 2^{-n} B_i) & \text{if } d \geqq 5. \end{cases} \tag{28}$$

For $d = 4$, the condition (25) implies that $\operatorname{cap}_0(F_n) \log 2^n \leqq 1/2$ for all n sufficiently large. Thus, both for $d = 4$ and $d \geqq 5$, we get from (27) and (28) that

$$\mathbb{N}_0(\mathscr{R} \cap \tilde{F}_n \neq \emptyset) \leqq 2\, \beta\, p\, 2^{n(d-4)} \operatorname{cap}_{d-4}(F_n) \tag{29}$$

for all n sufficiently large. From (26) and (29), we see that the condition (25) implies $\mathbb{N}_0(\mathscr{R}^* \cap F \neq \emptyset) < \infty$.

5.3. Wiener's test: the sufficiency part

We keep the notation of the previous subsection, but we will now prove that under the condition

$$\sum_{n=0}^{\infty} 2^{n(d-2)} \operatorname{cap}_{d-4}(F_n) = \infty, \tag{30}$$

one has $\mathbb{N}_0(\mathscr{R}^* \cap F \neq \emptyset) = \infty$.

Recall the notation $M = \sup\{\zeta_s, s \geqq 0\}$. Then,

$$\mathbb{N}_0(\mathscr{R}^* \cap F \neq \emptyset) \geqq \sum_{n=1}^{\infty} \mathbb{N}_0(\mathscr{R}^* \cap F_n \neq \emptyset, 2^{-2n} < M \leqq 2^{-2n+2}). \tag{31}$$

Let us fix a positive integer n. Notice that $2^{n-1}F_n \subset B(0,1)\backslash B(0,\frac{1}{2})$. From the scaling properties of $\mathbb{N}_0$ and Lemma 9, we get

$$\begin{aligned}&\mathbb{N}_0(\mathscr{R}^* \cap F_n \neq \emptyset, 2^{-2n} < M \leqq 2^{-2n+2})\\ &\quad = 2^{2(n-1)}\,\mathbb{N}_0(\mathscr{R}^* \cap (2^{n-1}F_n) \neq \emptyset, \tfrac{1}{4} < M \leqq 1)\\ &\quad \geqq 2^{2(n-1)}\gamma\,\mathrm{cap}_{d-4}(2^{n-1}F_n).\end{aligned} \tag{32}$$

When $d \geqq 5$, $\mathrm{cap}_{d-4}(2^{n-1}F_n) = 2^{(d-4)(n-1)}\mathrm{cap}_4(F_n)$, and when $d = 4$, $\mathrm{cap}_0(2^{n-1}F_n) \geqq \mathrm{cap}_0(F_n)$. In both cases, the desired result follows from (31) and (32).

5.4. Connections with partial differential equations

In this subsection, we prove Theorem 4. From [15], Proposition 5.3, the function

$$u(y) = 4\,\mathbb{N}_y(\mathscr{R} \cap D^c \neq \emptyset), \quad y \in D,$$

is the maximal nonnegative solution of equation $\Delta u = u^2$ in D. Therefore the proof reduces to checking that x is super-regular for $F = D^c$ if and only if

$$\lim_{D \ni y \to x} \mathbb{N}_y(\mathscr{R} \cap F \neq \emptyset) = \infty. \tag{33}$$

When $d \leqq 3$, we know that every $x \in \partial D$ is super-regular for F. On the other hand it is also very easy to check that (33) holds. Indeed, a scaling argument gives

$$\mathbb{N}_y(\mathscr{R} \cap F \neq \emptyset) \geqq \mathbb{N}_y(x \in \mathscr{R}) = c\,|y-x|^{-2},$$

with a constant $c > 0$.

From now on, we take $d \geqq 4$. We first assume that x is super-regular for F, or equivalently that (4) holds. Let $N \geqq 1$ be an integer, and let $y \in B(x, 2^{-N-1}) \cap D$. We have

$$\mathbb{N}_y(\mathscr{R} \cap F \neq \emptyset) \geqq \sum_{n=1}^{N} \mathbb{N}_y(\mathscr{R} \cap F \neq \emptyset, 2^{-2n} < M \leqq 2^{-2n+2}).$$

We can use Lemma 9 as in the derivation of (32) to get for every $n \in \{1, \dots, N\}$,

$$\mathbb{N}_y(\mathscr{R} \cap F \neq \emptyset, 2^{-2n} < M \leqq 2^{-2n+2}) \geqq \gamma 2^{2(n-1)}\,\mathrm{cap}_{d-4}(2^{n-1}F_n(x)).$$

Hence,

$$\mathbb{N}_y(\mathscr{R} \cap F \neq \emptyset) \geqq \gamma \sum_{n=1}^{N} 2^{2(n-1)}\,2^{(d-4)(n-1)}\mathrm{cap}_{d-4}(F_n(x))$$

and the desired result (33) follows from (4).

Conversely, assume that (33) holds. Fix $N \geqq 1$ and choose $\alpha \in (0, 1/2]$ such that

$$\mathbb{N}_y(\mathscr{R} \cap D^c \neq \emptyset) \geqq N$$

for every $y \in B(x, \alpha) \cap D$. Let $\varepsilon > 0$ and set $T(\varepsilon) = \inf\{s \geqq 0;\ \zeta_s \geqq \varepsilon\}$. Note that the law of $W_{T(\varepsilon)}$ under $\mathbb{N}_x(\cdot \mid T(\varepsilon) < \infty)$ is the law of a Brownian path started at x and stopped at time ε. Then, using the strong Markov property at time $T(\varepsilon)$ and then Proposition 2.5 of [15], we have

$$\begin{aligned}
\mathbb{N}_x(\mathscr{R}^* \cap D^c \neq \emptyset) &\geqq \mathbb{N}_x(T(\varepsilon) < \infty\,, \mathbb{E}^*_{W_{T(\varepsilon)}}(\mathscr{R}^* \cap D^c \neq \emptyset)) \\
&\geqq \mathbb{N}_x\left(T(\varepsilon) < \infty, \left(1 - \exp -2 \int_0^{\varepsilon} dt\, \mathbb{N}_{W_{T(\varepsilon)}(t)}(\mathscr{R} \cap D^c \neq \emptyset) \right) \right) \\
&\geqq (2\varepsilon)^{-1} E_x \left(1 - \exp -2N \int_0^{\varepsilon} dt\, 1_{]0,\alpha[}(|B_t - x|) \right) .
\end{aligned}$$

By letting $\varepsilon \to 0$, we conclude that $\mathbb{N}_x(\mathscr{R}^* \cap D^c \neq \emptyset) \geqq N$, which completes the proof since N was arbitrary.

6 The special case of thorns

In this section, we give a more explicit form of the condition of Theorem 3 when F is a thorn with vertex $x = 0$ (cf. [24, p. 68] for the classical case of Brownian motion in $\mathbb{R}^d$). We consider a function h from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $h(r) = o(r)$ as $r \to 0$ and $r^{-1}h(r)$ is nondecreasing, at least for $r > 0$ sufficiently small. The thorn T_h is then defined by

$$T_h = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d;\ x_d \geqq 0,\ (x_1^2 + \cdots + x_{d-1}^2)^{1/2} \leqq h(x_d)\}.$$

The following result should be compared with the classical case.

Theorem 14. (i) *If* $d \leqq 4$, 0 *is always super-regular for* T_h.

(ii) *If* $d = 5$, 0 *is super-regular for* T_h *if and only if*

$$\int_{0+} \left(\log \frac{r}{h(r)} \right)^{-1} \frac{dr}{r} = \infty.$$

(iii) *If* $d \geqq 6$, 0 *is super-regular for* T_h *if and only if*

$$\int_{0+} \left(\frac{h(r)}{r} \right)^{d-5} \frac{dr}{r^3} = \infty.$$

For $h(r) = r^\alpha$, $\alpha > 1$, we see that 0 is super-regular for T_h if $d \leqq 5$, or if $d \geqq 6$ and $\alpha \leqq (d-3)/(d-5)$.

Proof. Suppose first that $d \leqq 4$. Then it suffices to consider the case $h = 0$, and the corresponding thorn T_0 is the half-line $\{x_1 = \cdots = x_{d-1} = 0, x_d \geqq 0\}$. A scaling argument immediately shows that $\mathbb{N}_0(\mathscr{R}^* \cap T_0 \neq \emptyset) = \lambda\, \mathbb{N}_0(\mathscr{R}^* \cap T_0 \neq \emptyset)$ for every $\lambda > 0$. Moreover $\mathbb{N}_0(\mathscr{R}^* \cap T_0 \neq \emptyset) > 0$ because points are not $\mathscr{R}$-polar in dimension $d \leqq 3$. It follows that $\mathbb{N}_0(\mathscr{R}^* \cap T_0 \neq \emptyset) = \infty$.

When $d \geqq 5$, we use the following estimate on the capacity of cylinders.

Lemma 15. *There exist two positive constants δ_1 and δ_2 such that, if $\mathscr{C}$ denotes the cylinder $\mathscr{C} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d,\ 0 \leqq x_d \leqq \lambda,\ (x_1^2 + \cdots + x_{d-1}^2)^{1/2} \leqq r\}$, where $0 < r \leqq 1$ and $\lambda > 2r$,*

$$\text{if } d = 5, \quad \delta_1 \frac{\lambda}{\log \frac{\lambda}{r}} \leqq \operatorname{cap}_{d-4}(\mathscr{C}) \leqq \delta_2 \frac{\lambda}{\log \frac{\lambda}{r}} \ ;$$

$$\text{if } d \geqq 6, \quad \delta_1 \left(\frac{\lambda}{r}\right) r^{d-4} \leqq \operatorname{cap}_{d-4}(\mathscr{C}) \leqq \delta_1 \left(\frac{\lambda}{r}\right) r^{d-4}.$$

Proof. From the scaling properties of $\operatorname{cap}_{d-4}(\cdot)$, we can restrict our attention to the case $r = 1$. For the lower bound, we let m denote Lebesgue measure on $\mathscr{C}$ and consider the probability measure $v = (m(\mathscr{C}))^{-1} m$. Elementary calculations show that

$$\int_{\mathscr{C}} \int_{\mathscr{C}} dy\, dz\, |y - z|^{4-d} \leqq C\, \lambda \left(1 + \int_1^{\lambda} du\, u^{4-d}\right)$$

and it follows that

$$\int_{\mathscr{C}} \int_{\mathscr{C}} v(dy)\, v(dz)\, |y - z|^{4-d} \leqq \begin{cases} C'\, \lambda^{-1} \log \lambda & \text{if } d = 5, \\ C'\, \lambda^{-1} & \text{if } d \geqq 6. \end{cases}$$

The desired lower bound then follows from the definition of $\operatorname{cap}_{d-4}(\mathscr{C})$.

Let us now turn to the proof of the upper bound. If $d \geqq 6$, this is very easy. Let $\mathscr{C}_i = \{x \in \mathscr{C}, i \leqq x_d < i + 1\}$. Then, by the subadditivity property of $\operatorname{cap}_{d-4}(\cdot)$ (see e.g. Landkov [13, p. 141]), we have

$$\operatorname{cap}_{d-4}(\mathscr{C}) \leqq \sum_{0 \leqq i \leqq \lambda} \operatorname{cap}_{d-4}(\mathscr{C}_i) \leqq (\lambda + 1) \operatorname{cap}_{d-4}(\mathscr{C}_1).$$

Finally, if $d = 5$, we use the following potential-theoretic fact (see [13, Ch. 2]). For every compact subset K of $\mathbb{R}^d$, there exists a finite measure μ_K supported on K (the equilibrium measure of K with respect to the kernel $|x|^{4-d}$) such that

$$\int \mu_K(dy)\, |x - y|^{4-d} \leqq 1, \quad \forall x \in \operatorname{supp} \mu_K \tag{34}$$

and $\operatorname{cap}_{d-4}(K) = \langle \mu_K, 1 \rangle$. A simple argument using the triangle inequality then gives

$$\int \mu_K(dy)\, |x - y|^{4-d} \leqq 2^{d-4}, \quad \forall x \in \mathbb{R}^d,$$

Hence, by taking $d=5$, $K=\mathscr{C}$ and $b=(0,\ldots,0,1)$, we get for a constant $c>0$,

$$2\,\lambda \geqq \int_0^\lambda du \int \mu_{\mathscr{C}}(dy)\,|y-ub|^{-1}$$
$$= \int \mu_{\mathscr{C}}(dy) \int_0^\lambda du\,|y-ub|^{-1} \geqq c\,\mathrm{cap}_{d-4}(\mathscr{C})\,\log\lambda,$$

which completes the proof of Lemma 15. □

We now complete the proof of Theorem 14. As previously, take

$$F_n = \{x \in T_h,\, 2^{-n} \leqq |x| < 2^{-n+1}\}.$$

We may for every n sufficiently large find two cylinders $\mathscr{C}_n^+$, $\mathscr{C}_n^-$ with respective radii $r_n^+ = h(2^{-n+1})$, $r_n^- = h(2^{-n})$ and respective lengths $\lambda_n^+ = 2^{-n+1}$, $\lambda_n^- = 2^{-n-1}$, such that

$$\mathscr{C}_n^- \subset F_n \subset \mathscr{C}_n^+.$$

Suppose first that $d \geqq 6$. From Lemma 15 we get for n sufficiently large,

$$\delta_1 \left(\frac{2^{-n-1}}{h(2^{-n})}\right) h(2^{-n})^{d-4} \leqq \mathrm{cap}_{d-4}(F_n) \leqq \delta_2 \left(\frac{2^{-n+1}}{h(2^{-n+1})}\right) h(2^{-n+1})^{d-4}.$$

By Theorem 3, we see that 0 is super-regular for T_h if and only if

$$\sum_{n=1}^{\infty} 2^{(d-3)n}\, h(2^{-n})^{d-5} = \infty.$$

Since the function $r^{-1}h(r)$ is nondecreasing for r small, the latter condition is clearly equivalent to the one stated in Theorem 14 (iii). The case $d=5$ is similar. □

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