Extension of the 1981 Arakawa and Lamb Scheme to Arbitrary Grids

Chris Eldred and David Randall
Department of Atmospheric Science
Colorado State University
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AL81 Scheme- Desirable Properties

Desirable Properties

1. Steady geostrophic modes
2. No spurious vorticity production
3. Energy-conserving pressure gradient and Coriolis forces
4. Mass, PV, **total energy and potential enstrophy conservation**
5. PV advection is consistent with mass advection
6. *Fully explicit (no global solve)*
7. *Good wave dipersion properties if* $\frac{\lambda}{d} > 1$
8. No spurious stationary modes or *extra branches of dispersion relationship*

Note: AL81 is discrete exterior calculus based scheme (Thuburn et. al 2012), also Hamiltonian (Salmon 2004)

AL81 Scheme- Limitations and Extensions

Shortcomings

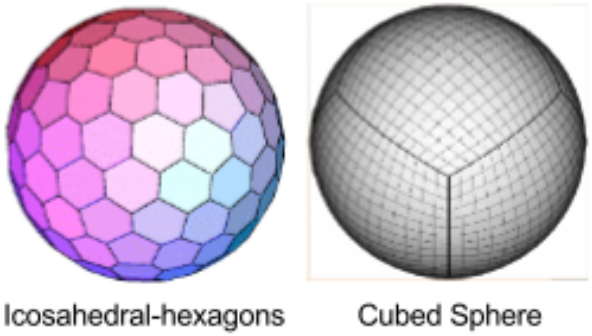
1. Restricted to logically square, orthogonal grids
2. Poor wave dispersion properties if $\frac{\lambda}{d} << 1$
3. Multilevel versions can suffer from Hollingsworth instability
4. 2nd order (can get quasi-4th order on uniform grids)

TRiSK: Ringler, Skamarok, Klemp, Thuburn, Cotter, Dubos

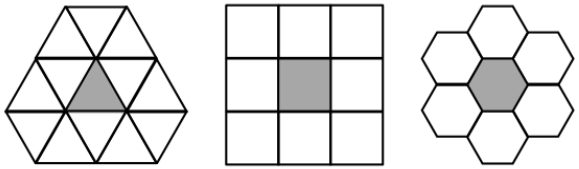
1. Discrete Exterior Calculus based
2. General, non-orthogonal polygonal grids
3. **Choose between total energy and potential enstrophy conservation**
4. Spurious wave dispersion branches on non-quadrilateral meshes

Generalized C Grid Discretization: TRiSK

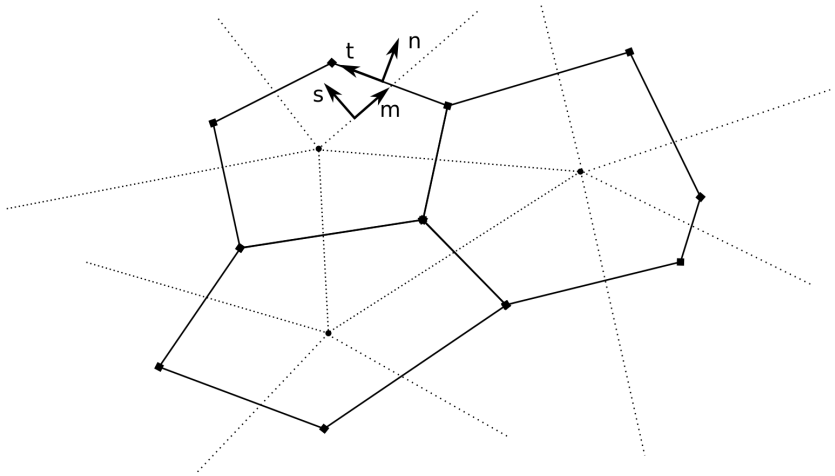
1. Extends AL81 to arbitrary, non-orthogonal polygonal meshes
2. **Must choose between total energy and potential enstrophy conservation**



Spherical Grids



(Uniform) Planar Grids



General non-orthogonal primal-dual polygonal mesh

Generalized C Grid Discretization: Hamiltonian-DEC

- Discrete variables are $m_i = \int h dA$ (discrete primal 2-form) and $u_e = \int \vec{u} \cdot \vec{dl}$ (discrete dual 1-form)
- C grid staggering (m_i at cell centers, u_e at edges)
- General formulation is:

$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & \mathbf{Q} \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2}(C_e, u_e)_{\mathbf{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I}\Phi_i \\ \mathbf{H} C_e \end{pmatrix}$$

$$q_v = \frac{\bar{D}_2 u_e + f_v}{m_v} = \frac{\zeta_v + f}{m_v} = \frac{\eta_v}{m_v}$$

$$m_v = \mathbf{R} m_i$$

$$m_e = \phi \mathbf{I} m_i \qquad C_e = m_e u_e$$

$$\Phi_i = g m_i + K_i \qquad K_i = \phi^T \frac{u_e \mathbf{H} u_e}{2}$$

Poisson Bracket Form

$$\frac{d\mathcal{A}}{dt} = \{\mathcal{A}, \mathcal{H}\}_Q + \{\mathcal{A}, \mathcal{H}\}_R$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2}(F_e, u_e)_{\mathbf{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I}\Phi_i \\ \mathbf{H}F_e \end{pmatrix}$$

$$\{\mathcal{A}, \mathcal{B}\}_Q = \sum_e \frac{\delta \mathcal{A}}{\delta u_e} \mathbf{Q} \frac{\delta \mathcal{B}}{\delta u_e}$$

$$\{\mathcal{A}, \mathcal{B}\}_R = \sum_i \frac{\delta \mathcal{A}}{\delta m_i} D_2 \frac{\delta \mathcal{B}}{\delta u_e} + \sum_e \frac{\delta \mathcal{A}}{\delta u_e} \bar{D}_1 \frac{\delta \mathcal{B}}{\delta m_i}$$

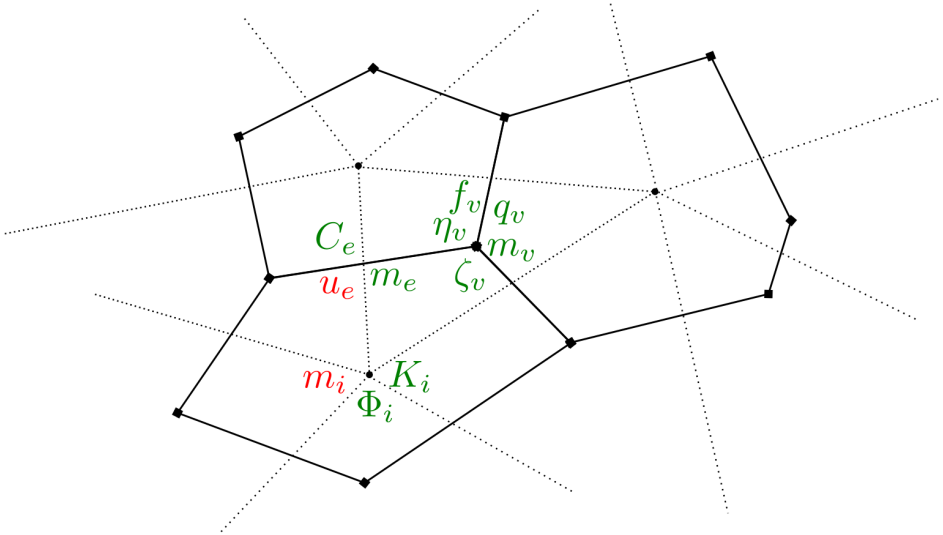


Diagram of grid staggering for generalized C grid

Requirements on Q

- PV Consistency:

$$\{\mathcal{A},\mathcal{B}\}_Q \rightarrow_{q_v=c} c * \{\mathcal{A},\mathcal{B}\}_W$$

- Total Energy Conserving:

$$\{\mathcal{A},\mathcal{B}\}_Q = -\{\mathcal{B},\mathcal{A}\}_Q$$

- Potential Enstrophy Conserving :

$$\{\mathcal{A},\mathcal{Z}\}_Q + \{\mathcal{A},\mathcal{Z}\}_R = 0 \qquad \forall \mathcal{A}$$

$$\mathcal{Z}_\mathcal{C} = \frac{1}{2}(\zeta_v, q_v)_\mathbf{J} \qquad \frac{\delta \mathcal{Z}_\mathcal{C}}{\delta \vec{x}} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$

General Form of **Q**

Following Salmon 2004, set

$$\{\mathcal{A},\mathcal{B}\}_Q = \sum_i \left(\sum_{(e,e')\in EP(i)} \sum_{v\in VC(i)} q_v \alpha_{e,e',v} \frac{\delta(\mathcal{A},\mathcal{B})}{\delta(e,e')} \right)$$

Note that

$$\{\mathcal{A},\mathcal{B}\}_Q = -\{\mathcal{B},\mathcal{A}\}_Q$$

by construction since $\frac{\delta(\mathcal{A},\mathcal{B})}{\delta(e,e')} = -\frac{\delta(\mathcal{B},\mathcal{A})}{\delta(e,e')}$. Additional requirements on $\{\mathcal{A},\mathcal{B}\}_Q$ lead to a system of equations for $\alpha_{e,e',v}$.

Potential Enstrophy Conservation

Chain rule yields

$$\{m_i, \mathcal{Z}_\mathcal{C}\}_Q + \{m_i, \mathcal{Z}\}_R = 0 \qquad \forall m_i$$

$$\{u_e, \mathcal{Z}_\mathcal{C}\}_Q + \{u_e, \mathcal{Z}\}_R = 0 \qquad \forall u_e$$

Plug in $\mathcal{Z}_\mathcal{C}$ to get

$$D_2 D_1 q_v = 0 \qquad \forall q_v$$

and

$$-\bar{D}_1 \mathbf{R}^T \frac{q_v^2}{2} + \mathbf{Q} D_1 q_v = 0 \qquad \forall q_v$$

Latter leads to an overdetermined system of linear systems (solved via least-squares).

Solution Procedure

1. Can rewrite last equation as

$$\mathbf{A}\vec{\alpha} = \vec{b}$$

where $\mathbf{A}$ is a rectangular matrix that comes from $\mathbf{Q}D_1$, $\vec{\alpha}$ is the vector of $\alpha_{e,e',v}$'s and $\vec{b}$ comes from $\bar{D}_1\mathbf{R}^T$.

2. Enough free parameters in solution of $\mathbf{A}\vec{\alpha} = \vec{b}$ that original equations can be split into:

$$\mathbf{A}_i\vec{\alpha}_i = \vec{b}_i$$

which applies for each cell, independently!

3. $$\{\mathcal{A},\mathcal{B}\}_Q \rightarrow_{q_v=c} c * \{\mathcal{A},\mathcal{B}\}_W$$

happens "automatically" for all grids tested (uniform square, uniform hex and geodesic).

Summary

Summary

- Combined Salmon 2004 (Hamiltonian) and Thuburn, Cotter and Dubos 2012 (DEC) approaches to develop an extension of Arakawa and Lamb 1981 to arbitrary polygonal grids
- Preserves all desirable properties of AL81 (however has extra wave modes on non-quadrilateral meshes)

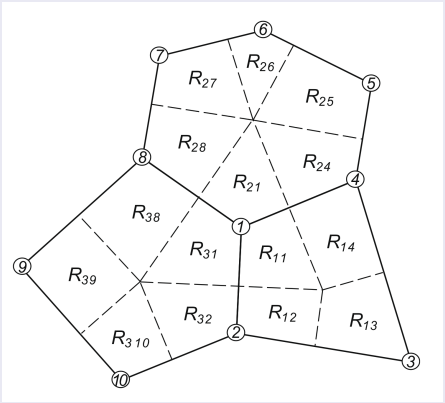
$\mathbf{Q}$ Variants

- $\mathbf{Q} = \mathbf{Q_e W}$ (Potential Enstrophy Conserving)
- $\mathbf{Q} = \frac{1}{2}\mathbf{Q_e W} + \frac{1}{2}\mathbf{W Q_e}$ (Total Energy Conserving)
- $\mathbf{Q} =$ complicated (Total Energy and Potential Enstrophy Conserving)
- $\mathbf{Q_e} = \phi q_v = \sum_{v \in VE(e)} \frac{q_v}{2}$

Description of Model Configuration

Operator Choices

- $\mathbf{I} = \frac{1}{A_i}$
- $\mathbf{J} = \frac{1}{A_v}$
- $\mathbf{H} = \frac{le}{de}$
- $m_e = \phi m_i = \sum_{i \in CE(e)} \frac{m_i}{2}$, K_i defined correspondingly



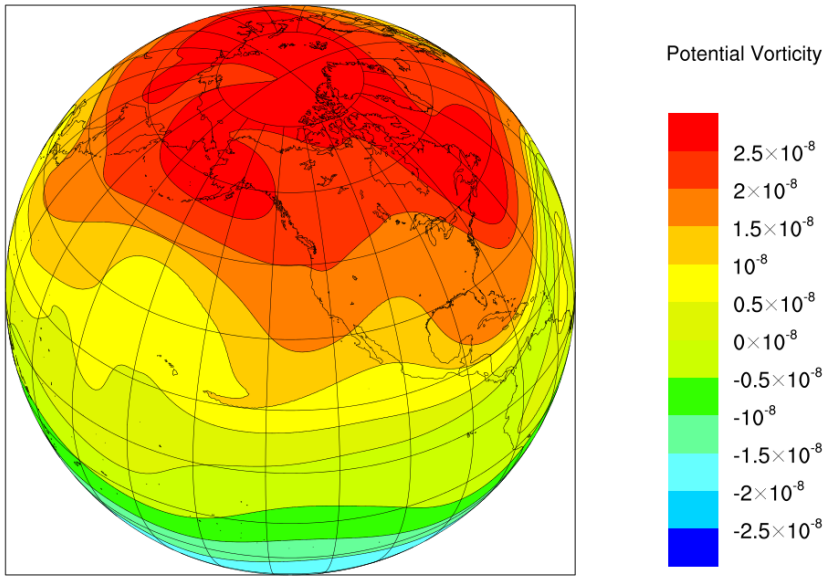
R Operator

Model Settings

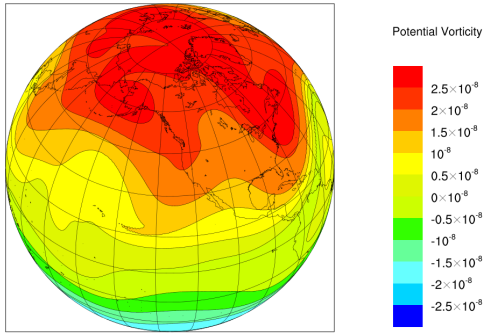
- 3rd Order Adams-Bashford Timestepping, 90s time step, run for 50 days
- G6 Geodesic Grid = 40962 cells (120km nominal resolution)
- $\nu \vec{\nabla}^2 \vec{v}$ dissipation added to momentum equation ($\nu = 1.0x10^5 m^2 s^{-1}$)- stable without

Flow over an Isolated Mountain (Williamson Test Case 5)

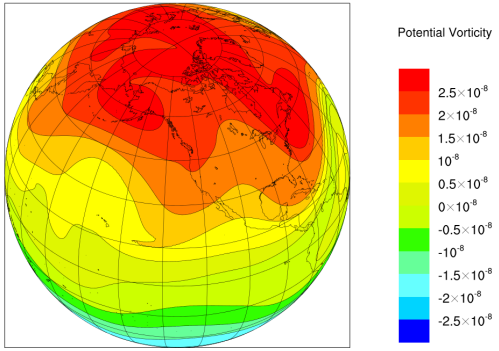
Potential Vorticity at Day 50, Doubly Conservative



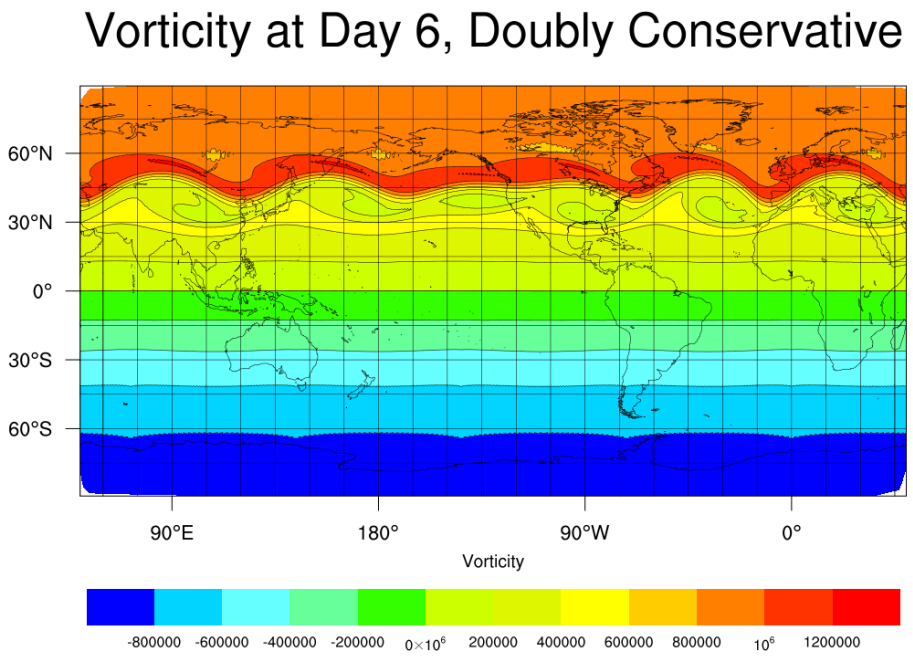
Potential Vorticity at Day 50, Enstrophy Conserving

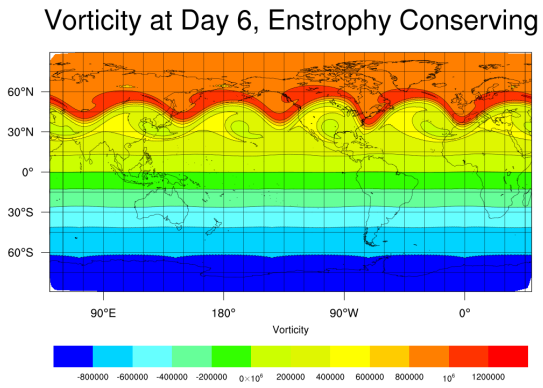


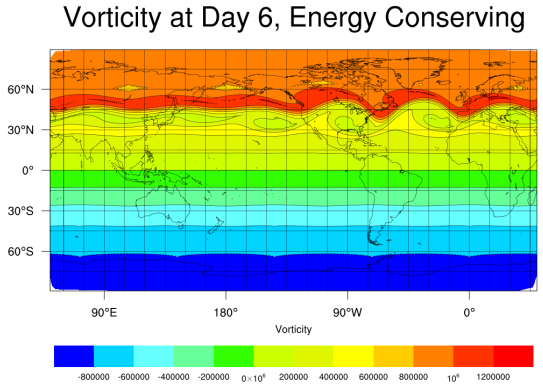
Potential Vorticity at Day 50, Energy Conserving



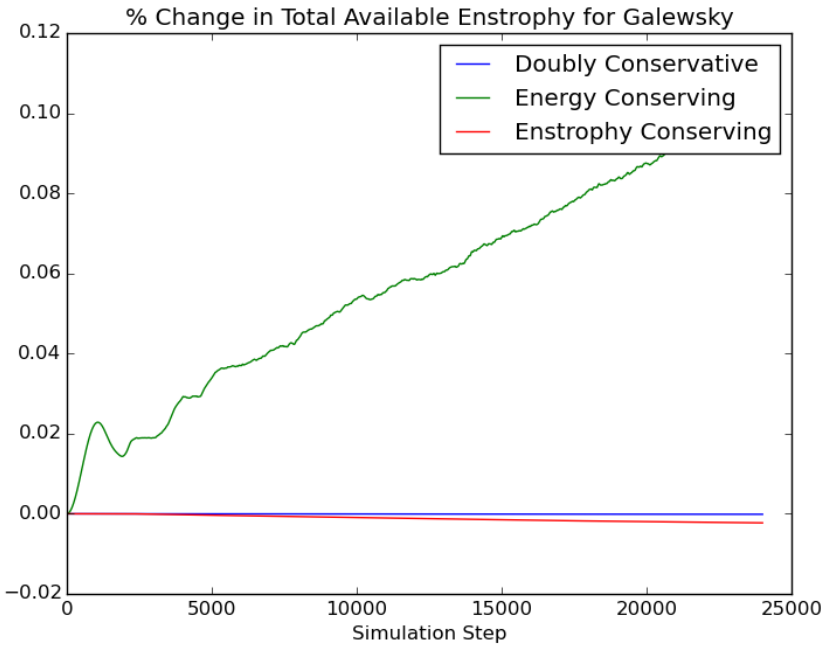
Galewsky et. al (Unstable Jet)

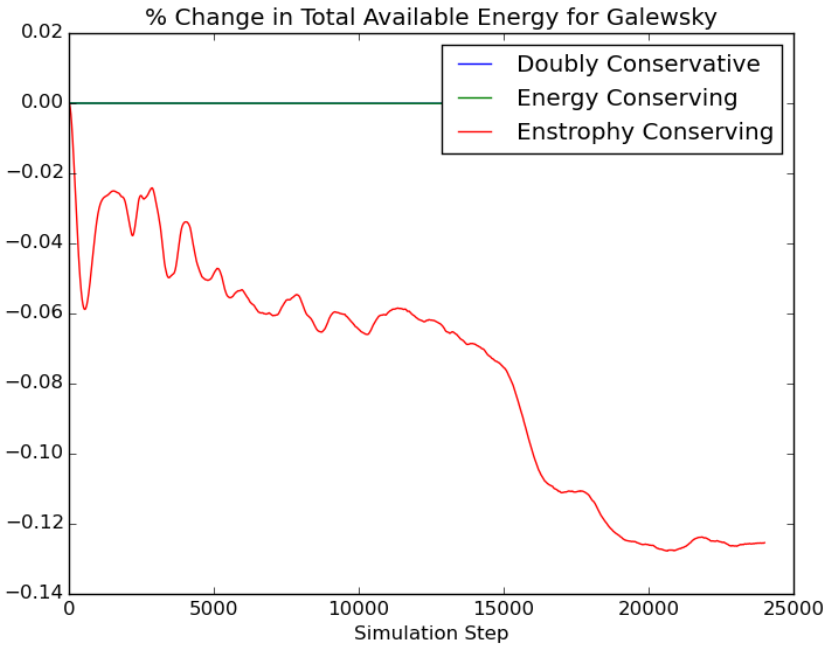






Conservation Properties (Galewsky Test Case)





These results are WITHOUT any added dissipation

Conclusions and Future Work

Conclusions

1. AL81 has been extended to arbitrary grids
2. Scheme offers comparable performance to existing schemes (preliminary results)

Future Work

1. Larger meshes- V7 (160K) and V8 (640K)
2. Detailed evaluation of scheme
3. Comparison to Z-grid scheme (Salmon 2007), other schemes
4. Extension to non-orthogonal cubed sphere grid, possibly Weller diamond grid and Healpix grid