Stationary and Propagating Modes in Discrete Models of the Linear Shallow Water Equations

Chris Eldred and David Randall

Department of Atmospheric Science, Colorado State University

April 7th, 2014

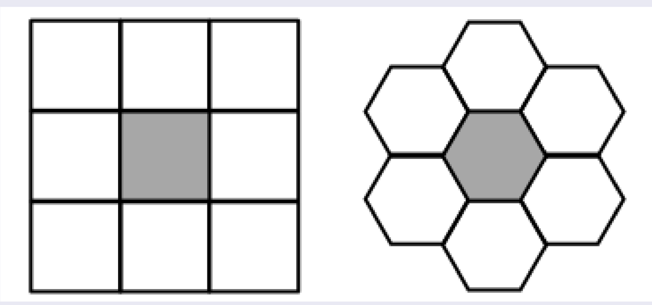
Abstract

- Shallow water equations are a useful analogue of the fully compressible Euler equations for atmospheric model development
- Linear properties (propagating and stationary modes) play an important physical role in the behaviour of the atmosphere
- Using the Atmospheric Dynamical Core Testbed (ADCoT, described below), the linear properties of three finite-difference schemes **(TRISK: Ringler et. al 2010; HR95: Heikes & Randall 1995 and NICAM: Tomita et. al 2001)** on the **f-plane/sphere** are compared to those of the continuous equations

ADCOT: Design & Implementation

- Horizontal meshes represented using MOAB mesh library
- Currently supported meshes: Perfect planar square and hexagonal; geodesic (tweaked) spherical meshes

Figure: Sample planar grids



- Variables (scalar, vector, vector component) placed arbitrarily on mesh elements
- Operators defined as sparse matrices (linear) or algebraic combinations of vector operators and (sparse) matrix multiplication (non-linear)

$$\vec{\nabla}(\frac{u^2}{2} + gh) \rightarrow \mathrm{G}(\mathrm{K}\vec{u}^2 + g\vec{h})$$

- Uses MOAB, PETSc and SLEPc to provide grid management, linear/eigenvalue solvers and I/O; main code written in Fortran 95
- Code generation using Cheetah enables fast prototyping and flexibility
- Analysis packages are written in Python/Fortran 95 using the NFFT, PyNGL, Numpy, Scipy and Matplotlib libraries
- Adams-Bashford and Runge-Kutta explicit time stepping
- TRISK, HR95 and NICAM horizontal discretizations
- Intended primarily for single moment discretizations

Linear Shallow Water Equations on an f-plane

- Momentum Form

$$\frac{\partial \vec{u}}{\partial t} = -f\hat{k} \times \vec{u} - g\vec{\nabla}h$$
$$\frac{\partial h}{\partial t} = -H(\vec{\nabla} \cdot u)$$

- Vorticity-Divergence Form

$$\frac{\partial \zeta}{\partial t} = -f\delta$$
$$\frac{\partial \delta}{\partial t} = f\zeta - g\vec{\nabla}^2 h$$
$$\frac{\partial h}{\partial t} = -H\delta$$

Results: Stationary Modes

- **Operator null spaces (stationary modes) calculated as $0 = \mathbb{L}\vec{x} \rightarrow 0 = \mathrm{A}\vec{x}$ (SVD problem)**
- **Momentum Stationary Modes** (C grid and A grid)
- Consider generalized system as

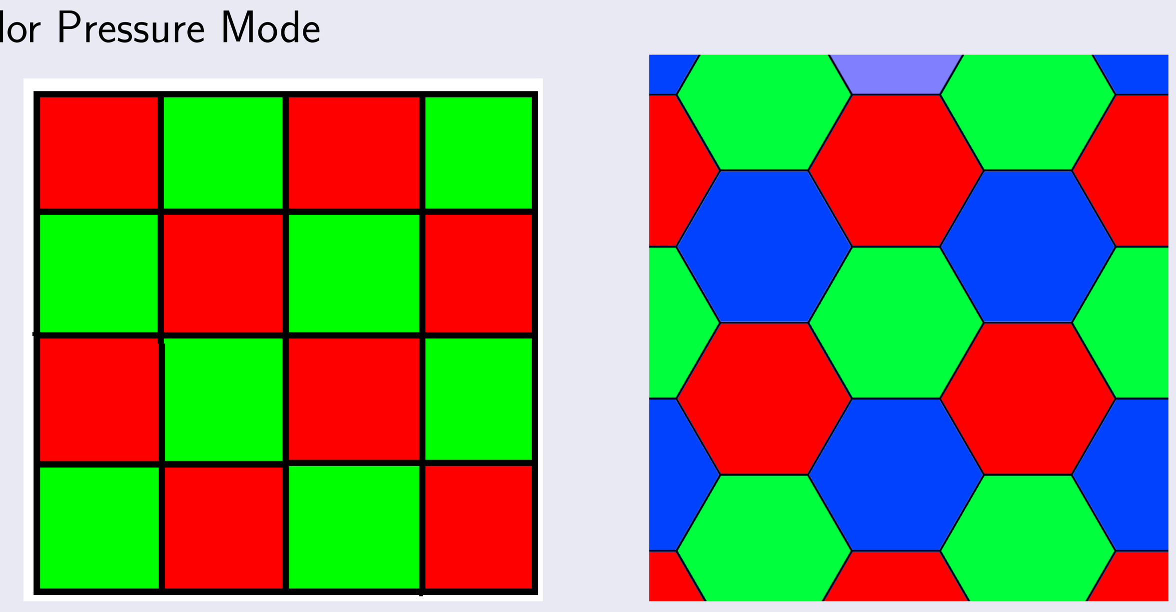
$$f\mathrm{T}\vec{u} + g\mathrm{G}\vec{h} = 0$$
$$H\mathrm{D}\vec{u} = 0$$

  - Geostrophic Modes: $\mathrm{D}\vec{u} = 0$ AND $f\mathrm{T}\vec{u} + g\mathrm{G}\vec{h} = 0$
  - Hydrostatic Modes: $\mathrm{G}h = 0$ with $\vec{h}$ = const,$\vec{u} = 0$
  - Pressure Modes (Spurious): $\mathrm{G}\vec{h} = 0$ with $\vec{h}$ = non-const,$\vec{u} = 0$
  - TD Modes (Spurious): $\mathrm{T}\vec{u} = 0$ AND $\mathrm{D}\vec{u} = 0$ with $\vec{h} = 0$,$\vec{u} =$ non-const
- In general, pressure modes occur only for A grid schemes, while TD modes occur only for C grid scheme
- When time discretization is introduced, additional stationary modes (such as inertial modes) can occur
- **Vorticity-Divergence Stationary Modes** (Z grid)
- Consider generalized system as

$$f\vec{\delta} = 0$$
$$f\vec{\zeta} - g\mathrm{L}\vec{h} = 0$$
$$H\vec{\delta} = 0$$

  - Geostrophic Modes: $f\vec{\zeta} - g\mathrm{L}\vec{h} = 0$ with $\vec{\delta} = 0$
  - Hydrostatic Modes: $\mathrm{L}h = 0$ with $\vec{h}$ =const,$\vec{\delta} = 0$,$\vec{\zeta} = 0$
  - Pressure Modes (Spurious): $\mathrm{L}\vec{h} = 0$ with $\vec{h}$ =non-const,$\vec{\delta} = 0$,$\vec{\zeta} = 0$
- Immediate observation: Vorticity-Divergence schemes have much simpler stationary mode structure than Momentum based schemes
- **TRISK (C grid) has no spurious stationary modes on PS, PH or geodesic meshes** (T operator has a large null space, but D does not)
- **HR95 (Z grid) has no spurious stationary modes on PS, PH or geodesic meshes** (L operator is well behaved)
- **NICAM (A grid) has a pressure mode on doubly periodic PS and PH meshes**: comes from incorrect null space in the gradient operator

Figure: Perfect Square and Hexagonal Grids, NICAM, Two and Three Color Pressure Mode



- **Actual gradient null spaces depend critically on grid size and periodicity assumptions** (basically, the tiling must be able to repeat an integer number of times and have correct boundary conditions)
- Example: PS 10x10 grid has spurious pressure modes for NICAM, but PS 11x11 grid does not
- **Important question: does NICAM still have a pressure mode on geodesic meshes?** (possible that "gradient correction" term and/or change in topology due to pentagonal cells will remove it)
- **Important question: can numerical dissipation remove or control the pressure mode on PS, PH and geodesic meshes?** (follow up: is this sensitive to the type of diffusion- divergence damping vs. hyperdiffusion?)

Results: Propagating Modes

- **Dispersion relationship calculated as $\frac{d\vec{x}}{dt} = \mathbb{L}\vec{x} \rightarrow i\omega\vec{x} = \mathrm{A}\vec{x}$ (eigenvalue problem)**
- Fourier transforms (NFFT package) are used to determine which spatial wavenumbers each eigenvector/eigenvalue pair is associated with
- TRiSK (C-grid), HR95 (Z-Grid) and NICAM (A-Grid) investigated on PS and PH meshes

Figure: Perfect Square Grid, NICAM, $\frac{\lambda}{d} = 2.0$

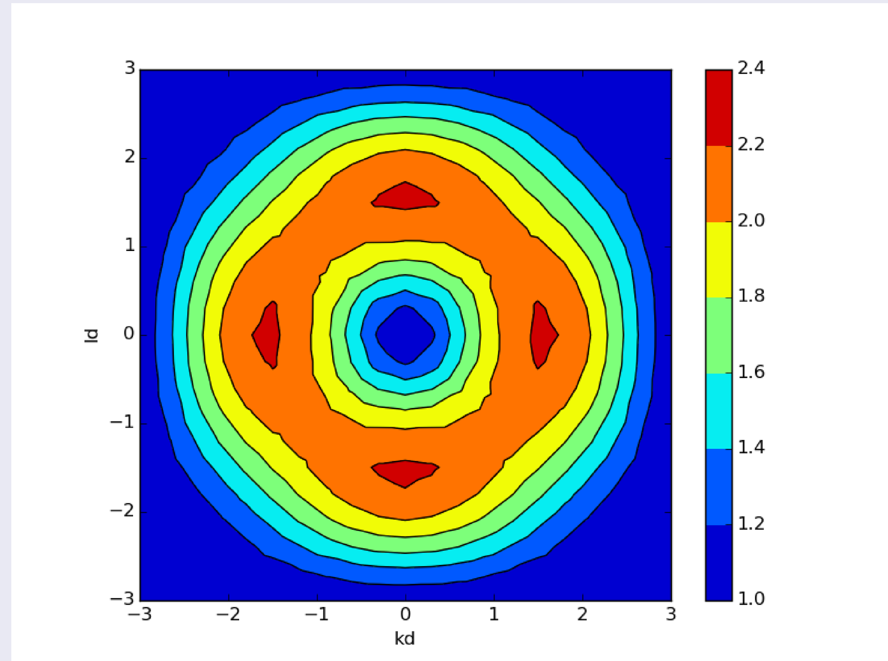
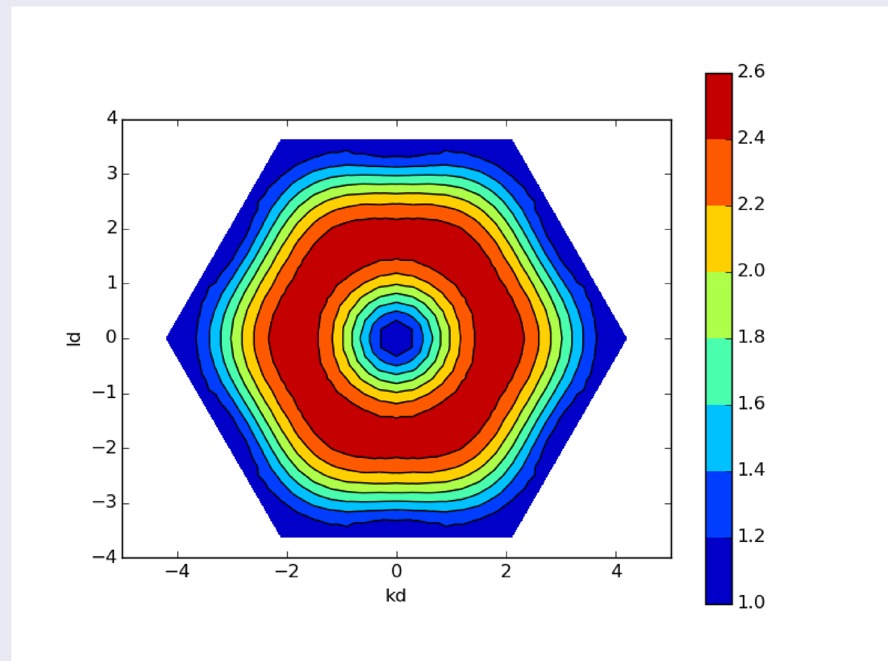


Figure: Perfect Hexagonal Grid, NICAM, $\frac{\lambda}{d} = 2.0$



- Results are similar for $\frac{\lambda}{d} = 0.1$ (not shown)
- Analytic dispersion relations have been calculated for PS and PH cases (numerical results match to within expected precision; not shown)
- Allowed wavenumbers on PS and PH also calculated (not shown)
- **All grid and scheme combinations have stationary geostrophic modes** (these become Rossby modes when f is variable)
- **All grid and scheme combinations have correct number of geostrophic and inertia-gravity wave modes except TRISK on PH grids** (has an extra geostrophic mode, that becomes a spurious Rossby mode when f is variable)
- **NICAM behaviour on PS and PH grids is very similar to Arakawa A grid behaviour on PS grids** (PH is more isotropic than PS, higher frequency modes still have wrong group velocity sign, qualitatively insensitive to $\frac{\lambda}{d}$)
- **Important question: what are the other effects of using numerical dissipation to damp high-frequency inertia-gravity waves in NICAM on PS, PH and geodesic meshes?**