# PARTIALLY MIXING AND LOCALLY RANK ONE SMOOTH TRANSFORMATIONS AND FLOWS ON THE TORUS $\mathbf{T}^d$, $d \geq 3$.

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Abstract. We give the first examples of $C^\infty$ measure preserving flows and diffeomorphisms on the torus that are partially mixing and have a simple spectrum.

## 1. Introduction.

1.1. We will denote by $\mathcal{C}$ the circle $\mathbf{R}/\mathbf{Z}$. When $d \geq 2$, we will denote by $\mathbf{T}^d$ the torus $\mathbf{R}^d/\mathbf{Z}^d$. Assume $R_\alpha$ is a minimal translation on $\mathbf{T}^d$ and consider on $\mathbf{T}^{d+1}$ the *irrational* translation flow:

$$\frac{dx}{dt} = (1, \alpha).$$

This flow is minimal and uniquely ergodic for the Haar measure $dx$ on $\mathbf{T}^{d+1}$. We will denote it by $\{R_{t(1,\alpha)}\}$.

1.2. Given $\phi \in C^\infty(\mathbf{T}^{d+1}, \mathbf{R}_+^*)$, we define the *reparametrization*, or *smooth time change, of* $\{R_{t(1,\alpha)}\}$, *with speed* $\frac{1}{\phi}$, to be the flow given by

$$\frac{d\theta}{dt} = \frac{\alpha}{\phi(\theta, s)}, \qquad \frac{ds}{dt} = \frac{1}{\phi(\theta, s)}.$$

The reparametrized flow is minimal and uniquely ergodic with invariant measure $\phi(x)dx$ (see for example [13]).

1.3. *Statements.* We will define in (1.1) (following [15]) an uncountable dense subset $Y \subset \mathbf{R}^2$, for which we will prove the following

**Theorem 1** (Flows). *Let $\tau \in [0, 1]$. For any $(\alpha, \alpha') \in Y$, there exists a strictly positive $C^\infty$ function $\phi(x, y, z)$ on the torus $\mathbf{T}^3$ such that the reparametrization of the irrational flow $\{R_{t(\alpha,\alpha',1)}\}$ by the function $\frac{1}{\phi}$ is $\tau$-partially mixing and has local rank $(1 - \tau)$.*

All definitions will be given in Section 2. The method used to construct the above flows allows us to obtain examples of diffeomorphisms with the same properties by simply considering some of their time-t maps:

**Theorem 2** (Diffeomorphisms). *Let $\tau \in [0, 1]$. For every flow $\{T^t\}$ constructed in Theorem 1, there exists a $G_\delta$ dense set of times $\mathcal{S}_{\alpha,\alpha'} \subset \mathbf{R}$*

*such that for any* $s \in \mathcal{S}_{\alpha,\alpha'}$*, the time-s map* $T^s$*, is* $\tau$*-partially mixing and has local rank* $(1-\tau)$*.*

When $0 < \tau < 1/2$, local rank $(1-\tau)$ implies a simple spectrum for the flow or the transformation (Cf. §2.2.5). We obtain therefore the following corollary of Theorems 1 and 2

**Corollary 1** (Simple spectrum)**.** *There exists measure preserving flows and diffeomorphisms of class* $C^\infty$ *on the torus that are partially mixing and have a simple spectrum.*

1.4. *Remarks.*

1.4.1. The notion of partial mixing was introduced by Ornstein and Freidman in [7] (Cf. Definition 2.3.1). In the mentioned article the authors produce examples of partially mixing and non-mixing transformations. Later, in [6] the same authors produce examples of $\tau$-mixing transformations for any $\tau \in [0,1]$. This latter result was then generalized by Fieldsteel and Freidman to flows [3]. Other constructions and examples followed, [5], [4]... But all of them were produced in the same abstract frame of measure theory with *cutting and stacking* methods inspired by some earlier works of Chacon and Ornstein.

Theorems 1 and 2 give an explicit $C^\infty$ confirmation of the results in [7], [6], and [3], in a frame that can reasonably be interpreted as natural: the frame of smooth reparametrizations of irrational flows on the torus $\mathbf{T}^d$, $d \geq 3$.

1.4.2. Our result does not use in any means *cutting and stacking* techniques, and partial mixing is exclusively produced by displaying *uniform stretch* for ceiling functions of special flows over translations ([14], [10], [1]).

1.4.3. The notion of local rank $\tau$ or *locally rank one* was first introduced by J-P. Thouvenot as a generalization of the rank one property (see the definitions in §2). It proved to be an interesting sufficient criterion implying the Loosely Bernoulli property [2].

1.4.4. That the spectral properties of a reparametrized translation flow of the torus might be different from the canonical characteristics of translations was already known from the results of Kolmogorov [11] and Shklover [14]. Theorem 1 gives automatically a continuum of non-equivalent $C^\infty$ reparametrizations for any irrational flow $\{R_{t(1,\alpha,\alpha')}\}$, $(\alpha,\alpha') \in Y$.

1.4.5. The method we use to insure local rank $(1-\tau)$ for the flow (and more especially for the time-$s$ diffeomorphism) forces some rigidity on the flow (for the definitions, see (2.2) and (2.3)). Precisely, the flows we obtain are $(1-\tau)-$rigid (i.e. $\text{rig}(T) = 1-\tau$). Since $\text{mix}(T) + \text{rig}(T) \leq 1$, for any transformation $T$, $\tau-$partially mixing is the best we could obtain under this constraint of rigidity. It was proved in [9] that whenever $\text{mix}(T) + \text{r}(T) > 1$ the transformation $T$ had two fold minimal self joinings. This is not the case here and there exists an example of $\tau$-mixing and $(1-\tau)$-rigid transformation that has a factor and fails hence to have the minimal self joining property [4].

1.5. *Special flows over translations.* Considering a Poincaré section, a smoothly reparametrized irrational flow of $\mathbf{T}^{d+1}$ can be viewed as a special flow constructed over a translation of $\mathbf{T}^d$ with a ceiling function $\varphi \in C^\infty(\mathbf{T}^d, \mathbf{R}_+^*)$. The exact definition of such special flows and the correspondence between them and reparametrizations of irrational flows will be described in more detail in §2.1.

When $d = 1$, improved Denjoy-Koksma inequalities for $C^1$ functions force a special flow, constructed over an irrational rotation of the circle with a smooth ceiling function, to be rigid. *Hence smooth reparametrizations of irrational flows on* $\mathbf{T}^2$ *are always rigid.*

When $d \geq 2$, it is a fact that for the *general* $\alpha \in \mathbf{R}^d$, Denjoy-Koksma inequalities remain valid for any smooth function $\varphi$. In this case special flows over $R_\alpha$ with $C^\infty$ ceiling functions as well as smooth reparametrizations of $\{R_{t(1,\alpha)}\}$ will be rigid. *General* here includes a $G_\delta$ dense set and a set of full Lebesgue measure (Cf. [11] and [1]). Still, in an appendix to his thesis, J-C. Yoccoz constructs, for an uncountable set of irrational vectors $(\alpha, \alpha') \in \mathbf{R}^2$, analytic functions $\varphi_{\alpha,\alpha'}$ that violate the Denjoy-Koksma inequality above $R_{\alpha,\alpha'}$ [15]. This is crucial for producing non rigid and even mixing smooth special flows over translations on $\mathbf{T}^2$ or on higher dimensional tori (Cf. [1]). To avoid Denjoy-Koksma inequalities above an irrational translation $R_{\alpha,\alpha'}$, one can turn to vectors $(\alpha, \alpha')$ with the property that the denominators of the convergents of $\alpha$ and $\alpha'$ are far apart. More precisely

**Definition 1.1.** *Let $Y$ be the set of couples $(\alpha, \alpha') \in \mathbf{R}^2 - \mathbf{Q}^2$, such that the denominators of the convergents of $\alpha$ and $\alpha'$, respectively $\{q_n\}_{n\in\mathbf{N}}$ and $\{q'_n\}_{n\in\mathbf{N}}$, satisfy the following: there exists $n_0 \in \mathbf{N}$ such that, for any $n \geq n_0$*

$$q'_n \geq e^{3q_n}, \tag{1}$$

$$q_{n+1} \geq e^{3q'_n}, \tag{2}$$

$$q'_n \wedge q_n = 1. \tag{3}$$

Here $p \wedge q = 1$ stands for $p$ and $q$ relatively prime. We will see in (1.1) how to obtain vectors satisfying (1)-(3). In [1] we were able to prove for $(\alpha, \alpha') \in \mathbf{R}^2$ satisfying (1) and (2), that the special flow constructed over $R_{\alpha,\alpha'}$ with the real analytic ceiling function introduced by Yoccoz

$$\varphi(x,y) = 1 + \text{Re}\left(\sum_{k=1}^{\infty} \frac{1}{e^{q_k}} e^{i2\pi q_k x} + \sum_{k=1}^{\infty} \frac{1}{e^{q'_k}} e^{i2\pi q'_k y}\right) \tag{4}$$

is mixing. The property underlying mixing for these flows is *uniform stretching* of the ceiling-function Birkhoff-sums,

$$\varphi_m(x,y) := \varphi(x,y) + \varphi(R_{\alpha,\alpha'}(x,y)) + ... + \varphi(R^{m-1}_{\alpha,\alpha'}(x,y)). \quad (5)$$

Due to the disposition of the denominators of the convergents of $\alpha$ and $\alpha'$, the Birkhoff sums $\varphi_m$, for any $m$ sufficiently large, will be *always* stretching (i.e. they have large derivatives), in one or in the other of the two directions $x$ or $y$, depending on whether $m$ is far from $q_n$ or far from $q'_n$. The stretch then implies, that when $t$ goes to infinity, the image of a small interval on the base $\mathbf{T}^2$, taken either in the $x$ or in the $y$ direction, will be increasingly expanded along the fibers of the special flow and equidistributed in the space by ergodicity of $R_{\alpha,\alpha'}$.

In the current paper, we will modify the function $\varphi$ in order to gain the property of locally rank one for the special flow, with the cost of losing some of the mixing feature and remain with partial mixing instead.

1.6. **Plan.** In Section 2 we give the definitions of the properties we are interested in and state useful criteria to obtain them. First, we define special flows and state an equivalent theorem to Theorem 1 involving special flows over translations of $\mathbf{T}^2$. In §2.2 we define *locally rank one* for flows and diffeomorphisms and give a criterion for special flows over rank one transformation that guarantees locally rank one. Starting with a sequence of *generating rank one towers* of the transformation on the base (Cf. §2.2.2), the idea in the criterion is that under a condition of flatness on the Birkhoff sums of the ceiling function, it is possible to "lift" a fixed proportion of each tower into *monochromatic* towers for the special flow. The next subsection is devoted to partial mixing and contains a criterion for a special flow over an irrational translation on $\mathbf{T}^d$, $d \geq 1$, that guarantees partial mixing. This Criterion is an adaptation of Proposition 3.3. on mixing given in [1]. It relies on the display of *uniform stretch* by the Birkhoff sums on a subsequent proportion of the base. In Section 3 we introduce for $R_{\alpha,\alpha'}$, $(\alpha,\alpha') \in Y$, the rank one towers that we will partially lift to towers for the special flow. It is in Section 4 that we get down to the construction of the ceiling function that should fulfill simultaneously Criterion 2.2.1 on local rank and Criterion 2.3.1 on partial mixing. The construction as we said is a modification of $\varphi$ given in (4). Nevertheless, the $y$ part of $\varphi$ is kept unchanged since it already satisfies the criterion on local rank with the choice we will make for the towers of $R_{\alpha,\alpha'}$. As for the $x$ part, each term $\cos(2\pi q_n x)$ will be replaced by a function with the same period $1/q_n$, essentially zero on one side of the interval $[0, 1/q_n]$ and affine on the other side. These sides will represent respectively the rigid side and the mixing side of the flow. As the delimitation between the sides is arbitrary we can have local rank $(1-\tau)$ and $\tau$-partial mixing for any $\tau \in [0,1]$; hence covering from mixing to totally rigid flows all the intermediate partially mixing ones.

### 1.7. **Notations.**

1.7.1. For $d \in \mathbf{N}^*$, we denote by $C^\infty(\mathbf{T}^d, \mathbf{R})$ the set of real functions on $\mathbf{R}^d$ of class $C^\infty$ and $\mathbf{Z}^d$-periodic. By $C^\infty(\mathbf{T}^d, \mathbf{R}^*_+)$ we will denote the subset of $C^\infty(\mathbf{T}^d, \mathbf{R})$ of strictly positive functions. We will also use the notation

$$||\varphi|| := \sup_{z \in \mathbf{T}^d} |\varphi(z)|.$$

1.7.2. For $r \in \mathbf{N}$, the notation $D^r_x \varphi$ is used for the derivative of order $r$ of $\varphi$ with respect to $x$.

1.7.3. Given a real number $x$, we will denote by $|||x|||$ the closest distance of $x$ to the integers, and by $\{x\}$ we will denote the fractional part of $x$.

## 2. Preliminaries and Definitions.

2.1. **Special flows.** We give first the definition of a special flow. Given a dynamical system $(M, T, \mu)$ and a real function $\varphi \in L^1(M, \mu)$, $\varphi > c > 0$, the *special flow constructed over* $(M, T, \mu)$ *and under the function* $\varphi$ is the quotient flow of the action

$$\begin{array}{rcl} M \times \mathbf{R} & \longrightarrow & M \times \mathbf{R} \\ (x, s) & \longrightarrow & (x, s+t) \end{array}$$

by the relation $(x, s + \varphi(x)) \sim (T(x), s)$. This flow acts on the space $M_{T,\varphi} = M \times \mathbf{R}/\sim$, and preserves the normalized Lebesgue measure on $M_{T,\varphi}$, i.e. the product of $\mu$ on the base $M$ with the Lebesgue measure on the fibers divided by the constant $\int_M \varphi(x)dx$. We will call this measure $\nu$ and denote the special flow by $\{T, \varphi\}$.

Consider now a reparametrization of a translation flow on $\mathbf{T}^3$, $R_{t(1,\alpha,\alpha')}$, by a $C^\infty$ function $\frac{1}{\phi}$. It is not hard to see, considering a Poincaré section that the reparametrized flow is isomorphic to the special flow constructed over $R_{\alpha,\alpha'}$ on $\mathbf{T}^2$ with the ceiling function defined by

$$\varphi(x, y) = \int_0^1 \phi(x + s\alpha, y + s\alpha', s)ds.$$

Conversely, given a special flow $\{R_{\alpha,\alpha'}, \varphi\}$, if there is a smooth function $\phi \in C^\infty(\mathbf{T}^3, \mathbf{R}^*_+)$ satisfying the linear equation above, the flow can be viewed as a reparametrization of $R_{t(1,\alpha,\alpha')}$ with velocity $\frac{1}{\phi}$. In [1], we were able to show, generalizing the one dimensional result of Shklover [14], that

**Lemma 2.1.1.** *For any irrational vector $(\alpha, \alpha') \in \mathbf{R}^2/\mathbf{Q}^2$ and any function $\varphi \in C^\infty(\mathbf{T}^2, \mathbf{R}_+^*)$, there exists a $C^\infty$ strictly positive function on $\mathbf{T}^3$, $\phi_{\alpha,\alpha'}$ such that*

$$\varphi(x,y) = \int_0^1 \phi_{\alpha,\alpha'}(x+s\alpha, y+s\alpha', s)ds.$$

In conclusion, Theorem 1 is equivalent to

**Theorem 3** (Special flows)**.** *Given $\tau \in [0,1]$ and $(\alpha,\alpha') \in Y$, there exists $\varphi \in C^\infty(\mathbf{T}^2, \mathbf{R}_+^*)$ such that the special flow $\{R_{\alpha,\alpha'}, \varphi\}$ is $\tau$-partially mixing and has local rank $1-\tau$.*

This is the theorem we will prove in the sequel. We will then derive from it Theorem 2.

## 2.2. **Locally rank one, definitions and criteria.**

2.2.1. *Rokhlin Lemma for measure preserving transformations.* Let $(X,T,\mu)$ be an ergodic system, then for any $\varepsilon > 0$ and any $n \in \mathbf{N}$ there exists a measurable subset $A \subset X$ such that $A, T(A), ..., T^{n-1}(A)$ are disjoint and

$$\mu\left(A \bigcup T(A) \bigcup ... \bigcup T^{n-1}(A)\right) > 1-\varepsilon.$$

We call $A \bigcup T(A) \bigcup ... \bigcup T^{n-1}(A)$ an $(n, 1-\varepsilon)$-*tower* for $T$.
More generally, whenever $A$ is measurable and $A, T(A), ..., T^{n-1}(A)$ are disjoint with $\mu(A \bigcup T(A) \bigcup ... \bigcup T^{n-1}(A)) > \tau$, we will talk of an $(n,\tau)-$tower for $T$. The set $A$ is the *base* of the tower and $n$ is its *height*. Every $T^k(A)$, $k \leq n-1$, is called a *level* of the tower. We will use the notation $\mathcal{T} := A \bigcup T(A) \bigcup ... \bigcup T^{n-1}(A)$, to designate the tower as well as the partial partition whose atoms are the levels $T^k(A)$, $k \leq n-1$.

Given a partition $\mathcal{P}$ and an $\epsilon > 0$, we say that a level $T^k(A)$ of the tower is $(1-\epsilon) -$ *monochromatic* with respect to $\mathcal{P}$ if a proportion strictly larger than $(1-\epsilon)$ of this level is included in one of the atoms of $\mathcal{P}$.

2.2.2. *Definition of locally rank one for transformations.* Let $(X,T,\mu)$ be a dynamical system. Given $0 < \tau \leq 1$, we say that $T$ has *local rank* $\tau$ if for any measurable partition $\mathcal{P}$ of $X$ and for every $\epsilon > 0$ arbitrarily small, there is an $(n, \tau-\epsilon)$ tower for $T$ with a proportion larger than $1-\epsilon$ of its steps being $(1-\epsilon)-$monochromatic with respect to $\mathcal{P}$. A system having local rank $\tau > 0$ is said to be *locally rank one*. If $\tau = 1$, we recover the definition of rank one. In the latter case, there exists for $(X,T,\mu)$ a sequence of towers $\mathcal{T}_n$ generating the sigma algebra of finite partitions of $(X,\mu)$.
Rokhlin's Lemma can be stated for flows:

2.2.3. *Rokhlin Lemma for flows.* Let $(T^t, X, \nu)$ be an ergodic flow. For any positive $h$ and $\varepsilon$, we can represent $\{T^t\}$ as a special flow over a system $(M, T, \mu)$ with a ceiling function $f$ such that:

(i) $f(x) \leq h$ for every $x$ in $M$,
(ii) $f(x) = h$ on a subset $B \subset M$ of measure $\mu(B) \geq 1 - \varepsilon$.

This flow version of Rokhlin's lemma was first introduced by Ornstein in [12].
We call $\bigcup_{t=0}^{h} T^t(B)$ an $(h, 1-\varepsilon)$-tower for $\{T^t\}$ with base $B$ and height $h$. Every $T^t(B)$, $t \leq h$, is a *horizontal level* of the tower.

Given a finite partition $\mathcal{P}$ of the initial space $(X, \mu)$ and an $\epsilon > 0$, we say that a horizontal level $T^s(B)$, $s \leq h$ of the tower is $(1-\epsilon)-$ *monochromatic* with respect to $\mathcal{P}$ if a proportion not less than $1-\epsilon$ of the $T^s_*(\mu)$-measure of this horizontal level is included in one of the atoms of $\mathcal{P}$.

We can generalize in a natural way the definition of local rank $\tau$ to flows. We give the definition as stated by P. Zeitz in [16]:

2.2.4. *Definition of locally rank one for flows.* Let $(T^t, X, \nu)$ be a dynamical system. Given $0 < \tau \leq 1$, we say that $T^t$ has *local rank* $\tau$ if for any finite measurable partition $\mathcal{P}$ of $X$ and for every $\epsilon < 1$, there is an $(h, \tau - \epsilon)$ tower for $\{T^t\}$ with at least a proportion $(1-\epsilon)$ of its horizontal levels being $(1-\epsilon)-$monochromatic with respect to $\mathcal{P}$.

2.2.5. *Incidence on the spectral multiplicity.* It is now a classical result in ergodic theory [8] that

**Lemma 2.2.1 (Simple spectrum).** *If a transformation (or a flow) is $\tau$-locally rank one with $\tau > \frac{1}{2}$, then it has a simple spectrum, i.e. there exists a function $\xi \in L^2(X, \mu, \mathbf{C})$ such that the cyclic subspace spaned by $\xi$, $\xi o T$,... is equal to $L^2(X, \mu, \mathbf{C})$.*

2.2.6. *A criterion that guaranties locally rank one for special flows over rank one transformations.* The criterion involves the Birkhoff-sums of the special flow's ceiling-function and allows us to "lift" a proportion of a rank one tower on the base to a monochromatic $(1-\tau)$-tower for the flow.

**Proposition 2.2.1 (Criterion for locally rank one).** *Let $\tau \in (0,1)$. Assume that $(M, T, \nu, \varphi)$ is a special flow of rank one, constructed with a bounded ceiling function $\varphi$. If there exists a generating sequence of Rokhlin towers for $T$, $B_n \bigcup ... \bigcup T^{h_n}(B_n)$, $n \in \mathbf{N}$, and if $\varphi$ satisfies*

$$(6) \qquad \sup_{m \leq h_n} \sup_{z, z' \in C_n} |\varphi_m(z') - \varphi_m(z)| \longrightarrow 0,$$

*where $C_n \subset B_n$ is such that $\liminf \frac{\mu(C_n)}{\mu(B_n)} \geq 1 - \tau$, then the flow $(M, T, \nu, \varphi)$ has local rank $1-\tau$.*

*Proof.* Define $\overline{h}_n = \varphi_{h_n}(z_0)$ for some fixed $z_0 \in C_n$ and consider the tower $\mathcal{T}_n = \bigcup_{t=0}^{\overline{h}_n} T^t(C_n)$. Since $\varphi$ is smooth and $T$ is ergodic (rank one implies ergodicity), we have $\liminf \nu(\mathcal{T}_n) \geq \tau$, where $\nu$ is the probability measure invariant by the special flow. On the other hand the fact that we considered a generating sequence of towers for $T$ and hypothesis (6) imply that the horizontal levels of $\mathcal{T}_n$ are increasingly monochromatic when $n$ goes to infinity, with regard to any fixed finite partition of the space $M_{T,\varphi}$. □

## 2.3. **Partial mixing and Rigidity, definitions and criteria.**

2.3.1. *Partial mixing.* A flow $(T^t, X, \nu)$ is partially mixing if there exists a constant $\gamma \in ]0,1]$ such that, for any measurable subsets $A$ and $B$, one has for large $t$

$$\nu(T^t(A) \bigcap B) \geq \gamma \nu(A)\nu(B)$$

The largest number $\gamma$ for which this holds is called the *mixing number* of $\{T^t\}$ and denoted by $\text{mix}(T^t)$. By definition, $\text{mix}(T^t) = 1$ corresponds to a mixing flow.

2.3.2. *Rigidity.* A counterpart for the mixing number is the *rigidity number*, defined to be the largest $\rho \in [0,1]$ for which, there is a sequence of times $t_n \to \infty$ such that for any measurable subset $A$, for $n$ large enough

$$\nu(T^t(A) \bigcap A) \geq \rho \nu(A)$$

When $\text{rig}(T^t) = 1$, we say the flow is *rigid*.
Clearly, we always have $\gamma \leq 1 - \rho$, hence if $\rho$ is positive, the flow is not mixing.
*Remark.* The same definitions exist of course for discrete transformations.

2.3.3. *A criterion for Partial mixing for special flows over translations.* In the statement of the criterion we need two definitions.

**Definition 2.1.** *Given $\varepsilon > 0$, and $K > 0$; we say that a real function $f$, defined on some interval including a segment $[a,b]$, is $(\varepsilon, K)$-uniformly stretching on $[a,b]$ if $|f(b) - f(a)| \geq K$, and if for any $f(a) \leq u \leq v \leq f(b)$, the interval $I_{u,v} = \{s \in [a,b] / u \leq f(s) \leq v\}$ has Lebesgue measure or length*

$$(1-\varepsilon)\frac{v-u}{f(b)-f(a)}(b-a) \leq \lambda(I_{u,v}) \leq (1+\varepsilon)\frac{v-u}{f(b)-f(a)}(b-a).$$

**Lemma 2.3.1** (A Criterion for uniform stretch)**.** *If*

$$\begin{aligned} \inf_{s\in[a,b]} |f'(s)||b-a| &\geq K, \\ \sup_{s\in[a,b]} |f''(s)||b-a| &\leq \varepsilon \inf_{s\in[a,b]} |f'(s)|, \end{aligned}$$

*then $f$ is $(\varepsilon, K)$-uniformly stretching on $[a, b]$*

The proof of this very easy Lemma can be found in [1].

**Definition 2.2.** *A partial partition on the circle will be a collection of disjoint measurable sets of $\mathcal{C}$. We say that a sequence of partial partitions $\{\mathcal{P}_n\}_{n\in\mathbf{N}}$ on the circle has capacity $\tau$ if given any measurable subset $A$, and any $\epsilon > 0$, there exists $N$ such that for any $n \geq N$, there is a subset $A_n$, union of atoms from $\mathcal{P}_n$ such that*

$$\lambda(A \bigcap A_n) \geq (\tau - \epsilon)\lambda(A).$$

We can state now the following proposition that guarantees partial mixing for a special flow over a minimal translation of the torus, $R_{\alpha,\alpha'}$, $(\alpha, \alpha') \in \mathbf{R}^2$. For the statement, we assume that $\varphi$ is differentiable and $\int_{\mathbf{T}^2} \varphi(x, y)dxdy = 1$. The Birkhoff sums $\varphi_m$ are as in (5).

**Proposition 2.3.1 (Criterion for partial mixing).** *Let $0 < \tau \leq 1$. If there exists a sequence of partial partitions $\{\mathcal{P}_n\}$ of the circle with capacity at least $\tau$, the atoms of each $\mathcal{P}_n$ being intervals with lengths going uniformly to zero; and if there exists a sequence of integers $U_n \to \infty$ such that for any $n \in \mathbf{N}$ large enough, we have*
• *For all $m \in [\frac{U_{2n}}{2}, 2U_{2n+1}]$, for any interval $C^{(2n)} \in \mathcal{P}_{2n}$, for any $y \in \mathcal{C}$,*

$$\varphi_m(., y) \text{ is } (\frac{1}{n}, n) - uniformly\ stretched\ on\ C^{(2n)} \times \{y\},$$

• *For all $m \in [\frac{U_{2n+1}}{2}, 2U_{2n+2}]$, for any interval $C^{(2n+1)} \in \mathcal{P}_{2n+1}$, for any $x \in \mathcal{C}$,*

$$\varphi_m(x, .) \text{ is } (\frac{1}{n}, n) - uniformly\ stretched\ on\ \{x\} \times C^{(2n+1)},$$

*then the special flow $\{R_\alpha, \varphi\}$ is $\tau$-partially mixing.*

The proof of Proposition 2.3.1 on partial mixing is very similar to the one on mixing used in [1], with the only difference that it uses in addition Definition 2.2. The notion of uniform stretch of the Birkhoff sums that underlies mixing for suspension flows over one-dimensional rigid ergodic transformations, was introduced by Kocergin in [10]. The concept was already implicitly used by Shklover in his seminal construction of weak mixing flows over rotations [14]. In our situation, the stretch is obtained alternatively in the $x$ direction and the $y$ direction, and mixing follows from a Fubini argument.

## 3. The translation on the base.

3.1. **The choice of the translation on $\mathbf{T}^2$.** The importance of (1) and (2) in the choice of $(\alpha, \alpha')$ was mentioned in the introduction: it is the mechanism of alternation between the $q_n$ and $q'_n$ that is behind uniform stretch. On the other hand, our technique to obtain locally rank one flows requires generating rank one towers for the transformation on the base. To have nice rank one towers for $R_{\alpha,\alpha'}$ on $\mathbf{T}^2$ we need the extra condition (3) on $\alpha$ and $\alpha'$. Indeed,

**Proposition 3.1.1.** *We can choose $\alpha$ and $\alpha'$ such that in addition to (1) and (2), $q_n$ and $q'_n$ should be prime.*

**Proof.** We will construct $\alpha$ and $\alpha'$ by induction on the sequences of partial quotients of their continued fraction expansion. By definition

$$q_n = a_n q_{n-1} + q_{n-2} \quad for \quad n \geq 2,\ q_0 = 1,\ q_1 = a_1$$
$$q'_n = a'_n q'_{n-1} + q'_{n-2} \quad for \quad n \geq 2,\ q'_0 = 1,\ q'_1 = a'_1$$

It is easy to see, by recurrence, that regardless to our choice of the $a_i$, $q_{n-1}$ and $q_n$ are always relatively prime. Suppose now the numbers $a_i$ and $a'_i$ are chosen for $i \leq n$. By Dirichlet's theorem the arithmetical progression $q_{n-1}$, $q_{n-1}+q_n$, $q_{n-1}+2q_n$,..., contains infinitely many prime numbers; hence we can choose $a_{n+1}$ such that (2) is true ($q_{n+1} \geq e^{3q'_n}$), and $q_{n+1}$ be prime. In a similar fashion, we choose $a'_{n+1}$ such that (1) is true and $q'_{n+1}$ is prime. □

3.2. **The towers for $R_{\alpha,\alpha'}$.** Consider on the two torus the rectangle $B_n = (1/nq_n, (1-1/n)1/q_n) \times (1/nq'_n, (1-1/n)1/q'_n)$. Then we can state

**Proposition 3.2.1.** *The translation $R_{\alpha,\alpha'}$ is rank one. Further, the rectangle $B_n$ is the base of a $(q_n q'_n - 1, 1 - \frac{1}{n^2})$ tower for $R_{\alpha,\alpha'}$.*

**Proof.** For each integer $n$, we consider on the two torus the rectangle $(0, \frac{1}{q_n}) \times (0, \frac{1}{q'_n})$. Because $q_n$ and $q'_n$ are relatively prime, the first $q_n q'_n - 1$ iterates of this rectangle by $R_{p_n/q_n, p'_n/q'_n}$ are disjoint and they fill up the torus in measure.
Let $||.||$ denote here the Euclidean distance on $\mathbf{T}^2$. Recall that

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}},$$
$$\left|\alpha' - \frac{p'_n}{q'_n}\right| \leq \frac{1}{q'_n q'_{n+1}}.$$

Hence, for any $(x,y) \in \mathbf{T}^2$, and $m \leq q_n q'_n$,

$$\begin{aligned} ||R^m_{\alpha,\alpha'}(x,y) - R^m_{\frac{p_n}{q_n},\frac{p'_n}{q'_n}}(x,y)|| &\leq \frac{m}{q_n q_{n+1}} \\ &\leq \frac{q'_n}{q_{n+1}}. \end{aligned}$$

Condition (2) implies that this last quantity is less than $1/nq_n$, and from the foregoing on the rational translation it follows that the first $q_n q'_n - 1$ iterates of the rectangle $B_n$ under $R_{\alpha,\alpha'}$ are disjoint. The total measure of the tower is clearly $(1-\frac{2}{n})^2$. These towers generate the $\sigma$ algebra of finite measurable partitions since their steps are small rectangles. We say the translation is rank one by rectangles. □

## 4. Construction of the ceiling function.

From now on the translation $R_{\alpha,\alpha'}$ is fixed, as well as the number $\tau \in [0,1]$. The towers constructed in the precedent section will be denoted by $\mathcal{T}_n := B_n \bigcup R_{\alpha,\alpha'}B_n ... \bigcup R_{\alpha,\alpha'}^{q_n q'_n -1}B_n$. As we said in the introduction we want to modify $\varphi$ given in (4) into a function that satisfies Proposition 2.2.1 on local rank $\tau$ over the towers $\mathcal{T}_n$.

In [1], mixing was obtained for the special flow $\{R_{\alpha,\alpha'},\varphi\}$ since the criterion of Proposition 2.3.1 was valid for a sequence of partial partitions of capacity one. Specifically, when $m \in [e^{2q_n}, e^{2q'_n}]$, $\varphi_m$ was uniformly stretched in the $x$-direction on the typical subinterval of $[k/q_n, (k+1)/q_n]$, and when $m \in [e^{2q'_n}, e^{2q_{n+1}}]$, the same was true in the $y$-direction on the typical small subinterval of $[k/q'_n, (k+1)/q'_n]$. The stretch for the corresponding range of values of $m$ comes respectively from the terms $\cos(2\pi q_n x)/e^{q_n}$ and $\cos(i2\pi q'_n y)/e^{q'_n}$, the contribution of the other terms of $\varphi$ being negligible at this level.

It appears clearly that by this mechanism, the criterion for locally rank one above the towers $\mathcal{T}_n$ is violated on the larger side of $B_n$, i.e. the $x$-side. We will therefore replace $e^{i2\pi q_n x}/e^{q_n}$ by a $1/q_n$-periodic term that essentially vanishes on $[\tau/q_n, 1/q_n]$, relegating uniform stretch to the other part of the interval $[0,1/q_n]$, where the new term is taken to be affine. The same Liouvillian behavior of $R_\alpha$ will be behind the flatness of the Birkhoff side and the strong stretch on the other.

As for the $y$-side of $B_n$, the stretch due to $D_y\varphi_m$, as it is defined in (4), over an interval of size $1/q'_n$ will not appear before $m = e^{q'_n} \gg q_n q'_n$, the height of the tower. Hence, we keep the "y part" of $\varphi$ unchanged.

### 4.1. Construction of the "y part" of the ceiling function.

Define

$$\psi(y) := \mathrm{Re}\left(\sum_{k=1}^{\infty} \frac{1}{e^{q'_k}} e^{i2\pi q'_k y}\right).$$

The function $\psi$ is the $y$ part of the function $\varphi(x,y)$ that we will use to prove Theorem 3. In this subsection, Birkhoff sums with regard to $R_{\alpha,\alpha'}$ will only involve $R_{\alpha'}$ and we will write

$$\psi_m(y) = \sum_{k=0}^{m-1} \psi(y + k\alpha').$$

To prepare for the checking of criteria 2.2.1 and 2.2.1 we will prove the following propositions with respect to $\psi_m(y)$

**Proposition 4.1.1.** *One has*

$$\sup_{m\leq q_n q_n'} ||D_y\psi_m|| = o(n^{-10}q_n').$$

*Proof.* Let $p\in\mathbf{N}$, if $\chi^p(y):=\cos 2\pi py$, we have for any $m\in\mathbf{N}$,

$$\text{For any } p\in\mathbf{N},\quad ||D_y\chi^p_m||\leq 2\pi pm. \tag{7}$$

$$\text{For any } |p|<q_n',\quad ||D_y\chi^p_m||\leq \pi p q_n'. \tag{8}$$

First, (7) is trivial since $||D_y\chi^p||=2\pi p$. For (8) we write

$$D_y\chi^p_m(y)=\mathrm{Re}\left(i2\pi p\frac{1-e^{i2\pi mp\alpha'}}{1-e^{i2\pi p\alpha'}}e^{i2\pi py}\right),$$

but, by definition of the best approximations of $\alpha'$, one has for all $|p|<q_n'$

$$|||p\alpha'|||\geq |||q_{n-1}'\alpha'|||\geq\frac{1}{2q_n'},$$

where $|||.|||$ denotes the closest distance to the integers; hence

$$\begin{aligned}||D_y\chi^p_m|| &\leq \frac{2\pi p}{|\sin(2\pi|||p\alpha'|||)|}\\ &\leq \pi p q_n'.\end{aligned}$$

Proposition 4.1.1 is then obtained by using (8) for the lower order terms in the expression of $\psi$, $k\leq n-1$, and (7) for the higher order terms, $k\geq n$. □

The foregoing proposition will be helpful when checking the local rank criterion. As for partial mixing, we have the following proposition on uniform stretch that is exactly similar to the one obtained in [1], Proposition 3.4.

First, for $n\in\mathbf{N}$, let $J_n:=\{y\in\mathbf{T}^1\ /\ \{q_{n-1}'y\}\in[\frac{1}{n},\frac{1}{2}-\frac{1}{n}]\bigcup[\frac{1}{2}+\frac{1}{n},1-\frac{1}{n}]\}$. Next, define $\mathcal{P}_{2n}$ to be a partition of $J_n$ by intervals of lengths varying between $e^{-q_{n-1}'}$ and $2e^{-q_{n-1}'}$, then the capacity of $\{\mathcal{P}_{2n}\}_{n\in\mathbf{N}}$ is clearly equal to one and we have

**Proposition 4.1.2.** *For all $m\in[\frac{e^{2q_{n-1}'}}{2},2e^{2q_n}]$, for any interval $C^{(2n)}\in\mathcal{P}_{2n}$*

$$\psi_m \text{ is } \left(\frac{n}{q_{n-1}'},\frac{q_{n-1}'}{2n}\right)-\text{uniformly stretched on } C^{(2n)}.$$

*Proof.* The proof relies on the fact that since $|||q_{n-1}'\alpha'|||\sim 1/q_n'$, one has for $m\ll q_n'$, $D_y\chi_m^{q_{n-1}'}(y)\sim -m2\pi q_{n-1}'\sin(2\pi q_{n-1}'y)$. This fact, together with (8) and (7) yields the following Lemma that can be found in [1], Proposition 3.4.

**Lemma 4.1.1.** *For any* $y \in J_n$, *for any* $m \in [\frac{e^{2q'_{n-1}}}{2}, 2e^{2q_n}]$, *the following holds*

$$|D_y\psi_m(y)| \geq \frac{m}{e^{q'_{n-1}}}\frac{q'_{n-1}}{n} \geq \frac{q'_{n-1}}{2n}e^{q'_{n-1}}. \tag{9}$$

The proof of uniform stretch then follows from Lemma 2.3.1 since clearly $D_y^2\psi_m = o(m)$. □

4.2. **Construction of the "x part".** There exists on $\mathbf{R}$ a $C^\infty$ real function, $0 \leq \theta \leq 1$, such that

$$\theta(x) = 0 \quad for \quad x \in [-\infty, 1],$$

$$\theta(x) = 1 \quad for \quad x \in [2, +\infty].$$

From now on $\tau \in (0,1)$ is fixed. For every $n \in \mathbf{N}$, define on the interval $[0,1]$ the $C^\infty$ function

$$\theta^n(x) := [\theta(nx) - \theta(n(x-\tau))]\, x. \tag{10}$$

**Lemma 4.2.1.** *The following holds for sufficiently large* $n$

$$\theta^n(x) = 0, \qquad for \qquad x \in [0, \frac{1}{n}], \tag{11}$$

$$D_x\theta^n(x) = 1, \qquad for \qquad x \in [\frac{2}{n}, \tau], \tag{12}$$

$$\theta^n(x) = 0, \qquad for \qquad x \in [\tau + \frac{2}{n}, 1]. \tag{13}$$

*Proof.* The proof of this Lemma is straightforward. □

From (11) and (13), $\theta^n$ can be considered as a $C^\infty$ function on $\mathcal{C}$. Define then, the $C^\infty$ function on $\mathcal{C}$

$$\varsigma^n(x) := \frac{1}{e^{q_n}}\theta^n(q_n x). \tag{14}$$

Our first candidate to replace $\cos(2\pi q_n x)/e^{q_n}$ is the function $\varsigma^n$. Preparing for locally rank one and partial mixing we have

**Proposition 4.2.1.** *For every* $m \leq q_n q'_n$, *and for every* $x \in \mathcal{C}$ *such that* $\{q_n x\} \in [\tau + \frac{3}{n}, 1 - \frac{1}{n}]$, *we have*

$$D_x\varsigma^n_m(x) = 0.$$

Furthermore,

**Proposition 4.2.2.** *For every* $m \in [\frac{e^{2q_n}}{2}, 2e^{2q'_n}]$, *and for every* $x$ *such that* $\{q_n x\} \in [\frac{3}{n}, \tau - \frac{1}{n}]$, *we have*

$$D_x\varsigma^n_m(x) = \frac{mq_n}{e^{q_n}}.$$

*Proof.* One has

$$\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n q_{n+1}}. \tag{15}$$

Hence, for any $m \leq q_n q'_n$,

$$\{m q_n \alpha\} \leq \frac{q'_n}{q_{n+1}} = o(\frac{1}{n}),$$

so, if $\{q_n x\} \in [\tau + \frac{3}{n}, 1 - \frac{1}{n}]$, for any $m \leq q_n q'_n$ it is true that $\{q_n(x + m\alpha)\} \in [\tau + \frac{2}{n}, 1]$, which implies by (13) that $D_x \varsigma^n{}_m(x) = 0$. Proposition 4.2.1 is proved and Proposition 4.2.2 follows in a similar fashion using this time the fact that $e^{2q'_n} = o(q_{n+1})$. □

By virtue of the foregoing propositions, the new functions $\varsigma^n$ have the required feature of combining uniform stretch on one side and flatness on the other for $\varsigma^n_m$. Hence, we would want to replace each $e^{i2\pi q_n x}/e^{q_n}$ by $\varsigma^n$. Still, we are not sure that for the corresponding values of $m$, $\varsigma^n_m$ will prevail among the other terms whose contribution to the Birkhoff sums at this level is not proved to be negligible. The higher order terms, $\varsigma_k$, $k > n$ are not problematic since they are too small. But the presence of high frequencies in the expression of the lower order terms, $\varsigma_k$, $k < n$, might jeopardize the validity of the criterion on local rank as well as the one on uniform stretch. Therefore we have to introduce truncations in order to obtain a similar estimate as in (8).

Now, $\varsigma_n$ is a $C^\infty$ function on the torus, so we write its Fourier expansion

$$\varsigma^n(x) = \sum_{p=-\infty}^{+\infty} \sigma^{n,p} \chi^p(x),$$

where $\chi^p(x) = e^{i2\pi p x}$. For every $n \in \mathbf{N}$, let

$$\hat{\varsigma}^n(x) = \sum_{p=-q_{n+1}+1}^{q_{n+1}-1} \sigma^{n,p} \chi^p(x).$$

We need the following lemma

**Lemma 4.2.2.** *For any $r \leq r_0$, $r_0$ given, we have that $||D^r_x \hat{\varsigma}^n||$ decreases geometrically with $n$.*

*Proof.* Since $\sigma^{n,p}$ are the Fourier coefficients of the $C^\infty$ function $\varsigma^n$ , it is true that

$$|\sigma^{n,p}| \leq \frac{1}{(2\pi p)^{r_0+2}} ||D^{r_0+2}_x \varsigma^n||. \tag{16}$$

Hence, for any $r \leq r_0$

$$D^r_x \hat{\varsigma}^n(x) = \sum_{p=-q_{n+1}+1}^{q_{n+1}-1} \sigma^{n,p} (2i\pi|p|)^r \chi^p(x)$$

implies

$$\begin{aligned} ||D_x^r \hat{\varsigma}^n|| &\leq \sum_{p\in[-q_{n+1}+1,q_{n+1}-1]/\{0\}} \frac{1}{(2\pi p)^2}||D_x^{r_0+2}\varsigma^n|| \\ &\leq \frac{1}{12}||D_x^{r_0+2}\varsigma^n||. \end{aligned}$$

The lemma then follows from (14). □

One consequence of this lemma is that the function

$$\kappa(x) := \mathrm{Re}\left(\sum_{k=1}^{\infty}\hat{\varsigma}^k(x)\right)$$

is of class $C^\infty$.

4.3. **The properties of $\kappa_m(x)$.** With the control we will obtain on the Birkhoff sums of the lower terms in the expression of $\kappa(x)$, and the good approximation of $\hat{\varsigma}^n$ by $\varsigma^n$, we will be able to derive from Propositions 4.2.1 and 4.2.2 the following

**Proposition 4.3.1.** *For every $m \leq q_n q'_n$, and for every $x \in \mathbf{T}^1$ such that $\{q_n x\} \in [\tau + \frac{3}{n}, 1 - \frac{1}{n}]$, we have*

$$D_x\kappa_m(x) = o(n^{-10}q_n).$$

As for uniform stretch, consider the subset of the circle $I_n = \{x \in \mathbf{T}^1 / \{q_n x\} \in [\frac{3}{n}, \tau - \frac{1}{n}]\}$ and let $\mathcal{P}_{2n+1}$ be a partition of $I_n$ with intervals of lengths varying between $e^{-q_n}$ and $2e^{-q_n}$. Clearly, the capacity of the sequence $\{I_n\}_{n\in\mathbf{N}}$ is equal to $\tau$. We have

**Proposition 4.3.2.** *For all $m \in [\frac{e^{2q_n}}{2}, 2e^{2q'_n}]$, for any interval $C^{(2n+1)} \in \mathcal{P}_{2n+1}$*

$$\kappa_m \; is \; (\frac{4}{q_n}, \frac{q_n}{4})-uniformly\ stretched\ on\ \ C^{(2n+1)}.$$

The proof of Propositions 4.3.1 and 4.3.2 will require some lemmas. The truncation of $\varsigma^k$ enables us to assess, depending on $k$, a uniform bound of $||D_x\hat{\varsigma}_m^k||$. With this we obtain the required control on the lower order terms in the expression of $\kappa_m$

**Lemma 4.3.1.** *For any $m \in \mathbf{N}$, for any $n \in \mathbf{N}$, one has*

$$||\sum_{k=1}^{n-1} D_x\hat{\varsigma}_m^k|| = o(n^{-10}q_n). \tag{17}$$

*Proof.* We will first prove for any $m$ and $k$

$$||D_x\hat{\varsigma}_m^k|| \leq \frac{1}{12}||D_x^3\varsigma^k||q_{k+1}.$$

As for (8) we have for any $|p| < q_{k+1}$, and any $m \in \mathbf{N}$,

$$||D_x \chi_m^p|| \leq 2\pi p q_{k+1}.$$

Therefore,

$$\begin{aligned} ||D_x \hat{\varsigma}_m^k|| &= ||\sum_{p=-q_{k+1}+1}^{q_{k+1}-1} \sigma^{k,p} D_x \chi_m^p|| \\ &\leq \sum_{p=-q_{k+1}+1}^{q_{k+1}-1} |\sigma^{k,p}| 2\pi p q_{k+1}, \end{aligned}$$

using (16) once more we have then

$$\begin{aligned} ||D_x \hat{\varsigma}_m^k|| &\leq \sum_{p=-q_{k+1}+1}^{q_{k+1}-1} \frac{1}{(2\pi p)^2} ||D_x^3 \varsigma^k|| q_{k+1} \\ &\leq \frac{1}{12} ||D_x^3 \varsigma^k|| q_{k+1}. \end{aligned}$$

But it follows from (14) that

$$||D_x^3 \varsigma^k|| \leq \frac{k^3 q_k^3}{e^{q_k}} ||D_x^3 \theta||, \tag{18}$$

and the proof of the lemma follows. □

This lemma provides the required control on the contribution of the lower terms $\hat{\varsigma}^k$ to the Birkhoff sums of $\kappa$. On the other hand a straightforward majorization shows that the contribution of the higher order terms is negligible:

**Lemma 4.3.2.** *For any $m \leq q_{n+1}$,*

$$||\sum_{k=n+1}^{\infty} D_x \hat{\varsigma}_m^k|| = o(\frac{1}{q_{n+1}}). \tag{19}$$

*Proof.* For any $k$, $||D_x \hat{\varsigma}_m^k||$ is at most of the order of $mk^3 q_k^3 / e^{q_k}$. If we sum over $k > n$, having $m \leq q_{n+1}$, (19) immediately follows. □

Finally, $\hat{\varsigma}^n$ is close to $\varsigma^n$ and we have

**Lemma 4.3.3.** *For any $n \in \mathbf{N}$ large enough,*

$$||D_x \hat{\varsigma}^n - D_x \varsigma^n|| = o(\frac{1}{q_{n+1}}). \tag{20}$$

*Proof.* One has

$$D_x \hat{\varsigma}^n - D_x \varsigma^n = \sum_{p \in \mathbf{N} / ]-q_{n+1}, q_{n+1}[} 2i\pi \sigma^{n,p} \chi^p.$$

Hence, using (16) it follows that

$$||D_x\hat{\varsigma}^n - D_x\varsigma^n|| \leq \sum_{p\in\mathbf{N}/]-q_{n+1},q_{n+1}[} \frac{1}{(2\pi p)^2}||D_x^3\varsigma^n||.$$

The proof of (20) follows from (18). □

*Proof of Propositions 4.3.1 and 4.3.2.* With Lemmas 4.3.1–4.3.3, Proposition 4.3.1 easily follows from Proposition 4.2.1, while Proposition 4.2.2 implies that for every $m \in [\frac{e^{2q_n}}{2}, 2e^{2q_n'}]$ and for every $x$ such that $\{q_n x\} \in [\frac{3}{n}, \tau - \frac{1}{n}]$, we have

$$D_x\kappa_m(x) \geq \frac{1}{2}\frac{mq_n}{e^{q_n}},$$

which in its turn leads to the uniform stretch claimed in Proposition 4.3.2 in light of Lemma 2.3.1 and the fact that $||D_x^2\kappa_m(x)|| \leq m$.

## 5. Proof of Theorem 1.

We will actually prove the equivalent theorem, Theorem 3. Define on $\mathbf{T}^2$ the $C^\infty$ and strictly positive function

$$\varphi(x,y) := C + \psi(y) + \kappa(x),$$

where the constant $C$ is chosen such that $\int_{\mathbf{T}^2}\varphi = 1$.

We want to prove that the special flow constructed over $R_{\alpha,\alpha'}$ with the ceiling function $\varphi(x,y)$ is $\tau$-partially mixing and has local rank $(1-\tau)$.

*Application of Criterion 2.3.1 to obtain Partial mixing.* We take $U_{2n} = e^{2q_{n-1}'}$, and $U_{2n+1} = e^{2q_n}$. Let $\mathcal{P}_{2n}$ and $\mathcal{P}_{2n+1}$ be the partial partitions of the circle defined before Propositions 4.1.2 and 4.3.2. The capacity of the sequence $\{\mathcal{P}_n\}$ is $\tau$ and it follows from the latter propositions that the criterion in the Proposition 2.3.1 is satisfied by $\varphi$ with our choice of $U_n$ and $\mathcal{P}_n$.

*Application of Criterion 2.2.1 to obtain local rank* $(1-\tau)$. We consider the rectangle $C_n = [(\tau + \frac{3}{n})/q_n, (1-\frac{1}{n})/q_n] \times [1/nq_n', (1-\frac{1}{n})/q_n'] \subset B_n$. The criterion for local rank $(1-\tau)$ holds over $C_n$. Indeed, $\liminf \mu(C_n)/\mu(B_n) = (1-\tau)$ and it follows from Propositions 4.1.1 and 4.3.1 that

$$\sup_{m\leq q_nq_n'}\sup_{z,z'\in C_n}|\varphi_m(z') - \varphi_m(z)| = o(n^{-10}).$$

*Remark.* Because the return of $C_n$ after $q_nq_n'$ iterations by $R_{\alpha,\alpha'}$ is almost periodic, the above inequality could be extended to

$$\sup_{m\leq n^3q_nq_n'}\sup_{z,z'\in C_n}|\varphi_m(z') - \varphi_m(z)| = o(n^{-7}). \tag{21}$$

This fact will be helpful in the upcoming section.

## 6. Proof of Theorem 2.

In this section the flow $\{T^t,\nu\}$ is the special flow $\{R_{\alpha,\alpha'},\varphi\}$ of Theorem 3, and $M_\varphi$ denotes the space where it acts. For $t_0 \in \mathbf{R}$, we will denote by $T^{t_0}$ the diffeomorphism time-$t_0$ of $\{T^t\}$. Since the flow is $\tau-$partially mixing, the diffeomorphism $T^{t_0}$ is also $\tau-$partially mixing for any choice of $t_0$. The rank $(1-\tau)$ property however, is not necessarily inherited by $T^{t_0}$. We will prove the existence of a dense set of times $t_0$ such that the local rank property holds for $T^{t_0}$. Theorem 2 will follow since the property of local rank $\tau$ is a "$G_\delta$ property", in our case the exact statement is: the set $\mathcal{R}(\tau)$ of times $t_0$ such that $T^{t_0}$ has local rank $\tau$ is a $G_\delta$ subset of $\mathbf{R}$. To see this, one uses the definition 2.2.2 of the local rank $\tau$ property. For a fixed partition $\mathcal{P}_i$ and a fixed precision $\frac{1}{k}$, the property of having a $(\tau-1/k)$ tower for the time map $T^s$ with at least a proportion $(1-1/k)$ of its steps being $(1-1/k)-$monochromatic with respect to $\mathcal{P}_i$ is clearly an open condition on $s$. Hence, the set $R_{i,k}(\tau)$ of times $s$ for which the map $T^s$ satisifes the above condition is open. Therefore $\mathcal{R}(\tau)=\bigcap_k\bigcap_i T_{i,k}(\tau)$ is a $G_\delta$.

From Section 5 we recall that $C_n=[(\tau+\frac{3}{n})1/q_n]\times[1/nq'_n,(1-\frac{1}{n})1/q'_n]$ is the base of a $q_nq'_n-1$ tower for $R_{\alpha,\alpha'}$ of capacity approaching $\tau$ as $n$ approaches infinity. By ergodicity, this implies

$$\lim_{n\to\infty}\nu\left(\hat{\mathcal{T}}_n\right)=\tau, \qquad \text{for } \hat{\mathcal{T}}_n:=\bigcup_{t=0}^{\varphi_{q_nq'_n}(r_n)} T^tC_n, \tag{22}$$

where $r_n$ is any point in $C_n$.

Now, for an adequate choice of $t_0$, we will define for each $n$ a set $\Delta_n$ of $M_\varphi$ and construct above it a tower $\mathcal{T}_n$ for $T^{t_0}$ such that:

a) The symmetrical difference $\nu\left(\mathcal{T}_n\triangle\hat{\mathcal{T}}_n\right)\to 0$ as $n$ goes to infinity.

b) As $n$ goes to infinity, the levels of the tower $\mathcal{T}_n$ become increasingly monochromatic with respect to any fixed partition of $M_\varphi$.

Under these conditions the sequence $\{\mathcal{T}_n\}_{n\in\mathbf{N}}$ will be a generating rank $(1-\tau)$ sequence of towers for the diffeomorphism $T^{t_0}$.

We will look for $\Delta_n$ under the form

$$\Delta_n:=\bigcup_{t=0}^{\varepsilon_n}T^t(C_n).$$

The choice of $t_0$ and of the numbers $\varepsilon_n$ will be given by the forthcoming lemma where $r_n$ is any fixed point in $C_n$. The projection of $\Delta_n$ on the base is $C_n$. By the Liouvillian nature of $R_{\alpha,\alpha'}$ the first returns of $\Delta_n$ under the action of $T^{t_0}$ to the fibers over $C_n$ will have their projection almost coinciding with $C_n$. The idea is to choose $t_0$ such that the first return over $C_n$ stacks exactly over $\Delta_n$.

**Lemma 6.1.** *For any $t\in\mathbf{R}$, and any $\epsilon>0$, there exists $t_0\in(t-\epsilon,t+\epsilon)$ for which there exist sequences*

◇ $u_n \underset{n\to\infty}{\longrightarrow} +\infty$,

◇ $\frac{1}{n^2} < \varepsilon_n < \frac{1}{n}$,

*such that*

$$u_n t_0 = \varphi_{q_n q'_n}(r_n) + \varepsilon_n. \tag{23}$$

*Proof.* Given $t \in \mathbf{R}$, it is easy to construct the sequence $u_n \to \infty$ such that for large $n$ the sets

$$A_n := [\frac{\varphi_{q_n q'_n}(r_n)}{u_n} + \frac{1}{n^2 u_n}, \frac{\varphi_{q_n q'_n}(r_n)}{u_n} + \frac{1}{n u_n}]$$

satisfy $A_{n+1} \subset A_n \subset (t-\varepsilon, t+\varepsilon)$. We take $t_0 = \bigcap A_n$. □

Define

$$\mathcal{D}_n := \bigcup_{t=0}^{t_0} T^t(C_n). \tag{24}$$

It clearly follows from (23) that

$$\nu\left( (\bigcup_{i=0}^{u_n-1} T^{it_0}\mathcal{D}_n) \triangle \hat{\mathcal{T}}_n \right) = o(1), \tag{25}$$

where the symbol $\triangle$ denotes the symmetrical difference.
Let $k_n := [\frac{t_0}{\varepsilon_n}]$, where $[.]$ denotes the integer part. From Lemma 6.1 we have $k_n \le n^3$, and from (21) it follows that the first $k_n u_n$ iterates of $\Delta_n$ by $T^{t_0}$ are almost “cubes”, “translated” from $\Delta_n$. They are therefore increasingly monochromatic with respect to any fixed partition of $M_\varphi$ as $n$ goes to infinity as required in b). Furthermore, for the same reasons as in (21), we have for any $k \le k_n$,

$$\begin{aligned} k u_n t_0 = k\varphi_{q_n q'_n}(r_n) + k\varepsilon_n &= \varphi_{kq_n q'_n}(r_n) + k\varepsilon_n + o(\varepsilon_n) \\ &= \varphi_{kq_n q'_n}(x,y) + k\varepsilon_n + o(\varepsilon_n), \end{aligned}$$

for any $(x,y) \in C_n$. From this we obtain that

$$\begin{aligned} T^{ku_n t_0}(x,y,s) &= (R_{\alpha,\alpha'}^{kq_n q'_n}(x,y), s + ku_n t_0 - \varphi_{kq_n q'_n}(x,y)) \\ &= (R_{\alpha,\alpha'}^{kq_n q'_n}(x,y), s + k\varepsilon_n + o(\varepsilon_n)) \\ &\sim (x,y,s+k\varepsilon_n). \end{aligned}$$

It follows that $\Delta_n$, $T^{u_n t_0}(\Delta_n)$,...,$T^{(k_n-1)u_n t_0}(\Delta_n)$ are essentially disjoint and stack one above each other to fill $\mathcal{D}_n$. With (25) this implies that as $n$ runs through the integers, the towers

$$\mathcal{T}_n := \bigcup_{j=0}^{(k_n-1)u_n} T^{jt_0}(\Delta_n)$$

form a rank $(1-\tau)$ sequence of towers for $T^{t_0}$. □

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