

# RIGIDITY RESULTS FOR QUASIPERIODIC $SL(2, \mathbb{R})$ COCYCLES

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ABSTRACT. In this paper we introduce a new technique that allows to investigate reducibility properties of smooth  $SL(2, \mathbb{R})$ -cocycles above irrational rotations of the circle beyond the usual Diophantine conditions on these rotations.

For any given irrational angle on the base, we show that if the cocycle has bounded fibered products and if its fibered rotation number belongs to a set of full measure  $\Sigma(\alpha)$ , then the matrix map can be perturbed in the  $C^\infty$  topology yielding a  $C^\infty$ -reducible cocycle. Moreover the cocycle itself is *almost rotations-reducible* in the sense that it can be conjugated arbitrarily close to a cocycle of rotations. In case the rotation on the circle is of super-Liouville type, the same result holds if we only assume that the cocycle is  $L^2$ -conjugated to a cocycle of rotations.

If the base rotation is Diophantine, we show that if the cocycle is  $L^2$ -conjugated to a cocycle of rotations and if its fibered rotation number belongs to a set of full measure, then it is  $C^\infty$ -reducible. This extends a result proven in [4].

As an application, we obtain that given any smooth  $SL(2, \mathbb{R})$  cocycle above irrational rotations of the circle, it is possible to perturb the matrix map in the  $C^\infty$  topology so that the upper Lyapunov exponent becomes strictly positive. The latter result is generalized, based on different techniques, by Avila in [1] to quasiperiodic  $SL(2, \mathbb{R})$ -cocycles over higher dimensional tori.

Also, in the course of the paper we are lead to give a quantitative version of a theorem by L. H. Eliasson, a proof of which is given in the Appendix. This motivated the introduction of a quite general K.A.M. scheme allowing to treat bigger losses of derivatives for which we prove convergence.

## 1. INTRODUCTION

We will be interested in smooth quasi-periodic cocycles on  $\mathbb{T} \times SL(2, \mathbb{R})$  where  $\mathbb{T}$  denotes the circle  $\mathbb{R}/\mathbb{Z}$  (the base) and  $SL(2, \mathbb{R})$  the set of 2 by 2 real matrices with determinant 1 (the fiber). For  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and  $A \in C^r(\mathbb{T}, SL(2, \mathbb{R}))$ ,  $r \in \mathbb{N} \cup \{\infty, \omega\}$  we define the cocycle  $(\alpha, A)$  as a diffeomorphism on the product  $\mathbb{T} \times SL(2, \mathbb{R})$

$$\begin{aligned} (\alpha, A) : \mathbb{T} \times SL(2, \mathbb{R}) &\rightarrow \mathbb{T} \times SL(2, \mathbb{R}), \\ (\theta, y) &\mapsto (\theta + \alpha, A(\theta)y). \end{aligned}$$

We denote by  $C_0^r(\mathbb{T}, SL(2, \mathbb{R}))$  the set of maps  $A \in C^r(\mathbb{T}, SL(2, \mathbb{R}))$  that are homotopic identity.

We say that a  $C^r$ -cocycle  $(\alpha, A)$  is  $C^r$ -*reducible* if there exists  $B \in C^r(\mathbb{R}/2\mathbb{Z}, \mathbb{P}SL(2, \mathbb{R}))$  and  $A_* \in SL(2, \mathbb{R})$  such that

$$B(\theta + \alpha)^{-1}A(\theta)B(\theta) = A_*, \quad \forall \theta \in \mathbb{R}/2\mathbb{Z}.$$

Reducibility is an important issue in the study of quasi-periodic cocycles and its applications (to the spectral theory of Schrödinger operators for example). An obvious obstruction to reducibility is a non-uniformly hyperbolic behavior of the cocycle, that is a non-uniform exponential increase almost everywhere of the fibered products of matrices above the base rotation. On the other hand, when certain parametrized families of cocycles are considered, Kotani's theory asserts essentially that there is an almost sure dichotomy between non-uniform hyperbolicity and  $L^2$  rotations-reducibility (conjugacy to a cocycle with values in  $SO(2, \mathbb{R})$ , see exact definition below). More precisely, to any cocycle  $(\alpha, A)$  one can associate a *fibered Lyapunov exponent*  $LE(\alpha, A)$  (we refer to section 2.1 for the definitions and basic results). Now let  $Q_\varphi(\cdot) = R_\varphi A(\cdot)$  where  $R_\varphi \in SO(2, \mathbb{R})$  is the rotation matrix  $\begin{pmatrix} \cos 2\pi\varphi & -\sin 2\pi\varphi \\ \sin 2\pi\varphi & \cos 2\pi\varphi \end{pmatrix}$ . Then, for Lebesgue almost every  $\varphi \in [0, 1]$  either  $LE(\alpha, A) > 0$  or  $(\alpha, R_\varphi A)$  is  $L^2$ -rotation reducible (see [4] for references). This dichotomy also holds for Lebesgue almost every value of  $E \in \mathbb{R}$  in the setting of Schrödinger cocycles

$$S_{V,E} = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

( $V \in C^r(\mathbb{T}, \mathbb{R})$ ), which, combined with rigidity results that we now describe gives information on the spectrum of the corresponding Schrödinger operator.

In [4] the latter dichotomy was pushed further to a global analytic rigidity result (also true in the smooth category). By analytic (resp. smooth) rigidity we mean the fact by which weak regularity information ( $L^2$ ) on the conjugating map is enough to guarantee its analyticity (resp. smoothness) under some additional assumptions. An important role is played by the fibered rotation number of the cocycle (see section 2.2 for its definition).

**THEOREM A.** [4] *If  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  is recurrent-Diophantine and if  $A \in C_0^\omega(\mathbb{T}, SL(2, \mathbb{R}))$  satisfies : (i)  $\rho_f(\alpha, A)$  is Diophantine with respect to  $\alpha$  and (ii)  $(\alpha, A)$  is  $L^2$ -conjugated to a cocycle of rotations, then  $(\alpha, A)$  is  $C^\omega$ -reducible. As a consequence, for Lebesgue a.e.  $\varphi \in [0, 1]$  (resp. if  $V \in C^\omega(\mathbb{T})$ , for Lebesgue a.e.  $E \in \mathbb{R}$ ) either  $(\alpha, R_\varphi A)$  (resp.  $(\alpha, S_{V,E})$ ) is  $C^\omega$ -reducible or it is non-uniformly hyperbolic.*

The set of Recurrent-Diophantine numbers is by definition the full Lebesgue-measure set of irrationals with the property that the iteration of  $\alpha$  by the Gauss map  $G : (0, 1) \rightarrow (0, 1)$ ,  $G(x) = \{x^{-1}\}$ , falls infinitely many times in some Diophantine set with fixed constant and exponent. Also, for any fixed  $\alpha$ , numbers which are Diophantine w.r.t.  $\alpha$  form a set of full Lebesgue measure. See section 2.3 for precise definitions. One aim of this note is to extend in a smooth setting the result of [4] to all Diophantine frequencies as follows.

*Almost-sure dichotomy for every Diophantine  $\alpha$ .*

**THEOREM 1.** *If  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  is Diophantine, there exists a set  $\Sigma(\alpha) \subset \mathbb{T}$  of measure 1 such that if  $A \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  satisfies : (i)  $\rho_f(\alpha, A)$  is in  $\Sigma(\alpha)$  and (ii)  $(\alpha, A)$  is  $L^2$ -conjugated to a cocycle of rotations, then  $(\alpha, A)$  is  $C^\infty$ -reducible.*

In the analytic case this theorem is also true, by different techniques, as a consequence of [3], [2], [5] where, in addition, the Diophantine condition on  $\alpha$  can be relaxed to be  $\lim_{n \rightarrow \infty} \frac{\log q_{n+1}}{q_n} = 0$  where  $q_n$  denotes the denominator of the  $n$ -th continuous fractions expansion of  $\alpha$  (see section 2.3).

As in [4], Theorem 1 yields a global dichotomy that generalizes to all Diophantine frequencies Theorem A above (in the  $C^\infty$  category).

**COROLLARY 1.** *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  be Diophantine and let  $A \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  (resp.  $V \in C^\infty(\mathbb{T}, \mathbb{R})$ ). Then, for Lebesgue almost every  $\varphi \in [0, 1]$  (resp.  $E \in \mathbb{R}$ ), the cocycle  $(\alpha, R_\varphi A)$  (resp.  $(\alpha, S_{V,E})$ ) is either non-uniformly hyperbolic or  $C^\infty$ -reducible.*

### Almost rigidity

It is clear that the notion of reducibility for cocycles defined above Liouvillean frequencies is too restrictive since in that case even an  $\mathbb{S}^1$  or  $\mathbb{R}$ -valued cocycle is in general not reducible. The good notion in this case is rotations-reducibility. We say that  $(\alpha, A)$  is  $L^2$ -conjugated, (resp.  $C^r$ -conjugated), to a *cocycle of rotations* (for short we shall say  $L^2$  (resp.  $C^r$ ) rotation-reducible) if there exists a measurable  $B : \mathbb{T} \rightarrow SL(2, \mathbb{R})$  such that  $\|B(\cdot)\| \in L^2(\mathbb{T}, \mathbb{R})$ , (resp.  $B \in C^r(\mathbb{T}, SL(2, \mathbb{R}))$ ), and

$$B(\theta + \alpha)^{-1} A(\theta) B(\theta) \in SO(2, \mathbb{R}), \quad \forall \theta \in \mathbb{T}.$$

The extension of Theorem A or Corollary 1 to any irrational number provided reducibility is replaced by rotations-reducibility is indeed an interesting and important problem.

Here, we introduce and study the following notions of almost-reducibility and almost-rotations reducibility: a cocycle  $(\alpha, A)$  is *almost reducible* (resp. *almost rotations-reducible*) if there exist sequences  $B_{(n)} \in C^r(\mathbb{R}/2\mathbb{Z}, SL(2, \mathbb{R}))$  and  $A_{(n)} \in SL(2, \mathbb{R})$  (resp.  $A_{(n)} \in C^r(\mathbb{T}, SO(2, \mathbb{R}))$ ) such that  $B_{(n)}^{-1}(\theta + \alpha) A(\theta) B_{(n)}(\theta) - A_{(n)}$  goes to 0 in the  $C^r$ -topology as  $n$  goes to  $\infty$ .

We prove the following "almost rigidity" results

**THEOREM 2.** *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ . There exists a set  $\Sigma(\alpha) \subset \mathbb{T}$  of measure 1 such that if  $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  satisfies:*

- (i)  $\rho_f(\alpha, A) \in \Sigma(\alpha)$
- (ii)  $(\alpha, A)$  is  $C^0$ -rotations-reducible;

*then  $A$  is  $C^\infty$  almost rotations-reducible and is  $C^\infty$ -accumulated by functions  $\tilde{A}(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  such that  $(\alpha, \tilde{A})$  is  $C^\infty$ -reducible.*

**REMARK 1.** Note that from [14], it follows that (ii) is equivalent to the fact that the fibered products of  $A(\cdot)$  be  $C^0$ -bounded: if we use the notation  $(\alpha, A(\cdot))^n = (n\alpha, A_n(\cdot))$  ( $n \in \mathbb{Z}$ ), this means  $\sup_{n \in \mathbb{Z}} \|A_n(\cdot)\|_{C^0(\mathbb{T})} < \infty$ .

This theorem can be strengthened in two cases, the first one is when the frequency in the base is Diophantine, in which case as stated in Theorem 1 (ii) can be replaced by  $L^2$ -rotations-reducible and the conclusion is reducibility. The other case where we will see that Theorem 2 can be strengthened is when  $\alpha$  is very well approximated by rational numbers: We shall say that  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  is *super Liouville* if

$$\limsup_{n \rightarrow \infty} \frac{\log \log q_{n+1}}{\log q_n} = \infty$$

where  $q_n$  denotes the denominator of the  $n$ -th continuous fractions expansion of  $\alpha$  (see section 2.3).

**THEOREM 3.** *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  be super Liouville. There exists a set  $\Sigma(\alpha) \subset \mathbb{T}$  of measure 1 such that if  $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  satisfies :*

- (i)  $\rho_f(\alpha, A) \in \Sigma(\alpha)$
- (ii)  $(\alpha, A)$  is  $L^2$  rotations-reducible.

*then  $A$  is  $C^\infty$  almost rotations-reducible and is  $C^\infty$ -accumulated by functions  $\tilde{A}(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  such that  $(\alpha, \tilde{A})$  is  $C^\infty$ -reducible.*

#### *Density of cocycles with positive Lyapunov exponent*

As a corollary of Theorems 1, 2 we have the following result:

**THEOREM 4.** *For any fixed irrational  $\alpha \in \mathbb{T}$  the set of  $A \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  such that  $(\alpha, A)$  has positive Lyapunov exponent is dense for the  $C^\infty$ -topology in  $C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$*

*Proof.* The proof relies on Kotani's theory and follows the lines of [13].

Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and  $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$ . Then, by a theorem of De Concini and Johnson [6], either for any  $\delta > 0$ , there exists  $\epsilon_0 \in (-\delta, \delta)$  for which the Lyapunov exponent of  $(\alpha, A(\cdot)R_{\epsilon_0})$  is positive, or  $(\alpha, A(\cdot)R_\epsilon)$  has  $C^0$ -bounded fibered products for any  $\epsilon \in (-\delta, \delta)$ , in which case we also have that the continuous function  $\epsilon \in (-\delta, \delta) \rightarrow \rho_f(\alpha, A(\cdot)R_\epsilon)$  is not constant.

As mentioned earlier, we know from [14] that  $C^0$ -boundedness of the fibered products is equivalent to the fact that the cocycle is  $C^0$ -conjugated to rotations. Since the fibered rotation number of this cocycle is continuous and not constant as  $\epsilon$  varies in  $(-\delta, \delta)$ , we can choose  $\epsilon_0$  such that, depending on  $\alpha$ , hypothesis (i) of Theorem 2 or of Theorem 1 be satisfied. We conclude that the cocycle  $(\alpha, A(\cdot)R_{\epsilon_0})$  is accumulated by reducible cocycles.

Theorem 4 then follows if we make the following observation (*cf.* [13])

**LEMMA 1.** *Any constant  $A_0$  in  $SL(2, \mathbb{R})$  is  $C^\infty$ -accumulated by functions  $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  such that  $(\alpha, A(\cdot))$  is hyperbolic (in the fiber).*

□

**REMARK 2.** In fact, as proven in [1], a stronger result holds: Theorem 4 is true for  $SL(2, \mathbb{R})$ -cocycles above translation on tori of any dimension (and even more general dynamics).

Let us describe briefly the novelty of this paper. There are usually two techniques to attack the reducibility problem for quasiperiodic systems: K.A.M. theory and Renormalization. In the case of  $SL(2, \mathbb{R})$  cocycles a third approach to tackle reducibility, based on localization and Aubry duality, proved to be fruitful in the works of [] and [].

Renormalization is usually used to reduce to a local (perturbative) situation where K.A.M. techniques are applicable. This naturally requires Diophantine conditions on both the rotation number on the base and on the fibered rotation number of some renormalized system. The crucial observation in the present paper is that if the Diophantine condition on the base fails, it is indeed possible to take advantage of this fact and conjugate the cocycle closer to rotations (instead of rotation!) with an error that will be even smaller as the involved small divisor is small (*cf.* Theorem 2 and 3). This remark is also useful to treat the Diophantine case in full

generality (*cf.* Theorem 1). We refer to sections 3.2 and 4.1 for a more detailed discussion concerning each case.

*A quantitative version of Eliasson theorem and a generalized KAM scheme* \*\*\*

Given an analytic cocycle  $(\alpha, A)$ ,  $\alpha$  Diophantine and  $A$  sufficiently close to a constant (as a function of  $\alpha$ ), Eliasson proved in [8] a theorem that guarantees reducibility of  $(\alpha, A)$ , provided the fibered rotation number satisfies some (weak) Diophantine condition.

We will prove a precise version of Theorem 1 (see Theorem 8 below) that uses an extension of Eliasson's reducibility theorem to the smooth case and also provides estimates, involving the Diophantine constants, on the required closeness to constants. This is our Theorem 5.

As pointed to us by the referee (and as will be explained in §4.1) a usual KAM scheme with estimates would be sufficient for the purpose of proving Theorem 1. However, proving the quantitative version of Eliasson's theorem presents its own interest and allows for a slightly more general result on quasi-reducibility (*cf.* §4.1).

Thus, we present a proof of Theorem 5 in the Appendix together with a quite general KAM scheme that we think may be of broader utility especially in problems where small divisors cause losses of derivatives that are proportional to the order of differentiability.

## 2. DEFINITIONS AND PRELIMINARIES.

**2.1. The fibered Lyapunov exponent.** Given a cocycle  $(\alpha, A)$ , for  $n \in \mathbb{Z}$ , we denote the iterates of  $(\alpha, A)$  by  $(\alpha, A)^n = (n\alpha, A_n(\cdot))$  where for  $n \geq 1$

$$\begin{cases} A_n(\cdot) = A(\cdot + (n-1)\alpha) \cdots A(\cdot) \\ A_{-n}(\cdot) = A(\cdot - n\alpha)^{-1} \cdots A(\cdot - \alpha)^{-1} \end{cases}$$

We call fibered products of  $(\alpha, A)$  the matrices  $A_n(\cdot)$  for  $n \in \mathbb{N}$ .

The fibered Lyapunov exponent is defined as the limit

$$L(\alpha, A) := \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\theta \in \mathbb{T}} \log \|A_n(\theta)\| d\theta,$$

which by the subadditive theorem always exists (similarly the limit when  $n$  goes to  $-\infty$  exists and is equal to  $L(\alpha, A)$ ).

**2.2. The fibered rotation number.** Assume that  $A(\cdot) : \mathbb{T} \rightarrow SL(2, \mathbb{R})$  is continuous and homotopic to the identity; then the same is true for the map

$$F : \mathbb{T} \times \mathbb{S}^1 \rightarrow \mathbb{T} \times \mathbb{S}^1$$

$$(\theta, v) \mapsto \left( \theta + \alpha, \frac{A(\theta)v}{\|A(\theta)v\|} \right),$$

therefore  $F$  admits a continuous lift  $\tilde{F} : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$  of the form  $\tilde{F}(\theta, x) = (\theta + \alpha, x + f(\theta, x))$  such that  $f(\theta, x+1) = f(\theta, x)$  and  $\pi(x + f(\theta, x)) = A(\theta)\pi(x)/\|A(\theta)\pi(x)\|$ , where  $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ ,  $\pi(x) = e^{i2\pi x} := (\cos(2\pi x), \sin(2\pi x))$ . In order to simplify the terminology we shall say that  $\tilde{F}$  is a lift for  $(\alpha, A)$ . The map  $f$  is independent of

the choice of the lift up to the addition of a constant integer  $p \in \mathbb{Z}$ . Following [9] and [10] we define the limit

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tilde{F}^k(\theta, x)),$$

that is independent of  $(\theta, x)$  and where the convergence is uniform in  $(\theta, x)$ . The class of this number in  $\mathbb{R}/\mathbb{Z}$ , which is independent of the chosen lift, is called the *fibered rotation number* of  $(\alpha, A)$  and denoted by  $\rho_f(\alpha, A)$ . Moreover  $\rho_f(\alpha, A)$  is continuous as a function of  $A$  (with respect to the uniform topology on  $C_0^0(\mathbb{T}, SL(2, \mathbb{R}))$ ).

**2.3. Continued fraction expansion. Diophantine conditions.** Define as usual for  $0 < \alpha < 1$ ,

$$a_0 = 0, \quad \alpha_0 = \alpha,$$

and inductively for  $k \geq 1$ ,

$$a_k = [\alpha_{k-1}^{-1}], \quad \alpha_k = \alpha_{k-1}^{-1} - a_k = G(\alpha_{k-1}) = \left\{ \frac{1}{\alpha_{k-1}} \right\},$$

where  $[\ ]$  denotes the integer part and  $G(\cdot)$  the fractional part (the gauss map). We also set,

$$\beta_k = \prod_{j=0}^k \alpha_j.$$

For all  $k \geq 0$ ,

$$\beta_k = (-1)^k (q_k \alpha - p_k), \tag{2.1}$$

$$\frac{1}{q_{k+1} + q_k} < \beta_k < \frac{1}{q_{k+1}}, \tag{2.2}$$

$$\beta_k = \frac{1}{q_{k+1} + \alpha_{k+1} q_k}. \tag{2.3}$$

We use the notation

$$\|x\| = \inf_{p \in \mathbb{Z}} |x - p|.$$

Recall that

$$\forall 1 \leq k < q_n, \quad \|k\alpha\| \geq \|q_{n-1}\alpha\|. \tag{2.4}$$

We say that  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  satisfies a *Diophantine condition*  $DC(\gamma, \tau)$ ,  $\gamma > 0$ ,  $\tau > 0$  if for every  $p \in \mathbb{Z}, q \in \mathbb{N}^*$

$$|q\alpha - p| \geq \frac{\gamma^{-1}}{q^{1+\tau}}.$$

Let  $\alpha \in \mathbb{R}$  and  $\theta > 0$ . We say that  $\rho \in \mathbb{R}/\mathbb{Z}$  is  *$\theta$ -Diophantine with respect to  $\alpha$*  if there exist  $C$  such that

$$|2\rho - k\alpha - l| > C(1 + |k| + |l|)^{-\theta}, \quad (k, l) \in \mathbb{Z}^2.$$

For short we shall say that  $\rho$  is *Diophantine w.r.t.  $\alpha$*  (no mention to  $\theta$  is made) when it is  $\theta$ -Diophantine w.r.t.  $\alpha$  with  $\theta = 2$ . Note that the set of Diophantine numbers with given exponent  $\tau > 0$  :  $\cup_{\gamma > 0} DC(\gamma, \tau)$  is a set of full Lebesgue measure. Note also that given any  $\alpha \in \mathbb{R}$ , the set of  $\rho \in \mathbb{T}$  that are Diophantine with respect to  $\alpha$  is a set of full Haar measure on the circle.

**2.4. Eliasson's local theorem on reducibility.** We will need the following *quantitative* extension of a result by Eliasson [8] (where the case of analytic continuous quasi-periodic Schrödinger cocycles is considered).

**THEOREM 5.** *There exist two constants  $C_1, C_2 > 0$  such that the following holds. Let  $\hat{A} \in SL(2, \mathbb{R})$  and  $\tau > 0$ , be fixed. Then there exists  $\epsilon(\tau, \hat{A}) > 0$  such that if a cocycle  $(\alpha, A) \in \mathbb{R} \times C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  satisfies*

- (1)  $\alpha \in DC(\gamma, \tau)$ ;
- (2)  $\rho_f(\alpha, A)$  is Diophantine with respect to  $\alpha$ ;
- (3)  $\|A - \hat{A}\|_{C^0} \leq \gamma^{-d_0} \epsilon$  and  $\|A - \hat{A}\|_{C^{s_0}} \leq 1$ , where  $d_0 = C_1(\tau + 1)$ ,  $s_0 = [C_2(\tau + 1)]$

then  $(\alpha, A)$  is  $C^\infty$  reducible.

**REMARK 3.** Assume that only  $\tau$  is fixed and  $\hat{A}$  varies in some compact subset  $K$  of  $SL(2, \mathbb{R})$ . Then  $\epsilon$  can be taken uniform with respect to  $\hat{A}$ .

A sketch of proof of Theorem 5, based on Eliasson's original proof, is given in Appendix B.

**2.5.  $\mathbb{Z}^2$ -actions** [13],[4]. Let  $\Omega^r = \mathbb{R} \times C^r(\mathbb{R}, SL(2, \mathbb{R}))$  be the subgroup of  $\text{Diff}(\mathbb{R} \times \mathbb{R}^2)$  made of skew-product diffeomorphisms  $(\alpha, A) : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R} \times \mathbb{R}^2$

$$(\alpha, A)(x, w) = (x + \alpha, A(x)w).$$

A  $C^r$  fibered  $\mathbb{Z}^2$ -action is a homomorphism  $\Phi : \mathbb{Z}^2 \rightarrow \Omega^r$ . We denote  $\Lambda^r$  the space of such actions. Since  $\Phi(1, 0)$  and  $\Phi(0, 1)$  determine  $\Phi$  we often write for short  $\Phi = ((\Phi(1, 0), \Phi(0, 1)))$ . Let  $\pi_1 : \mathbb{R} \times C^r(\mathbb{R}, SL(2, \mathbb{R})) \rightarrow \mathbb{R}$ ,  $\pi_2 : \mathbb{R} \times C^r(\mathbb{R}, SL(2, \mathbb{R})) \rightarrow C^r(\mathbb{R}, SL(2, \mathbb{R}))$  be the coordinate projections. Let also  $\gamma_{n,m}^\Phi = \pi_1 \circ \Phi(n, m)$  and  $A_{n,m}^\Phi = \pi_2 \circ \Phi(n, m)$ .

We let  $\Lambda_0^r$  be the set of  $\Phi \in \Lambda^r$  such that  $\gamma_{1,0}^\Phi = 1$  and  $\gamma_{0,1}^\Phi \in [0, 1]$ .

We say that an action is

- *constant* if for all  $(n, m) \in \mathbb{Z}^2$ , the maps  $A_{n,m}^\Phi$  are constants;
- *normalized* if  $\Phi(1, 0) = (1, Id)$  and in that case if  $\Phi(0, 1) = (\alpha, A)$  the map  $A \in C^r(\mathbb{R}, SL(2, \mathbb{R}))$  is clearly  $\mathbb{Z}$ -periodic.

If  $\Phi \in \Lambda^\infty$ ,  $r \in \mathbb{N}$  and  $I \subset \mathbb{R}$  is an interval or  $I = \mathbb{T}$  we denote by  $\|\Phi\|_{r,I}$  the quantities

$$\|\Phi\|_{r,I} = \max(\|\partial^r A_{1,0}^\Phi\|_{C^0(I)}, \|\partial^r A_{0,1}^\Phi\|_{C^0(I)}), \quad \|\Phi\|_{r,I}^{\max} = \max_{0 \leq s \leq r} \|\Phi\|_{s,I}.$$

We define

$$d_{r,I}(\Phi_1, \Phi_2) = \max(\|\partial^r (A_{1,0}^{\Phi_1} - A_{1,0}^{\Phi_2})\|_{C^0(I)}, \|\partial^r (A_{0,1}^{\Phi_1} - A_{0,1}^{\Phi_2})\|_{C^0(I)}).$$

and a distance  $d_{r,I}$  on  $\Lambda^r$  : if  $\Phi_1, \Phi_2 \in \Lambda^r$  we set

$$d_{r,I}^{\max}(\Phi_1, \Phi_2) = \max_{0 \leq s \leq r} d_{s,I}(\Phi_1, \Phi_2).$$

When  $I$  is the interval  $[0, T]$  ( $T \in \mathbb{R}$ ) we denote  $\|\cdot\|_{r,I}$  and  $d_{r,I}(\cdot, \cdot)$  by  $\|\cdot\|_{r,T}$  and  $d_{r,T}(\cdot, \cdot)$ .

**DEFINITION 1.** Two fibered  $\mathbb{Z}^2$  actions  $\Phi, \Phi'$  are said to be *conjugated* if there exists a smooth map  $B : \mathbb{R} \rightarrow SL(2, \mathbb{R})$  such that

$$\forall (n, m) \in \mathbb{Z}^2 \quad \Phi'(n, m) = (0, B) \circ \Phi(n, m) \circ (0, B)^{-1}$$

that is, if

$$\begin{cases} A_{n,m}^{\Phi'}(t) = B(t + \gamma_{n,m}^{\Phi}) A_{n,m}^{\Phi}(t) B(t)^{-1} \\ \gamma_{n,m}^{\Phi'} = \gamma_{n,m}^{\Phi}. \end{cases}$$

We write  $\Phi' := \text{Conj}_B(\Phi)$  and get from the Hadamard-Kolmogorov convexity estimates (see Appendix A equation 5.4) the following estimates :

$$\|\Phi'\|_{r,I} \leq K_r(I)(1 + \|B\|_{0,I})^3 (\|\Phi\|_{r,I}^{\max} \|B\|_{0,I} + \|\Phi\|_{0,I} \|B\|_{r,I}) \quad (2.5)$$

and similarly if  $\Phi'_j = \text{Conj}_B(\Phi_j)$ ,  $j = 1, 2$

$$d_{r,I}(\Phi'_1, \Phi'_2) \leq K_r(I)(1 + \|B\|_{0,I})^3 (d_{r,I}^{\max}(\Phi_1, \Phi_2) \|B\|_{0,I} + d_{0,I}(\Phi_1, \Phi_2) \|B\|_{r,I}) \quad (2.6)$$

where the constant  $K_r(I)$  is for any  $r$  a decreasing function of the length of  $I$  (and can be chosen equal to  $C^r$ , where  $C > 0$ , when  $I = \mathbb{T}$ ). We shall sometime denote  $K_r(I)$  by  $K_{r,I}$ ; when  $I = [0, T]$  we write  $K_{r,T}$  in place of  $K_{r,[0,T]}$  and when  $I = \mathbb{T}$  we simply write  $K_r$ .

A fibered action is said to be *reducible* if it is conjugated to a constant action.

A fibered  $\mathbb{Z}^2$ -action can always been conjugated to a normalized one :

LEMMA 2. *If  $\Phi \in \Lambda^r$  with  $\gamma_{1,0}^{\Phi} = 1$  then there exist a conjugation  $B \in C^r(\mathbb{R}, SL(2, \mathbb{R}))$  and a normalized action  $\tilde{\Phi}$  such that  $\tilde{\Phi} = \text{Conj}_B(\Phi)$ ; one then has the following estimate: Let  $s, T \in \mathbb{N}$  with  $0 \leq s \leq r$ .*

$$\|B\|_{C^s([0,T])} \leq K_s T^s (\|\Phi(1,0)\|_{0,T})^T \|\Phi(1,0)\|_{s,T}^{\max}.$$

*If, furthermore  $P := \max_{T \in \mathbb{Z}} \|\Phi(T,0)\|_{C^r(\mathbb{R})} < \infty$  then there exists  $K_s$  such that*

$$\|B\|_{C^s([0,T])} \leq K_s P^{4s} T^s \|\Phi(1,0)\|_{0,1}^2 \|\Phi(1,0)\|_{s,T}^{\max}.$$

*The same results are true if  $\Phi \in \Lambda^r(SO(2, \mathbb{R}))$ .*

We give the proof of Lemma 2 in Appendix A.

Finally, we observe that a normalized action  $\Phi = ((1, \text{Id}), (\alpha, A))$  is reducible if and only if the cocycle  $(\alpha, A)$  is reducible (cf. [13],[4]).

**2.6. Renormalization of actions.** A fundamental tool in this note will be the results on convergence of renormalizations of bounded cocycles obtained in [13, 4, 5]. We recall here, following [4], the scheme of renormalization of  $\mathbb{Z}^2$  actions introduced in [13].

Let  $\lambda \neq 0$ . Define  $M_\lambda : \Lambda^r \rightarrow \Lambda^r$  by

$$M_\lambda(\Phi)(n, m) = (\lambda^{-1} \gamma_{n,m}^{\Phi}, A_{n,m}^{\Phi}(\lambda \cdot)).$$

Let  $\theta_* \in \mathbb{R}$ . Define  $T_{\theta_*} : \Lambda^r \rightarrow \Lambda^r$  by

$$T_{\theta_*}(\Phi)(n, m) = (\gamma_{n,m}^{\Phi}, A_{n,m}^{\Phi}(\cdot + \theta_*)).$$

Let  $U \in GL(2, \mathbb{Z})$ . Define  $N_U : \Lambda^r \rightarrow \Lambda^r$  by

$$N_U(\Phi)(n, m) = \Phi(n', m'), \quad \begin{pmatrix} n' \\ m' \end{pmatrix} = U^{-1} \begin{pmatrix} n \\ m \end{pmatrix}.$$

Notice that these three operations commute. Also,  $T_{\theta_*}$  and  $N_U$  commute with  $\text{Conj}_B$  while  $M_\lambda \circ \text{Conj}_B = \text{Conj}_{B(\lambda \cdot)} \circ M_\lambda$ . Also, if  $\Phi_1, \Phi_2 \in \Lambda^\infty$

$$\|M_\lambda \Phi_1\|_{r, \lambda^{-1}T} = \lambda^r \|\Phi_1\|_{r,T}, \quad d_{r, \lambda^{-1}T}(M_\lambda \Phi_1, M_\lambda \Phi_2) = \lambda^r d_{r,T}(\Phi_1, \Phi_2) \quad (2.7)$$

Let

$$Q_n = \begin{pmatrix} q_n & p_n \\ q_{n-1} & p_{n-1} \end{pmatrix} \quad (2.8)$$

and define for  $n \in \mathbb{N}$  and  $\theta_* \in \mathbb{R}$  the *renormalized actions*

$$\mathcal{R}^n(\Phi) = M_{\beta_{n-1}} \circ N_{Q_n}(\Phi) \quad (2.9)$$

$$\mathcal{R}_{\theta_*}^n(\Phi) = T_{\theta_*}^{-1}[\mathcal{R}^n(T_{\theta_*}(\Phi))]. \quad (2.10)$$

Renormalization is a powerful tool to study reducibility due to the following elementary fact:

LEMMA 3. [13, 4] *If  $\mathcal{R}_{\theta_*}^n(\Phi)$  is  $C^r$ -reducible for some  $n$  and  $\theta_*$  then  $\Phi$  is  $C^r$ -reducible.*

**2.7. Fibered rotation number of non-degenerate  $\mathbb{Z}^2$ -actions.** It is possible to associate to any (non-degenerate) fibered  $\mathbb{Z}^2$ -action  $\Phi$  a notion of *fibered degree*. Since our initial quasi-periodic cocycle  $(\alpha, A)$  is homotopic to the identity, the degree of the associate  $\mathbb{Z}^2$ -action  $\Phi^A$  is zero (cf. [13], [4]) and in that case, once a lift is chosen for the corresponding quasi-periodic projective cocycle, we can define a *fibered rotation number*  $\text{rot}(\Phi)$  for the  $\mathbb{Z}^2$ -action  $\Phi^A$ . A change in the choice of the lift results in adding to the fibered rotation number an element of the frequency module  $\{k\gamma_{1,0}^\Phi + l\gamma_{0,1}^\Phi : (k, l) \in \mathbb{Z}^2\}$ . We refer the reader to [13], [4] for the definition of this rotation number and its behavior under renormalization. It is proven in the aforementioned references that

$$\text{rot}(\mathcal{R}^n \Phi) = (-1)^n \text{rot}(\Phi) / \beta_{n-1}. \quad (2.11)$$

**2.8. Convergence of the renormalized actions** [13, 4, 5]. Let  $\Phi$  be the action  $((1, \text{Id}), (\alpha, A))$ . Denote by  $((1, C_n^{(1)}), (\alpha_n, C_n^{(2)})) = \mathcal{R}_{\theta_*}^n(\Phi)$ . By definition

$$C_n^{(1)} = A_{(-1)^{n-1}q_{n-1}}(\beta_{n-1}\cdot), \quad C_n^{(2)} = A_{(-1)^n q_n}(\beta_{n-1}\cdot). \quad (2.12)$$

In [5], it was proved that

THEOREM 6 ([5]). *If  $(\alpha, A)$  is of degree 0 and is  $L^2$ -conjugated to a cocycle of rotations, then for almost every  $\theta_* \in \mathbb{T}$ , there is a constant matrix  $B$  and a subsequence of  $n$  such that if  $\text{Conj}_B(\Phi)$  denotes the conjugate action of  $\Phi$  by  $B$ , then  $\mathcal{R}_{\theta_*}^n(\text{Conj}_B(\Phi)) = ((1, \tilde{C}_n^{(1)}), (\alpha_n, \tilde{C}_n^{(2)}))$  with*

$$\tilde{C}_n^{(j)} = e^{U_n^{(j)}} R_{\rho_n^{(j)}}, \quad j = 1, 2$$

*such that for any  $r \in \mathbb{N}$ ,  $\|U_n^{(j)}\|_{C^r([0,1], sl(2, \mathbb{R}))} \rightarrow 0$ , and  $\rho_n^{(j)} \in \mathbb{R}$ .*

### 3. THE WELL APPROXIMATED CASE

#### 3.1. Statement of the result.

DEFINITION 2. A number  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  is said to be of Roth type if for every  $\epsilon > 0$  there exist finitely many integers  $q$  such that  $\|q\alpha\| \leq \frac{1}{q^{1+\epsilon}}$ .

We will say that a number  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  is *well approximated* if it is not of Roth type. In this case, there exist  $\epsilon > 0$  and an infinite set  $\mathcal{N}(\epsilon) \subset \mathbb{N}$  such that for any  $n \in \mathcal{N}(\epsilon)$  we have

$$\|q_n \alpha\| \leq \frac{1}{q_n^{1+\epsilon}}.$$

Notice that if  $\alpha$  is well-approximated there exists  $M \in \mathbb{N}$  such that for any  $n \in \mathcal{N}(\epsilon)$  sufficiently large

$$\alpha_n^M \leq \frac{1}{q_n}. \quad (3.1)$$

We say that  $\alpha \in \mathbb{T} - \mathbb{Q}$  is *super-Liouville* if  $\limsup_{n \rightarrow \infty} \frac{\log \log q_{n+1}}{\log q_n} = \infty$ . Equivalently, this means that for any  $A > 0$   $\limsup_{n \rightarrow \infty} \frac{\log q_{n+1}}{q_n^A} = \infty$ . In that case we denote by  $\mathcal{L} \subset \mathbb{N}$  an infinite set for which  $\lim_{n \in \mathcal{L}, n \rightarrow \infty} \frac{\log \log q_{n+1}}{\log q_n} = \infty$ .

In this section, we prove the following theorem that encompasses both Theorem 2 and Theorem 3.

**THEOREM 7.** *If  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  is well approximated and if  $A(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  satisfies one of the two following sets of assumptions*

(Case (I))

- (H1)  $\left\| \frac{2\rho_f(\alpha, A)}{\beta_{n_i-1}} \right\| \geq \rho_0 > 0$ , for some sequence  $n_i \rightarrow \infty$ ,  $n_i \in \mathcal{N}(\epsilon)$ ;
- (H2)  $(\alpha, A)$  is  $C^0$ -conjugated to a cocycle of rotations (or equivalently has uniformly bounded products);

(Case (II))

- (SL)  $\alpha$  is super-Liouville;
- (H1)  $\left\| \frac{2\rho_f(\alpha, A)}{\beta_{n_i-1}} \right\| \geq \rho_0 > 0$ , for some sequence  $n_i \rightarrow \infty$ ,  $n_i \in \mathcal{L}$ ;
- (H2')  $(\alpha, A)$  is  $L^2$ -conjugated to a cocycle of rotations;

then  $A(\cdot)$  is  $C^\infty$ -accumulated by functions  $\tilde{A}(\cdot) \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  such that  $(\alpha, \tilde{A})$  is  $C^\infty$ -reducible. Moreover  $(\alpha, A)$  is almost rotation-reducible.

**REMARK 4.** Given any sequence of numbers  $\beta_n \rightarrow 0$ , the set of numbers  $\rho$  for which there exist  $\rho_0 > 0$  and an infinite sequence  $n_i \in \mathcal{N}(\epsilon)$  (resp.  $n_i \in \mathcal{L}$ ) such that for all  $i$  we have  $\|\rho/\beta_{n_i}\| \geq \rho_0$ , is of full Lebesgue measure.

**3.2. Plan of the proof of Theorem 7.** We concentrate on the description of *Case (I)*. The basic observation for the proof of Theorem 7 is the following: let  $(\alpha, A)$  be a cocycle (homotopic to the identity) where  $A(\cdot)$  is close to some cocycle of rotations  $R_\varphi(\cdot) \in C^\infty(\mathbb{T}, SO(2, \mathbb{R}))$  and assume that  $\varphi$  is close to  $\varphi(0)$  that satisfies  $\min_{l \in \mathbb{Z}} \|\varphi(0) - (l/2)\| \geq \delta$  where  $\delta$  is not too small. On the other hand, let us assume that  $\alpha$  is very well approximated by rational numbers and consider the even worse case where  $\alpha$  is very small. The assumptions on  $A$  allows to find a conjugation  $B \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  that "diagonalizes"  $A(\cdot)$  in the usual algebraic sense:  $B(\cdot)^{-1}A(\cdot)B(\cdot) = R_{\varphi_1(\cdot)} \in C^\infty(\mathbb{T}, SO(2, \mathbb{R}))$ , the  $C^k$ -norms of  $B$  being under control. Though this conjugation relation is only algebraic it can be put in a "dynamical" form by writing

$$B(\cdot + \alpha)^{-1}A(\cdot)B(\cdot) = \left( B(\cdot + \alpha)^{-1}B(\cdot) \right) R_{\varphi_1(\cdot)}.$$

But,  $B(\cdot + \alpha)^{-1}B(\cdot) = I + O(\alpha \|\partial B\|)$  and since we have assumed that  $\alpha$  is very small this quantity will also be very small (as we said before, we have good estimates on the norms of  $B$ ). The virtue of this remark is to reduce by conjugation the situation to a perturbative case where the size of the perturbation is now related

to the "badness" of  $\alpha$ . This is the content of Lemma 4 below. Notice that this step can be iterated a finite number of times and this will allow us to prove Proposition 2, which is the main ingredient of the paper.

We now illustrate how this argument coupled with Renormalization gives the proof of Theorem 7. Consider now the case where  $\alpha$  is a Liouville number. If we assume that  $(\alpha, A)$  is  $C^0$ -rotations-reducible, Renormalization will converge to constants. This means (after normalizing the renormalized action) that there exists a cocycle  $(\alpha_n, \tilde{A}^{(n)})$ , with  $\alpha_n = G^n(\alpha)$  ( $G$  is the Gauss map  $G(x) = \{1/x\}$ ) and with  $\tilde{A}^{(n)}$  close to a constant rotation in  $C^k$ -topology for some fixed  $k$ , from which one can retrieve many of the dynamical properties of  $(\alpha, A)$ . If we make some assumption on the fibered rotation number of  $(\alpha, A)$  we can expect that  $\tilde{A}^{(n)}(0)$  will be not too close to  $I$ . Also, if  $\alpha$  is a Liouville number, there are renormalization times  $n_k$  where  $\alpha_{n_k}$  goes to zero much faster than any power of  $q_{n_k}^{-1}$ . If we apply the basic observation we have described in the previous paragraph, we can conjugate  $(\alpha_{n_k}, \tilde{A}^{(n_k)})$  to a cocycle  $(\alpha_{n_k}, \hat{A}^{(n_k)})$  that will be all the more close to an  $SO(2, \mathbb{R})$ -valued cocycles  $(\alpha_{n_k}, R_{\varphi_{n_k}})$  with  $\alpha_{n_k}$  small. Now invert the renormalization both for  $(\alpha_{n_k}, \hat{A}^{(n_k)})$  and  $(\alpha_{n_k}, R_{\varphi_{n_k}})$ : this basically means that we iterate each of these cocycles about  $q_{n_k}$  times. But these two cocycles are much closer that  $q_{n_k}^{-1}$ : this has as a consequence that the two cocycles obtained after applying inverse Renormalization will remain close. Since conjugation and renormalization commute (up to dilation that is of the order of  $q_{n_k}$  at the  $k^{\text{th}}$  step) this will give the desired result. Some technicalities about actions complicate a little bit the argument which essentially remains the same for *Case (II)*.

The same set of ideas and Proposition 2, coupled with the quantitative version of Eliasson's reducibility theorem will also be useful in the proof of Theorem 1 (cf. Subsection 4.1).

**3.3. The renormalized actions.** Theorem 6 (about the convergence of Renormalization) and Lemma 2 (about normalization) imply the following Proposition

**PROPOSITION 1.** *Let  $\Phi$  be the action  $((1, \text{Id}), (\alpha, A))$  and assume it is  $L^2$ -reducible. Then there exist a subsequence  $n_i$  and a sequence of matrices  $B_i$  such that for any  $r \in \mathbb{N}$ ,  $T \in \mathbb{R}$*

$$\|B_i\|_{r,T} \leq u_i(r, T), \quad (3.2)$$

where

$$u_i(r, T) = K_r T^r \|A\|_0^{q_{n_i}^{-1}(T+1)} \|A\|_r \quad (3.3)$$

and the action  $\text{Conj}_{B_i}(\mathcal{R}_{\theta_*}^{n_i}(\Phi))$  is of the form  $((1, \text{Id}), (\alpha_{n_i}, G_i))$  and satisfies

$$G_i(\cdot) = e^{U_i(\cdot)} R_{\rho_i}$$

with for any  $r \in \mathbb{N}$

$$U_i \in C^r(\mathbb{T}, sl(2, \mathbb{R})), \quad \|U_i\|_{C^r(\mathbb{T})} \rightarrow 0$$

and (cf. 2.11))

$$\rho_i = (-1)^{n_i} \frac{\rho_f(\alpha, A)}{\beta_{n_i-1}}.$$

Furthermore, if the fibered products  $A_n(\cdot)$  are uniformly  $C^0$ -bounded, namely if  $\max_{n \in \mathbb{Z}} \|A_n\|_{C^0(\mathbb{T})} < \infty$ , one can take  $u_i(r, T)$  to be equal to  $u(r, T)$

$$u(r, T) = \tilde{K}_r(A) T^r \quad (3.4)$$

where  $\tilde{K}_r(A)$  depends on  $A$ .

*Proof.* From §2.8, consider a subsequence of  $n_i$  (that we still call  $n_i$  to simplify notations) such that  $\mathfrak{R}_{\theta_*}^{n_i}(\text{Conj}_B(\Phi)) = ((1, \tilde{C}_i^{(1)}), (\alpha_{n_i}, \tilde{C}_i^{(2)}))$  with

$$\tilde{C}_i^{(j)} = e^{U_i^{(j)}} R_{\rho_i^{(j)}}, \quad j = 1, 2$$

and for any  $r \in \mathbb{N}$ ,  $\varepsilon_{i,r} := \max_{j=1,2} \|U_i^{(j)}\|_{C^r([0,1])} \rightarrow 0$  as  $i$  goes to infinity, and  $\rho_i^{(j)} \in \mathbb{R}$ .

Since  $\tilde{C}_i^{(1)} = \text{Ad}_B \cdot A_{(-1)^{n_i-1}q_{n_i-1}}(\beta_{n_i-1} \cdot)$ , from Lemma 8 of the Appendix A and (2.12) we deduce that for any  $T \in \mathbb{R}$

$$\|\partial^r \tilde{C}_i^{(1)}\|_{r,T} \leq K_r \beta_{n_i-1}^r q_{n_i-1}^r \|A\|_r \|A\|_0^{q_{n_i-1}} \quad (3.5)$$

$$\leq K_r \|A\|_r \|A\|_0^{q_{n_i-1}} \quad (3.6)$$

Using Lemma 2 this provides us with a normalizing conjugation  $\tilde{B}_i$  such that  $B_i := \tilde{B}_i B$  satisfies the required bound (3.2) and the conclusion of the Theorem.

If we now assume that  $\sup_{n \in \mathbb{Z}} \|A_n\|_0 < \infty$ , then the fibered products of  $\tilde{C}_i^{(1)}$  are  $C^0$ -bounded and we have

$$\|(1, \tilde{C}_i^{(1)})\|_{C^r([0,T])} \leq \beta_{n_i-1}^r q_{n_i-1}^r K_r \|A\|_{C^r(\mathbb{T})},$$

Using Lemma 2 concludes the proof.  $\square$

**3.4. Further reduction.** We now come to the crucial basic observation that allows us to conjugate the renormalized system which is close to a constant system to a cocycle which is a perturbation of a cocycle of rotations, the size of the perturbation being now explicit in terms of the new rotation number on the base  $(\alpha_{n_i})$ .

We use the notations of Proposition 1.

**PROPOSITION 2.** *There exists a bounded (in  $C^r(\mathbb{T}, SL(2, \mathbb{R}))$ -norms, for each  $r \in \mathbb{N}$ ) sequence of conjugacies  $D_i \in C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  such that the sequence  $\tilde{\Phi}_i^{(1)} = \text{Conj}_{D_i}((1, \text{Id}), (\alpha_{n_i}, G_i)) = ((1, \text{Id}), (\alpha_{n_i}, \tilde{G}_i))$  satisfies*

$$\tilde{G}_i(\cdot) = e^{\tilde{U}_i(\cdot)} R_{\tilde{\rho}_i(\cdot)} \quad (3.7)$$

with for any  $l, l' \in \mathbb{N}$

$$\lim_{i \rightarrow \infty} \frac{\|\tilde{U}_i\|_{C^l}}{\alpha_{n_i}^{l'}} = 0 \quad (3.8)$$

$$\lim_{i \rightarrow \infty} \|\tilde{\rho}_i(\cdot) - \rho_i\|_{C^{l'}} = 0. \quad (3.9)$$

Moreover, there exists constants  $\mu_s(\rho_0)$  depending only on  $s$  and  $\rho_0$  such that

$$\|D_i\|_{C^s(\mathbb{T})} \leq \mu_s(\rho_0).$$

Combining this Proposition with Proposition 1 we immediately get the following Corollary

**COROLLARY 2.** *If we define  $Z_i = D_i B_i$  we have  $\tilde{\Phi}_i = \text{Conj}_{Z_i}(\mathfrak{R}_{\theta_*}^{n_i}(\Phi)) = ((1, \text{Id}), (\alpha_{n_i}, \tilde{G}_i))$  and  $Z_i$  satisfies the following estimate for  $n_i \in \mathcal{N}(\varepsilon)$  (Case (I)) or  $n_i \in \mathcal{L}$  (Case (II))*

$$\|Z_i\|_{C^r([0,T])} \leq \mu_r(\rho_0) u_i(r, T). \quad (3.10)$$

Proposition 2 follows inductively from the following

LEMMA 4. *Let  $\rho_0 \in (0, 1/2)$  and  $M > 0$ . There exists  $\varepsilon_0 > 0$  and for any  $s \in \mathbb{N}$  constants  $C_s > 0$  such that for any  $U \in C^r(\mathbb{T}, sl(2, \mathbb{R}))$ ,  $\rho \in C^r(\mathbb{T}, \mathbb{R})$  satisfying  $\|U\|_{C^0(\mathbb{T})} \leq \varepsilon$ ,  $\inf_{l \in \mathbb{Z}, x \in \mathbb{T}} |2\rho(x) - l| \geq 2\rho_0$  and  $\|\rho\|_{C^s} \leq M$  there exists  $Y, \tilde{U} \in C^r(\mathbb{T}, sl(2, \mathbb{R}))$  and  $\tilde{\rho} \in C^r(\mathbb{T}, \mathbb{R})$  such that*

$$(0, e^{Y(\cdot)}) \circ (\alpha, e^{U(\cdot)} R_{\rho(\cdot)}) \circ (0, e^{Y(\cdot)})^{-1} = (\alpha, e^{\tilde{U}(\cdot)} R_{\tilde{\rho}(\cdot)})$$

with

$$\|\tilde{U}\|_{C^{s-1}} \leq C_s(\rho_0) \alpha \|U\|_{C^s} \quad (3.11)$$

$$\|\tilde{\rho} - \rho\|_{C^s} \leq C_s(\rho_0) \|U\|_{C^s}, \quad (3.12)$$

with a conjugacy  $e^{Y(\cdot)}$  satisfying

$$\|Y\|_{C^s} \leq C_s(\rho_0) \|U\|_{C^s}. \quad (3.13)$$

*Proof.* By simple algebra, there exists  $\varepsilon_0 > 0$  such that for any  $u \in sl(2, \mathbb{R})$ ,  $\rho \in \mathbb{R}$  satisfying  $\|u\| \leq \varepsilon_0$ ,  $\rho \in (\rho_0, 1/2 - \rho_0)M$  there exist  $y \in sl(2, \mathbb{R})$  and  $\tilde{\rho} \in \mathbb{R}$  such that

$$e^u R_\rho = e^{-y} R_{\tilde{\rho}} e^y.$$

Also,  $y = H_1(u, \rho)$ ,  $\tilde{\rho} = H_2(u, \rho)$  for some real analytic functions  $H_1, H_2$  defined on  $E_{\varepsilon_0, \rho_0, M} := \{u \in sl(2, \mathbb{R}) : \|u\| \leq \varepsilon_0\} \times (\rho_0, 1/2 - \rho_0)$  and we clearly have  $H_1(0, \rho) = 0$  and  $H_2(0, \rho) = \rho$  for any  $\rho \in (\rho_0, 1/2 - \rho_0)$ . We define  $Y(\cdot) = H_1(U(\cdot), \rho(\cdot))$  and  $\tilde{\rho}(\cdot) = H_2(U(\cdot), \rho(\cdot))$  so that

$$e^{U(\theta)} R_\rho(\theta) = e^{-Y(\theta)} R_{\tilde{\rho}(\theta)} e^{Y(\theta)}.$$

It is then a standard fact that  $\|Y\|_{C^s} \leq C_s \|U\|_{C^s}$ ,  $\|\tilde{\rho} - \rho\|_{C^s} \leq C_s \|U\|_{C^s}$ .

Next we write

$$e^{U(\theta)} R_\rho(\theta) = e^{-Y(\theta+\alpha)} e^{\tilde{U}(\theta)} R_{\tilde{\rho}(\theta)} e^{Y(\theta)}$$

where  $e^{\tilde{U}(\theta)} := e^{Y(\theta+\alpha)} e^{-Y(\theta)}$ . Now it can be proven (cf [12], Prop. A.2.3) that if  $\phi, f, u$  are smooth functions, for any  $s \in \mathbb{N}$  there exists a constant  $C_s$  such that

$$\|\phi \circ (f + u) - \phi \circ u\|_s \leq C_s \|\phi\|_s (1 + \|f\|_0)^s (1 + \|f\|_s) \|u\|_s.$$

If we choose  $\phi = \exp$ ,  $f = Y(\cdot)$ ,  $u = Y(\cdot + \alpha) - Y(\cdot)$ , (3.11) then follows from (3.13) if  $\varepsilon_0 > 0$  is chosen sufficiently small.  $\square$

**3.5. End of proof of Theorem 7.** We introduce the notation  $\Psi := \mathcal{JR}_{\theta_*}^n(\Psi')$  if  $\Psi' = \mathcal{R}_{\theta_*}^n(\Psi)$ . It is easy to check that

$$\mathcal{JR}_{\theta_*}^n(\Phi) = T_{\theta_*}^{-1} \circ N_{Q_n^{-1}} \circ M_{\beta_{n-1}^{-1}} \circ T_{\theta_*}(\Phi). \quad (3.14)$$

The derivation of Theorem 7 from Corollary 2 will be a consequence of the following Lemma the proof of which is given in the Appendix :

LEMMA 5. *Let  $\Phi_1, \Phi_2 \in \Lambda^r$ . Then, provided  $d_{0,T}(\Phi_1, \Phi_2) \leq (1/q_n^2)$  we have*

$$d_{r, \beta_{n-1}^{-1} T}(\mathcal{JR}^n(\Phi_1), \mathcal{JR}^n(\Phi_2)) \leq K_{r, \beta_{n-1}^{-1} T} q_n^{2r+1} \|\Phi_2\|_{0,T}^{q_n} \|\Phi_2\|_{r,T} d_{r,T}(\Phi_1, \Phi_2).$$

We can now conclude the derivation of Theorem 7 from Corollary 2. We use the notations of Section 3.4. We define  $\Phi_1 = \Phi := ((1, Id), (\alpha, A))$ ,  $\Phi_1^{(i)} = \mathcal{R}_{\theta_*}^{n_i}(\Phi_1)$ ,  $\tilde{\Phi}_1^{(i)} = \text{Conj}_{Z_i}(\Phi_1^{(i)}) = ((1, Id), (\alpha_{n_i}, \tilde{G}_i))$  and  $\tilde{\Phi}_2^{(i)} = ((1, Id), (\alpha_{n_i}, R_{\tilde{\rho}_i}))$ . We then set  $\Phi_2^{(i)} = \text{Conj}_{Z_i^{-1}}(\tilde{\Phi}_2^{(i)})$  and  $\Phi_{2,i} = \mathcal{J}\mathcal{R}^{n_i}(\Phi_2^{(i)})$ . From Proposition 2 we know that there exists  $K_{r,l'}$  ( $r = l$ ) such that

$$d_{r,T}(\tilde{\Phi}_1^{(i)}, \tilde{\Phi}_2^{(i)}) \leq K_{r,l'} \alpha_{n_i}'$$

for any  $r, T, l'$ . If we set  $\tilde{\Phi}_{1,i} = \text{Conj}_{Z_i^{-1}(\beta_{n_i-1}^{-1}, \cdot)}(\Phi_1)$ ,  $\tilde{\Phi}_{2,i} = \text{Conj}_{Z_i^{-1}(\beta_{n_i-1}^{-1}, \cdot)}(\Phi_{2,i})$  we have  $\tilde{\Phi}_{j,i} = \mathcal{J}\mathcal{R}^{n_i}(\tilde{\Phi}_j^{(i)})$  ( $j = 1, 2$ ). By Lemma 5, the fact that  $\tilde{\Phi}_2^{(i)}$  is in  $\Lambda^\infty(SO(2, \mathbb{R}))$  and since for  $n_i$  big enough  $d_{r,T}(\tilde{\Phi}_1^{(i)}, \tilde{\Phi}_2^{(i)}) \leq (1/q_{n_i}^2)$  (see equation (3.1)) we can write

$$d_{r, \beta_{n_i-1}^{-1} T}(\tilde{\Phi}_{1,i}, \tilde{\Phi}_{2,i}) \leq K_{r,l', \beta_{n_i-1}^{-1} T} q_{n_i}^{2r+1} \|\tilde{\Phi}_2^{(i)}\|_{r,T} \alpha_{n_i}'^{l'}. \quad (3.15)$$

Since  $\Phi_{j,i} = \text{Conj}_{Z_i(\beta_{n_i-1}^{-1}, \cdot)}(\tilde{\Phi}_{j,i})$ , using inequalities (2.6) and (3.10) we then get

$$d_{r, \beta_{n_i-1}^{-1} T}(\Phi_1, \Phi_{2,i}) \leq K_{r,l', \beta_{n_i-1}^{-1} T} (\mu(\rho_0) u_i(r, \beta_{n_i-1} T))^4 q_{n_i}^{3r+1} \|\tilde{\Phi}_2^{(i)}\|_{r,T} \alpha_{n_i}'^{l'}. \quad (3.16)$$

From Lemma 8 of the Appendix

$$\|\tilde{\Phi}_2^{(i)}\|_{r,T} \leq K_r q_{n_i}^{2r} \quad (3.17)$$

and thus,

$$d_{r, \beta_{n_i-1}^{-1} T}(\Phi_1, \Phi_{2,i}) \leq K_{r,l', \beta_{n_i-1}^{-1} T} (\mu(\rho_0) u_i(r, \beta_{n_i-1} T))^4 q_{n_i}^{5r+1} \alpha_{n_i}'^{l'}.$$

Denote by  $\varepsilon_{r,n_i}$  the quantity on the right hand side of the previous equation with  $T = \beta_{n_i-1}^{-2}$ , that is (notice that  $K_{r,l', \beta_{n_i-1}^{-1}} = K_{r,l'}$  since  $\beta_{n_i-1}^{-1} > 1$ )

$$\varepsilon_{r,n_i} = K_{r,l'} (\mu(\rho_0) u_i(r, \beta_{n_i-1}^{-1}))^4 q_{n_i}^{5r+1} \alpha_{n_i}'^{l'}. \quad (3.18)$$

In Case (I), it is enough to assume that  $\alpha$  is well-approximated to ensure that  $\varepsilon_{r,n_i}$  goes to 0 as  $n_i \rightarrow \infty$ ,  $n_i \in \mathcal{N}(\varepsilon)$ , since in that case  $\alpha_{n_i} \leq q_{n_i}^{-\frac{1}{M}}$  for some subsequence; In Case (II), the assumption  $\lim_{n \in \mathcal{L}, n \rightarrow \infty} (\log q_n)^{-1} \log \log q_{n+1} = \infty$  is a sufficient condition to ensure this convergence to 0.

The action  $\Phi_{2,i}$  is conjugated to an action in  $\Lambda_{SO(2, \mathbb{R})}^r$  and is  $\varepsilon_{r,n_i}$ -close to  $\Phi_1$  in the  $C^r$ -topology but it is not necessarily normalized. Since

$$\|Id - \Phi_{2,i}(1, 0)\|_{C^r([0,1])} = \|\Phi_1(1, 0) - \Phi_{2,i}(1, 0)\|_{C^r([0,1])} \leq \varepsilon_{r,n_i}$$

we can construct from Lemma 2 a conjugation  $\tilde{B}_i \in C^r(\mathbb{R}, SL(2, \mathbb{R}))$  such that  $\tilde{B}_i(\cdot + 1)\Phi_{2,i}(1, 0)\tilde{B}_i(\cdot)^{-1} = Id$  and satisfying

$$\|\tilde{B}_i - Id\|_{C^r([0,1])} \leq K_r \varepsilon_{r,n_i}. \quad (3.19)$$

As a consequence, from (2.6),

$$d_{r,1}(\Phi_{2,i}, \tilde{\Phi}_{2,i}) \leq K_r \varepsilon_{r,n_i} (1 + \varepsilon_{r,n_i})^3 \|\Phi_{2,i}\|_{r,1}$$

and from (3.17)

$$d_{r,1}(\Phi_{2,i}, \tilde{\Phi}_{2,i}) \leq 2K_r^2 \varepsilon_{r,n_i} q_{n_i}^{2r}.$$

The *normalized* action  $\bar{\Phi}_{2,i} := \text{Conj}_{\tilde{B}_i} \Phi_{2,i}$  is conjugated to an action in  $\Lambda^r(SO(2, \mathbb{R}))$  and satisfies

$$d_{r,1}(\Phi_1, \bar{\Phi}_{2,i}) \leq 3K_r^2 \varepsilon_{r,n_i} q_{n_i}^{2r}. \quad (3.20)$$

With the preceding notations observe that  $\bar{\Phi}_{2,i} = \text{Conj}_{\bar{B}_i} \text{Conj}_{Z_i(\beta_{n_i-1}^{-1})}(\tilde{\Phi}_{2,i})$  and that  $\tilde{\Phi}_{2,i}$  is in  $\Lambda_{SO(2,\mathbb{R})}^r$ . This can be rephrased by saying that for any  $i$  there exist  $\bar{A}_i \in C^r(\mathbb{T}, SL(2, \mathbb{R}))$ ,  $\varphi_i \in C^r(\mathbb{T}, \mathbb{R})$  and  $\bar{B}_i \in C^r(\mathbb{T}, SL(2, \mathbb{R}))$  where  $\bar{B}_i(\cdot) = \bar{B}_i(\cdot)Z_i(\beta_{n_i-1}^{-1}\cdot)$  satisfies (cf. Corollary 2 and (3.19))

$$\|\bar{B}_i\|_{C^r(\mathbb{T})} \leq K_r q_{n_i}^r \mu_r(\rho_0) u_i(r, \beta_{n_i-1}^{-1}), \quad (3.21)$$

such that

$$\|A(\cdot) - \bar{A}_i(\cdot)\|_{C^r(\mathbb{T})} \leq 3K_r^2 \varepsilon_{r,n_i} q_{n_i}^{2r} \quad (3.22)$$

and

$$(\alpha, \bar{A}_i) = (0, \bar{B}_i) \circ (\alpha, R_{\varphi_i}) \circ (0, \bar{B}_i)^{-1}. \quad (3.23)$$

*Proof of the fact that  $(\alpha, A)$  is accumulated by reducible cocycles.* Since in both case (I) and (II)  $\varepsilon_{r,n_i} q_{n_i}^{2r}$  goes to zero as  $n_i$  goes to infinity ( $n_i$  being in  $\mathcal{N}(\varepsilon)$  in case (I) and  $n_i$  being in  $\mathcal{L}$  in case (II)) equation (3.22) shows that  $(\alpha, A)$  is  $C^r$ -accumulated by cocycles which are  $C^r$ -conjugated to the  $C^r$ -cocycles  $(\alpha, R_{\varphi_i(\cdot)})$  with values in  $SO(2, \mathbb{R})$ . But such cocycles can be accumulated by reducible one : just truncate the Fourier series of  $\varphi_i$  far enough to get a trigonometric polynomial  $\bar{\varphi}_i(\cdot)$  and solve the usual cohomological equation  $\psi(\cdot + \alpha) - \psi(\cdot) = \bar{\varphi}_i(\cdot) - \int_{\mathbb{T}} \bar{\varphi}_i(x) dx$ .

*Proof of the fact that  $(\alpha, A)$  is almost rotations-reducible.* From (3.23), (3.22), (3.21) and the convexity inequalities we have

$$\|(0, \bar{B}_i)^{-1} \circ (\alpha, A) \circ (0, \bar{B}_i) - R_{\varphi_i}\|_{C^r(\mathbb{T})} \leq 3K_r^4 q_{n_i}^{4r} (\mu_r(\rho_0) u_i(r, \beta_{n_i-1}^{-1}))^2 \varepsilon_{r,n_i}. \quad (3.24)$$

But from (3.3) and (3.18) the quantity in the right hand side of the previous inequality goes to zero as  $n_i$  goes to infinity ( $n_i$  being in  $\mathcal{N}(\varepsilon)$  in case (I) and  $n_i$  being in  $\mathcal{L}$  in case (II)). This proves the almost rotations-reducibility and completes the proof of Theorem 7.

**3.6. Proofs of Theorems 2, 3.** Theorems 2, 3 now clearly follow from Theorem 7.

#### 4. THE DIOPHANTINE CASE.

In this section, we prove Theorem 1.

**4.1. Plan of the proof of Theorem 1.** We show in this subsection how the arguments given in section 3.2 can be used to prove Theorem 1; we now assume that  $(\alpha, A)$  is a smooth cocycle homotopic to the identity and that  $\alpha$  is a Diophantine number. We observed in section 3.2 that after renormalization and conjugation, we could associate to  $(\alpha, A)$  cocycles  $(\alpha_n, \hat{A}^{(n)})$  where  $\alpha_n = G^n(\alpha)$  and  $\hat{A}^{(n)}$  is  $\alpha_n^C$  close in  $C^r$ -topology ( $k$ -fixed) to some  $SO(2, \mathbb{R})$ -valued cocycle  $(\alpha_n, R_{\varphi_n})$  where  $C > 0$  is an arbitrarily large constant. Since  $\alpha$  is Diophantine,  $\alpha_n$  is also Diophantine with the same exponent but its Diophantine constant can be very small. The relevant observation is that there exist infinitely many times  $n_k$  such that the Diophantine constant of  $\alpha_{n_k}$  is not less than  $\text{cst} \cdot \alpha_{n_k}$ . This allows us, first to find a further conjugation that conjugates  $(\alpha_{n_k}, \hat{A}^{(n_k)})$ ,  $\alpha_{n_k}^{C'}$ -close ( $C' > 0$ ) in  $C^r$ -topology to a *constant* cocycle  $(\alpha_{n_k}, R_{n_k})$  and second, to apply a quantitative version of Eliasson's Theorem (Theorem 5, the proof of which can be found in the Appendix) that guarantees reducibility of  $(\alpha_{n_k}, R_{n_k})$ , provided the fibered rotation number

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satisfies some (weak) Diophantine condition. Since reducibility is preserved by renormalization, this proves that  $(\alpha, A)$  is reducible.

In fact, it was pointed to us by the referee, that a KAM local theorem on reducibility with estimates would suffice to prove Theorem 1. Indeed, in Theorem 1 it is supposed that the fibered rotation number should belong to a set  $\Sigma(\alpha)$  of full measure, while in Theorem 8, that is a precise version implying Theorem 1, the set  $\Sigma(\alpha)$  is partly described by the fact that the fibered rotation number  $\rho_f(\alpha, A)$  is supposed to be Diophantine w.r.t  $\alpha$ , a condition that is evidently of full measure. With this condition, a quantitative version of Eliasson's result is needed to obtain reducibility.

However, one can further restrict the set  $\Sigma(\alpha)$  and remain with a set of full measure ensuring nonetheless that, along a subsequence, the renormalized cocycle  $(\alpha_{n_k}, \hat{A}^{(n_k)})$  has the vector  $(\alpha_{n_k}, \rho_{n_k})$  (where  $\rho_{n_k}$  is the fibered rotation number of  $(\alpha_{n_k}, \hat{A}^{(n_k)})$ ) satisfying a Diophantine condition with constant  $\text{cst} \cdot \alpha_{n_k}$ . From there a usual KAM theorem with estimates would allow to conclude since, again,  $\hat{A}^{(n_k)}$  is  $\alpha_{n_k}^{C'}$  close to a constant, where  $C' > 0$  is arbitrarily large.

**4.2. Diophantine constants.** We will first explicit the condition on the fibered rotation number that will be used in the proof of Theorem 1. We need the following

LEMMA 6. *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  be a Diophantine number. Then there exists  $\nu > 0$  and  $C > 0$  that depend on  $\alpha$ , and an infinite set  $\mathcal{M} \subset \mathbb{N}$  such that, for  $n \in \mathcal{M}$  we have*

$$\forall k \geq 1, \quad \|k\alpha_n\| \geq \frac{C\alpha_n}{k^{1+\nu}} \quad (4.1)$$

that is,  $\alpha_n \in \text{DC}(C\alpha_n, \nu)$ .

*Proof.* Define for  $n \geq 1$ ,  $\nu_n > 0$  such that  $q_{n+1} = q_n^{1+\nu_n}$ . Since  $\alpha$  is Diophantine, there exists  $\nu_0 > 0$  such that for all  $n \geq 1$  we have  $\nu_n \leq \nu_0$ . We define

$$\nu = 10\nu_0 + 10. \quad (4.2)$$

Let  $\mathcal{M}$  be the subset of  $n \in \mathbb{N}$  such that

$$\forall l \geq n, \quad \nu_l \leq 2\nu_n. \quad (4.3)$$

The set  $\mathcal{M}$  is infinite because  $\nu_n$  is bounded and we will show that the assertion of the Lemma holds for every  $n \in \mathcal{M}$ , with  $C = \frac{1}{2^{4\nu+1}}$ .

Recall that  $\alpha_k = \beta_k/\beta_{k-1}$  and that  $\beta_k = (-1)^k(q_k\alpha - p_k)$ , hence for  $k, l \in \mathbb{N}$  we have

$$\begin{aligned} |k\alpha_n - l| &= \frac{1}{\beta_{n-1}} |(kq_n + lq_{n-1})\alpha - kp_n - lp_{n-1}| \\ &\geq q_n |(kq_n + lq_{n-1})\alpha - kp_n - lp_{n-1}|. \end{aligned}$$

Since

$$\frac{q_n}{2q_{n+1}} \leq \alpha_n \leq 2\frac{q_n}{q_{n+1}}$$

we have

$$\|k\alpha_n\| = |k\alpha_n - l(k)|, \text{ for some } l(k) \leq 3k\frac{q_n}{q_{n+1}}.$$

If  $1 \leq k < \frac{q_{n+1}}{4q_n}$  we have

$$\|k\alpha_n\| = k\alpha_n \geq \alpha_n.$$

If  $\frac{q_{n+1}}{4q_n} \leq k < \frac{q_{n+2}}{4q_n}$  we have  $kq_n + l(k)q_{n-1} < q_{n+2}$  hence (2.2) and (2.4) imply

$$\begin{aligned} \|k\alpha_n\| &= |k\alpha_n - l(k)| \geq q_n |(kq_n + lq_{n-1})\alpha - kp_n - lp_{n-1}| \\ &\geq \frac{q_n}{2q_{n+2}}, \end{aligned}$$

while we have

$$\frac{q_n}{2q_{n+2}} = \frac{1}{2q_n^{(1+\nu_n)(1+\nu_{n+1})-1}}.$$

Since  $k \geq \frac{q_{n+1}}{4q_n}$  we have

$$\frac{1}{k^{\nu+1}} \leq \frac{4^{\nu+1}}{q_n^{\nu(\nu+1)}}$$

hence the fact that  $\nu_n(\nu+1) \geq \nu_n\nu \geq (1+\nu_n)(1+\nu_{n+1})$  implies that

$$\|k\alpha_n\| \geq \frac{C}{k^{\nu+1}}.$$

More generally, for  $\frac{q_{n+i}}{4q_n} \leq k < \frac{q_{n+i+1}}{4q_n}$ ,  $i \geq 1$ , we have  $kq_n + l(k)q_{n-1} < q_{n+i+1}$  hence (2.2) and (2.4) imply

$$\|k\alpha_n\| \geq \frac{q_n}{2q_{n+i+1}}$$

or equivalently

$$\|k\alpha_n\| \geq \frac{1}{2q_n^{(1+\nu_n)\dots(1+\nu_{n+i+1})-1}}$$

to be compared with

$$\frac{1}{24^{\nu+1}} \left( \frac{4q_n}{q_{n+i}} \right)^{\nu+1} = \frac{1}{2q_n^{(\nu+1)[(1+\nu_n)\dots(1+\nu_{n+i})-1]}}$$

and our choice of  $\nu$  in (4.2) then implies

$$\|k\alpha_n\| \geq \frac{C}{k^{\nu+1}}.$$

Lemma 6 is proved.  $\square$

We can now state a precise version of Theorem 1

**THEOREM 8.** *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  be a Diophantine number and let  $\mathcal{M} = \mathcal{M}(\alpha)$  be the set of integers given by Lemma 6. Then if  $A \in C_0^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  satisfies*

- (D1)  $\rho_f(\alpha, A)$  is Diophantine w.r.t  $\alpha$ ;
- (D2)  $\left\| \frac{2\rho_f(\alpha, A)}{\beta_{n_i-1}} \right\| \geq \rho_0 > 0$ , for some sequence  $n_i \rightarrow \infty$ ,  $n_i \in \mathcal{M}$ ;
- (D3)  $(\alpha, A)$  is  $L^2$ -conjugated to a cocycle of rotations;

then  $(\alpha, A)$  is  $C^\infty$ -reducible.

This theorem implies Theorem 1 since the arithmetic conditions imposed on  $\rho_f(\alpha, A)$  are both clearly of full measure.

**4.3. Proof of Theorem 8.** We will need the following elementary fact.

LEMMA 7. *If  $\rho$  is Diophantine w.r.t.  $\alpha$ , then, for any  $n \in \mathbb{N}$ ,  $\frac{\rho}{\beta_{n-1}}$  is Diophantine w.r.t.  $\alpha_n$ .*

*Proof.* As in the proof of Lemma 6 we have

$$\begin{aligned} \left| 2\frac{\rho}{\beta_{n-1}} - k\alpha_n - l \right| &= \frac{1}{\beta_{n-1}} |2\rho - (kq_n + lq_{n-1})\alpha + kp_n + lp_{n-1}| \\ &\geq \frac{1}{\beta_{n-1}} \frac{C}{(1 + |kq_n + lq_{n-1}| + |kp_n + lp_{n-1}|)^2} \\ &\geq \frac{C_n}{(1 + |k| + |l|)^2} \end{aligned}$$

□

Let  $\mathcal{M}$  be the set given by Lemma 6 and consider a sequence  $n_i \rightarrow \infty$ ,  $n_i \in \mathcal{M}$ , such that (D2) holds for this sequence.

Since reducibility is invariant under renormalization and conjugation, it is enough to show that the action  $\Phi_i = ((1, \text{Id}), (\alpha_{n_i}, G_i))$  of Section 2.8 is reducible.

Due to (D2) we can apply Proposition 2 and further conjugate  $\Phi_i$  to  $((1, \text{Id}), (\alpha_{n_i}, \tilde{G}_i))$  such that  $\tilde{G}_i = e^{\tilde{U}_i(\cdot)} R_{\tilde{\rho}_i(\cdot)}$ , with for any  $l, l' \in \mathbb{N}$

$$\lim_{i \rightarrow \infty} \frac{\|\tilde{U}_i\|_{C^l}}{\alpha_{n_i}^{l'}} = 0 \quad (4.4)$$

$$\lim_{i \rightarrow \infty} \|\tilde{\rho}_i - \rho_i\|_{C^l} = 0 \quad (4.5)$$

where  $\rho_i = (-1)^{n_i} \rho_f(\alpha, A) / \beta_{n_i-1}$ .

From (4.5) we have that  $\lim_{i \rightarrow \infty} \|\tilde{\rho}_i - \int_{\mathbb{T}} \tilde{\rho}_i\|_{C^l} = 0$ . Since in addition,  $\alpha_{n_i} \in \text{DC}(C\alpha_{n_i}, \nu)$ , there exists  $h_i \in C^\infty(\mathbb{T}, \mathbb{R})$  such that  $\tilde{\rho}_i(\cdot) - \int_{\mathbb{T}} \tilde{\rho}_i = h_i(\cdot + \alpha_{n_i}) - h_i(\cdot)$  and  $\lim_{i \rightarrow \infty} \alpha_{n_i} \|h_i\|_{C^l} = 0$ .

Hence, due to (4.4), conjugating  $(1, \text{Id}), (\alpha_{n_i}, \tilde{G}_i)$  by  $R_{h_i(\cdot)}$  we get  $((1, \text{Id}), (\alpha_{n_i}, \bar{G}_i))$  where

$$\|\bar{G}_i - R_{\int_{\mathbb{T}} \tilde{\rho}_i}\|_{C^l} = o(\alpha_{n_i}^{l'}) \quad (4.6)$$

for any  $l, l' \in \mathbb{N}$  as  $i \rightarrow \infty$ .

Finally, we have that  $\rho_f(\alpha_{n_i}, \bar{G}_i) = \rho_f(\alpha_{n_i}, G_i) = (-1)^{n_i} \rho_f(\alpha, A) / \beta_{n_i-1}$  is Diophantine with respect to  $\alpha_{n_i}$  as is asserted by Lemma 7. So, the fact that  $\alpha_{n_i} \in \text{DC}(C\alpha_{n_i}, \nu)$  and (4.6) allow to use Eliasson's local theorem on reducibility, Theorem 5 (cf. Remark 3) to conclude that for  $i$  sufficiently large  $(1, \text{Id}), (\alpha_{n_i}, \bar{G}_i)$  is reducible, and so is the case for  $\Phi_i$  and the original action  $\Phi$ .

□

## 5. APPENDIX A

**5.1. Proof of Lemma 2.** Indeed, one just have to prove that given  $C \in C^\infty(\mathbb{R}, SL(2, \mathbb{R}))$  one can always find  $B \in C^\infty(\mathbb{R}, SL(2, \mathbb{R}))$  such that  $C(t) = B(t+1)B(t)^{-1}$ , which is not difficult in the  $C^\infty$ -category (it is more difficult in the analytic category; see lemma 4.1 of [4]) : indeed, one can find a  $B \in C^\infty([0, 1], SL(2, \mathbb{R}))$  such that  $B(0) = \text{Id}$ ,  $B(1) = C(0)$  and such that for any  $k \geq 1$   $\partial^k B(0) = 0$  and  $\partial^k B(1) =$

$\partial^k(CB)(0)$ ,  $\|B\|_{s,[0,1]} \leq K_s \|C\|_{s,[0,1]}$ . Setting  $B(t) = C(t-1) \cdots C(t-m)B(t-m)$  for  $t \in [m, m+1]$  completes the definition of  $B$ . One then has for any  $T \in \mathbb{N}$

$$\|B\|_{C^s([0,T])} \leq K_s \max_{0 \leq l \leq T} \left( \|C_l\|_{C^s([0,1])} \|C\|_{C^0([0,1])} + \|C_l\|_{C^0([0,1])} \|C\|_{C^s([0,1])} \right), \quad (5.1)$$

where  $(l, C_l(\cdot)) = (1, C(\cdot))^l$ . Lemma 2 will follow from Lemma 8 and 3 below

LEMMA 8. *Let  $(u_k)_{1 \leq k \leq n}$  a sequence of functions in  $C^\infty(\mathbb{T}, SL(2, \mathbb{R}))$  and let  $U_n = u_1 \dots u_n$ . Denote by  $M_s = \max_{1 \leq k \leq n} (\|u_k\|_s)$ . Then*

$$\|U_n\|_r \leq (nC)^r M_0^{n-1} M_r.$$

*If the functions are defined on  $[0, 1]$  one can replace  $C^r$  by some constant  $K_r$  depending only on  $r$  (and on the interval  $[0, 1]$ ).*

*Proof.*

$$\partial^r U_n(x) = \partial^r \left( \prod_{k=n-1}^0 u_k \right) (x) \quad (5.2)$$

which by Leibniz formula is a sum of  $n^r$  terms of the form ( $s \leq r$ )

$$\begin{aligned} I_{(i^*)}(x) &= \left( \prod_{l=n-1}^{i_1+1} u_l(x) \right) \cdot \partial^{m_1} u_{i_1}(x) \cdot \left( \prod_{l=i_1-1}^{i_2+1} u_l(x) \right) \cdot \\ &\quad \partial^{m_2} u_{i_2}(x) \cdot \left( \prod_{l=i_2-1}^{i_3+1} u_l(x) \right) \cdots \\ &\quad \partial^{m_s} u_{i_s}(x) \cdot \left( \prod_{l=i_s-1}^0 u_l(x) \right) \end{aligned} \quad (5.3)$$

where  $i^*$  runs through  $\mathcal{J} = \{0, \dots, n-1\}^{\{1, \dots, r\}}$  and where  $\{i_1, \dots, i_s\} = i^*(\{1, \dots, r\})$  satisfy  $n-1 \geq i_1 > i_2 > \dots > i_s \geq 0$  and  $m_l = \#(i^*)^{-1}(i_l)$  (notice that  $m_1 + \dots + m_s = r$ ).

From this and the convexity (Hadamard-Kolmogorov) inequalities [11]

$$\|\partial^m u\|_{C^0} \leq C \|u\|_0^{1-(m/r)} \|\partial^r u\|_{C^0}^{\frac{m}{r}}, \quad 0 \leq m \leq r, \quad (5.4)$$

we deduce (using  $\sum_{p=1}^s m_p = r$ )

$$\begin{aligned} \|I_{(i^*)}(x)\| &\leq M_0^{n-s} \prod_{p=1}^s \left( C M_0^{1-\frac{m_p}{r}} M_r^{\frac{m_p}{r}} \right) \\ &\leq C^s M_0^{n-1} M_r \end{aligned} \quad (5.5)$$

so that

$$\begin{aligned} \|U_n\| &\leq \sum_{i^* \in \mathcal{J}} \|I_{(i^*)}(x)\| \\ &\leq n^r C^r M_0^{n-1} M_r \end{aligned} \quad (5.6)$$

□

Lemma 8 has an immediate corollary on the growth of the products  $A_n$

J'ai mis  $U_n$  dans la formule (5.6) et j'ai rajouté la phrase qui dit que l'ex lemme 9 est un corollaire de lemme 8. A verifier

COROLLARY 3. *Let  $J \subset \mathbb{R}$  be some interval and define  $J_n = \bigcup_{l=0}^{n-1} (J + l\alpha)$ . If  $M_k(x) > 1$  is an upper bound of  $\|A_k(x)\|$  for  $x \in J$  then for any  $x \in J$*

$$\|\partial^r A_n(x)\| \leq K^r \|A_n(x)\|_{C^0(J)} M_0(x)^{2r} (M_0^2(x) + \cdots + M_{n-1}^2(x))^r \|\partial^r A(x)\|_{C^0(J_n)}.$$

Observe that when the fibered products  $C_l(t)$  are uniformly bounded for  $t \in \mathbb{R}$  ( $P := \sup_{l \in \mathbb{R}} \|C_l\|_{C^0(\mathbb{R})} < \infty$ ) we have  $\|C_l\|_{C^s(\mathbb{R})} \leq K_s (M^4 l)^s \|C\|_{C^s([0,l])} \|C\|_{C^0([0,1])}$ , for some constants  $K_s$ ,  $s \in \mathbb{N}$  and using (5.1) we get the estimate for  $B$ .

In the general case,

$$\|C_l\|_{C^s([0,1])} \leq K_s l^s \|C\|_{0,l}^{l-1} \|C\|_{s,l}.$$

This and (5.1) conclude the proof of Lemma 2.  $\square$

## 5.2. Proof of Lemma 5.

*Proof.* We use the following lemma :

LEMMA 9. *Let  $(\gamma, A)$  and  $(\gamma, \tilde{A}) \in \Omega^r$  and define for  $k \in \mathbb{Z}$   $(\gamma, A)^k = (k\gamma, A_k)$ ,  $(\gamma, \tilde{A})^k = (k\gamma, \tilde{A}_k)$ . Then, if we define  $I_n = \bigcup_{l=0}^{n-1} (I + l)$ ,  $M_s = \|A\|_{C^s(I_n)}$ ,  $H_s = \|A - \tilde{A}\|_{C^s(I_n)}$  one has,*

$$\|\partial^s A_n - \partial^s \tilde{A}_n\|_{C^0(I)} \leq K_s n^{s+1} (M_0 + H_0)^{n-2} \left( (M_s + H_s) H_0 + (M_0 + H_0) H_s \right).$$

*Proof.* Define  $H = \tilde{A} - A$  and set  $\Delta(\nu, t) = \prod_{l=n-1}^0 (A(t+l\gamma) + \nu H(t+l\gamma))$ . By the mean value theorem we clearly have

$$\|\partial^s A_n - \partial^s \tilde{A}_n\|_{C^0(I)} \leq \max_{0 \leq \nu \leq 1} \|\partial_\nu(\partial^s \Delta(\nu, \cdot))\|_{C^0(I)}.$$

But  $\partial_\nu(\partial^s \Delta(\nu, \cdot))$  is a sum of  $n$  terms of the form ( $0 \leq m \leq n-1$ )

$$\partial^s \left( \left( \prod_{l=n-1}^{m+1} (A(t+l\gamma) + \nu H(t+l\gamma)) \right) H(t+m\gamma) \left( \prod_{l=m-1}^0 (A(t+l\gamma) + \nu H(t+l\gamma)) \right) \right).$$

By the convexity inequalities and Lemma 3, the  $C^s$ -norm of this expression is less than

$$C_s n^{s+1} (M_0 + H_0)^{n-2} \left( (M_s + H_s) H_0 + (M_0 + H_0) H_s \right)$$

$\square$

Denote  $\Psi_i = M_{\beta_{n-1}^{-1}} \circ T_{\theta_*}(\Phi_i)$ ,  $i = 1, 2$  and  $\Gamma_n = T_{\theta_*}^{-1} \circ N_{Q_n^{-1}}$ . We clearly have

$$d_{r, \beta_{n-1} T}(\Psi_1, \Psi_2) = \beta_{n-1}^{-r} d_{r, T}(\Phi_1, \Phi_2),$$

and

$$\|\Psi_1\|_{r, \beta_{n-1} T} = \beta_{n-1}^{-r} \|\Phi_1\|_{r, T}.$$

From formula (3.14), the previous lemma and the definition of  $N_{Q_n}$  we then get

$$\begin{aligned} d_{r, \beta_{n-1}^2 T}(\Gamma_n(\Psi_1), \Gamma_n(\Psi_2)) &\leq K_{r, \beta_{n-1} T} q_n^{r+1} \|\Psi_1\|_{0, \beta_{n-1} T}^{q_n-2} \times \\ &\left( \|\Psi_1\|_{0, \beta_{n-1} T} d_{r, \beta_{n-1} T}(\Psi_1, \Psi_2) + \|\Psi_1\|_{r, \beta_{n-1} T} d_{0, \beta_{n-1} T}(\Psi_1, \Psi_2) \right) \end{aligned} \quad (5.7)$$

which in view of the previous formulae is the conclusion of lemma 5.  $\square$

6. APPENDIX B  
AN ABRIDGED PROOF OF THEOREM 5

The proof relies on a K.A.M. scheme (named after Kolmogorov, Arnold and Moser). Assume that  $\alpha \in DC(\gamma, \tau)$ , that is

$$\|q\alpha\| \geq \frac{\gamma^{-1}}{q^{1+\tau}}$$

and that  $A(\cdot) = e^{F_0(\cdot)}A_0$  where  $F_0 \in C^r(\mathbb{T}, sl(2, \mathbb{R}))$  where  $sl(2, \mathbb{R})$  is the Lie algebra of  $SL(2, \mathbb{R})$  which elements are traceless  $2 \times 2$  real matrices.

DEFINITION 3. For  $N, K > 0$ , we define the set

$$DS(N, K) := \{\beta \in \mathbb{R} \mid \inf_{l \in \mathbb{Z}, 0 < |k| \leq N} |\beta - k\alpha - l| \geq K^{-1}\}$$

It is easy to check the following

LEMMA 10. *Let  $\alpha \in DC(\gamma, \tau)$ ,  $N > 0$  and  $K \geq \gamma(2N)^{3(\tau+1)}$ . Then: 1) If  $\beta \notin DS(N, K)$ , there exists a unique  $k_0 \in \mathbb{Z} - \{0\}$ ,  $|k_0| \leq N$  such that  $\|\beta - k_0\alpha\| < 1/K$  and we have  $\beta - k_0\alpha \in DS(N^3, K)$ . 2) The following inclusion holds:  $\{|\beta| < K^{-1}\} \in DS(N^3, K)$ .*

This lemma will be used with sequences  $N_n$  and  $K_n$  given by  $N_n = L^{(1+\sigma)^n}$ ,  $K_n = \gamma(2N_n)^{3(\tau+1)}$ , where  $\sigma > 0$  and  $L > 1$  some constants that will be fixed later. The lemma shows that if  $\beta$  is almost *resonant* at step  $n$ , that is  $\beta \notin DS(N_n, K_n)$ , then it is possible, by replacing  $\beta$  by  $\tilde{\beta} = \beta - k_0\alpha$ , to get a number  $\tilde{\beta}$  that is non-resonant for a much longer period. The importance of the exponent in the Diophantine condition on  $\tilde{\beta}$  will be explained later.

If  $g$  in  $SL(2, \mathbb{R})$  is elliptic (that is conjugated to a rotation matrix or equivalently if the absolute value of its trace is less than 2) it can be written  $e^M$  where  $M \in sl(2, \mathbb{R})$  is elliptic (conjugated to an infinitesimal rotation). In that case  $\det M$  is positive and we denote by  $\beta(g) = \pi^{-1}\sqrt{\det M}$ . The eigenvalues of  $\text{Ad}(g) : sl(2, \mathbb{R}) \rightarrow sl(2, \mathbb{R})$  (which is defined by  $\text{Ad}(g) \cdot X = gXg^{-1}$ ) are 1 and  $\pm e^{2\pi i\beta(g)}$ . Notice that if  $(\alpha, A)$  is a cocycle with  $A$  constant and elliptic then its fibered rotation number  $\rho$  equals  $\beta(A)/2$ .

If  $f$  is a periodic  $C^r$  function we define its Fourier coefficients  $\hat{f}(k) = \int_{\mathbb{T}} f(\theta)e^{-2\pi ik\theta} d\theta$ , its truncation up to order  $N$ ,  $T_N f = \sum_{|k| \leq N} \hat{f}(k)e^{2\pi ik\theta}$  and its remainder at order  $N$ ,  $R_N f = f - T_N f$ . We have for any  $s, s' \in \mathbb{N}$ ,  $s' \geq s$ , the following estimates,

$$|T_N f|_{s'} \leq C_{s,s'} N^{s'-s+1} |f|_s, \quad |R_N f|_s \leq C_{s,s'} \frac{|f|_{s'}}{N^{s'-s-1}}. \quad (6.1)$$

Also, if  $Q$  is quadratic in  $(f, g)$  ( $Q$  being  $C^2$  and  $Q(0, 0) = 0$ ,  $DQ(0, 0) = 0$ ) we have

$$|Q(f, g)|_s \leq C_s (1 + |f|_0 + |g|_0)^{s+1} (|f|_0 + |g|_0) (|f|_s + |g|_s), \quad (6.2)$$

which simplifies to

$$|Q(f, g)|_s \leq C_s (|f|_0 + |g|_0) (|f|_s + |g|_s),$$

if  $|f|_0 + |g|_0$  is *a priori* bounded (by 1 for example).

The main procedure operated at every step of the KAM scheme is given by the following proposition.

PROPOSITION 3. If  $|A| \leq 1$  and  $N$  and  $K$  are such that  $K = \gamma(2N)^{3(\tau+1)}$  and  $N^{6(\tau+1)}|F|_0$  is small enough, then:

(i) if  $\beta(A) \in DS(N, K)$  there exist  $Y \in C^\infty(\mathbb{T}, sl(2, \mathbb{R}))$ ,  $A' \in sl(2, \mathbb{R})$ ,  $F' \in C^\infty(\mathbb{T}, sl(2, \mathbb{R}))$  such that

$$e^{Y(\cdot+\alpha)}(e^{F(\cdot)}A)e^{-Y(\cdot)} = e^{F'(\cdot)}A' \quad (6.3)$$

$$A' = e^{\hat{F}(0)}A,$$

$$|F'|_s \leq C_s \gamma^2 N^{6(\tau+1)} |F|_s |F|_0 + C_{s,s'} \frac{|F|_{s'}}{N^{s'-s-1}}.$$

$$|Y|_s \leq \gamma C_s N^{6(\tau+1)} |F|_s$$

$$\rho(\alpha, e^{F'(\cdot)}A') = \rho(\alpha, e^{F(\cdot)}A)$$

(ii) if there exists  $m \in \mathbb{Z}$ ,  $0 < |m| \leq N$  such that  $\|\beta(A) - m\alpha\| < K^{-1}$  then there exist  $B \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2, \mathbb{R}))$ ,  $A' \in sl(2, \mathbb{R})$ ,  $F' \in C^\infty(\mathbb{T}, sl(2, \mathbb{R}))$  such that

$$|\beta(A')| < K^{-1}$$

$$B(\cdot + \alpha)(e^{F(\cdot)}A)B(\cdot)^{-1} = e^{F'(\cdot)}A'$$

$$|F'|_s \leq C_{s,s'} \gamma^4 N^{20(\tau+1)} \left( N^s |F|_0^2 + |F|_s |F|_0 + \frac{N^{s'} |F|_0 + |F|_{s'}}{N^{3(s'-s-1)}} \right).$$

$$\rho(\alpha, e^{F'(\cdot)}A') = \rho(\alpha, e^{F(\cdot)}A) - m\alpha/2.$$

*Proof.* Case (i) Since  $\alpha$  is Diophantine and  $\beta(A) \in DS(N, K)$ , where  $K = \gamma(2N)^{3(\tau+1)}$ , it is easy to see going through Fourier coefficients that the so-called *linearized* equation

$$Y(\cdot + \alpha) - \text{Ad}(A)Y(\cdot) = -(T_N F - \hat{F}(0))$$

has a unique solution  $Y \in C^\infty(\mathbb{T}, sl(2, \mathbb{R}))$  such that  $\hat{Y}(0) = 0$ , and for  $k \in \mathbb{Z} - \{0\}$  one has  $\hat{Y}(k) = 0$  if  $|k| > N$  and

$$\hat{Y}(k) = \left( e^{2\pi i k \alpha} \text{Id} - \text{Ad}(A) \right)^{-1} \hat{F}(k)$$

if  $|k| \leq N$ . The eigenvalues of  $\left( e^{2\pi i k \alpha} \text{Id} - \text{Ad}(A) \right)^{-1}$  are  $(e^{2\pi i k \alpha} - 1)^{-1}$ ,  $(e^{2\pi i k \alpha} - e^{2\pi i \beta(A)})^{-1}$  and  $(e^{2\pi i k \alpha} - e^{-2\pi i \beta(A)})^{-1}$  and from this it is not difficult to get

$$|Y|_s \leq C_s \gamma^2 N^{6(\tau+1)} |F|_s. \quad (6.4)$$

If we define  $A' = e^{\hat{F}(0)}A$  and  $F'$  by (6.3) one can see that  $F'$  is the sum of an expression which is at least quadratic in  $(T_N F, Y, F)$ , and of a remainder of size comparable to  $R_N F = F - T_N F$ . The required estimates then follow from the application of formulas (6.1), (6.2) and (6.4). The equality on the fibered rotation numbers is due to the fact that this fibered rotation number is invariant under conjugations which are *homotopic to the identity*.

Case (ii) In that case the constant matrix  $A$  is elliptic and can be conjugated to a rotation  $\exp(\beta(A)\pi H)$ , where  $H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , the constant conjugation being of size bounded by  $C/\beta(A)^2 \leq C\gamma^2 N^{2(\tau+1)}$  (because for some  $0 < |k_0| \leq N$ ,  $\|\beta - k_0\alpha\| \leq 1/K < (\gamma N^{3(\tau+1)})^{-1}$  while  $\|k_0\alpha\| \geq (\gamma N^{\tau+1})^{-1}$ ).

Applying this conjugation to  $e^F A$  results in a new cocycle  $e^{\bar{F}} \exp(\pi\beta(A)H)$  where  $|\bar{F}|_s \leq C\gamma^2 N^{2(\tau+1)} |F|_s$ . To keep the notations simple we will still write  $A$  in place

of  $\exp(\pi\beta(A)H)$  and  $F$  in place of  $\bar{F}$ . We now perform a conjugation  $D(\cdot) = \exp(-\pi Hm\cdot) \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2, \mathbb{R}))$  : writing

$$D(\cdot + \alpha)(e^{F(\cdot)}A)D(\cdot)^{-1} = e^{\tilde{F}(\cdot)}\tilde{A}$$

we see that  $\tilde{A} = A - m\alpha H$  so that  $\beta(\tilde{A}) = \beta(A) - m\alpha$  and

$$|\tilde{F}|_s \leq C_s(N^s|F|_0 + |F|_s).$$

Notice that although  $D$  is only  $2\mathbb{Z}$ -periodic,  $\tilde{F}$  is  $\mathbb{Z}$ -periodic. But then we have seen that,  $\beta(\tilde{A}) \in DS(N^3, K)$ , so that *Case (i)* applies to  $e^{\tilde{F}(\cdot)}\tilde{A}$  with  $N^3$  in place of  $N$ : there exists a conjugation  $\tilde{Y} \in C^\infty(\mathbb{T}, sl(2, \mathbb{R}))$  such that if we set  $B(\cdot) = e^{\tilde{Y}(\cdot)}D(\cdot)$ , one has the desired estimates

$$|F'|_s \leq C_{s,s'}\gamma^2 N^{2(\tau+1)} \left( \gamma^2 N^{6(\tau+1)} (N^s|F|_0^2 + |F|_s|F|_0) + \frac{N^{s'}|F|_0 + |F|_{s'}}{N^{3(s'-s-1)}} \right)$$

The relation on the fibered rotation number is due to the fact if  $B : \mathbb{R}/2\mathbb{Z} \rightarrow SL(2, \mathbb{R})$  has topological degree  $-m$

$$\rho(\alpha, B(\cdot + \alpha)e^{F(\cdot)}AB(\cdot)^{-1}) = \rho(\alpha, e^{F(\cdot)}A) - m\alpha/2.$$

□

In view of the estimates of Proposition 3 we now apply Corollary 4 of Appendix C with  $a = 20(\tau + 1)$ ,  $\sigma_0 = 1$ ,  $M = 1$ ,  $m = 0$ ,  $(\mu_p, \bar{\mu}_p) \in \{(1, 0), (3, 1)\}$ . This provides us with  $0 < \sigma < \sigma_0$ ,  $0 < g$ ,  $s_0 \in \mathbb{N}$  (of the form  $s_0 = O(\tau)$ ) and  $L = \gamma^4 \bar{C}_{s_0}$ ,  $\bar{C}_{s_0} = \max_{s \leq s_0, s' \leq s_0} C_{s,s'}$  for which the conclusions of Corollary 4 are satisfied. Assuming that

$$|F_0|_{s_0} \leq \gamma^{-d_0} \varepsilon$$

where  $d_0 = 4s_0/g$ ,  $\varepsilon = (\bar{C}_{s_0})^{-s_0/g}$ , setting  $N_n = L^{(1+\sigma)^n}$  and applying inductively Proposition 3 (with at step  $n$ ,  $N_n, A_n, F_n$  in place of  $N, A, F$ ) this enables us to construct inductively sequences  $A_n \in sl(2, \mathbb{R})$ ,  $F_n \in C^\infty(\mathbb{T}, sl(2, \mathbb{R}))$ ,  $Z_n \in C^\infty(\mathbb{R}/2\mathbb{Z}, SL(2, \mathbb{R}))$ ,  $\tilde{m}_n \in \mathbb{Z}$  such that

$$Z_n(\cdot + \alpha)e^{F_n(\cdot)}A_n Z_n(\cdot)^{-1} = e^{F_{n+1}(\cdot)}A_{n+1}$$

$$2\rho(\alpha, e^{F_{n+1}(\cdot)}A_{n+1}) = 2\rho(\alpha, e^{F_n(\cdot)}A_n) - \tilde{m}_n\alpha$$

and which satisfy the following estimates: setting  $\varepsilon_{n,s} = |F_n|_s$ ,  $K_n = \gamma(2N_n)^{3(\tau+1)}$  then for all  $s \geq 0$

$$\varepsilon_{n,s} = O(N_n^{-\infty}).$$

Also,

- if  $\beta(A_n) \in DS(N_n, K_n)$  then  $\tilde{m}_n = 0$  and  $Z_n$  is of the form  $e^{Y_n}$  with  $Y_n \in C^\infty(\mathbb{T}, sl(2, \mathbb{R}))$  satisfying  $|Y_n|_s \leq \gamma C_s N_n^{6(\tau+1)} \varepsilon_{n,s}$
- if  $\beta(A_n) \notin DS(N_n, K_n)$  then  $0 \neq |\tilde{m}_n| \leq N_n$  and  $|\beta(A_{n+1})| < K_n^{-1}$ .

Notice that  $Z_n$  can be, depending on whether at step  $n$  we are in *Case (i)* or *Case (ii)* of Proposition 3, "small" or "big" according to  $C^\infty$ -topology.

We have thus proven *almost reducibility* without any condition on the fibered rotation number :

**THEOREM 9.** *If  $\alpha \in \text{DC}(\gamma, \tau)$  is Diophantine and  $A \in \text{SL}(2, \mathbb{R})$  there exists  $\epsilon$  (depending only on  $\tau$  and  $A$ ) and  $s_0 \in \mathbb{N}$  such that for any  $F \in C^\infty(\mathbb{T}, \text{sl}(2, \mathbb{R}))$  satisfying  $|F|_{s_0} \leq \gamma^{-d_0} \epsilon$  there exists a sequence  $W_n \in C^\infty(\mathbb{R}/2\mathbb{Z}, \text{SL}(2, \mathbb{R}))$  ( $W_n = Z_{n-1} \cdots Z_n$ ) for which  $W_n(\cdot + \alpha)(e^{F(\cdot)} A)W_n(\cdot)^{-1}$  is  $\mathbb{Z}$ -periodic and converges to a constant in the  $C^\infty$ -topology.*

Now to prove Theorem 5 under the assumption  $|F|_{s_0} \leq \gamma^{-d_0} \epsilon$  we just have to prove that if  $\rho(\alpha, e^{F(\cdot)} A)$  is Diophantine with respect to  $\alpha$  then for all  $n$  large enough  $\beta(A_n) \in \text{DS}(N_n, K_n)$ . We make the following remarks: at the  $n$ -th step of the iteration scheme the fibered rotation number of  $(\alpha, e^{F_n} A_n)$  satisfies:

$$2\rho(\alpha, e^{F(\cdot)} A) = 2\rho(\alpha, e^{F_n(\cdot)} A_n) + (\tilde{m}_0 + \dots + \tilde{m}_{n-1})\alpha,$$

and using the smallness of  $F_n$  we get

$$|2\rho(\alpha, e^{F(\cdot)} A) - \beta(A_n) - (\tilde{m}_0 + \dots + \tilde{m}_{n-1})\alpha| \leq \epsilon_{n,0}.$$

Now assume that there is an infinite sequence  $n_k$  such that  $\tilde{m}_{n_k-1} \neq 0$ ; then  $|\beta(A_{n_k})| < K_{n_k}^{-1}$  and we have

$$|2\rho(\alpha, e^{F(\cdot)} A) - (\tilde{m}_0 + \dots + \tilde{m}_{n_k-1})\alpha| \leq K_{n_k}^{-1} + \epsilon_{n_k,0}.$$

But since  $\rho(\alpha, e^{F(\cdot)} A)$  is Diophantine with respect to  $\alpha$  we get

$$\frac{C}{(N_0 + \dots + N_{n_k-1})^2} \leq K_{n_k}^{-1} + \epsilon_{n_k,0}.$$

This is clearly impossible for  $k$  large enough since  $K_n = \gamma(2N_n)^{3(\tau+1)}$  and  $\epsilon_{n,0} = O(N_n^{-\infty})$ .

We conclude that for  $n$  big enough  $\beta(A_n) \in \text{DS}(N_n, K_n)$  and consequently, the conjugation  $Z_n(\cdot)$  is of the form  $e^{Y_n(\cdot)}$  with  $Y_n \in C^\infty(\mathbb{T}, \text{sl}(2, \mathbb{R}))$  satisfying  $|Y_n|_s \leq \gamma C_s N_n^{6(\tau+1)} \epsilon_{n,s}$  (*Case (i)* of the iteration scheme). The product  $Z_n(\cdot) \cdots Z_0(\cdot)$  clearly converges in the  $C^\infty$ -topology: this is the required conjugation that transforms  $(\alpha, e^{F(\cdot)} A)$  to a constant cocycle.

## 7. APPENDIX C

### CONVERGENCE OF A GENERALIZED KAM SCHEME

In this section we present a proof of the convergence of a generalized KAM scheme. We insist on the fact that non constant losses of derivatives are allowed when solving the linearized equations; typically, a loss of  $ms$  derivatives is possible ( $m < 1$ ) if we are dealing with  $C^s$ -norms. Also, a loss in the truncated part is allowed (in the sequel of the text, this is governed by the constant  $\mu, \bar{\mu}$ ). We have essentially followed the presentation of [12], chap. 5 and [7], sec. 6 with some simplifications and generalizations (and corrections of some minor errors).

Let  $a > 0$ ,  $0 < \sigma_0 \leq 1$ ,  $M, m, \mu, \bar{\mu} > 0$  such that

$$\frac{M}{\mu} < 1 + \sigma_0, \quad \frac{m}{\mu} < 1, \quad 0 < 2\bar{\mu} < \mu$$

Then there exists  $g$  such that

$$\frac{1}{\mu - \bar{\mu}} < g < \min\left(\frac{1 + \sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar{\mu}}\right)$$

and  $0 < \sigma < \sigma_0$  such that

$$1 + \sigma < (\mu - \bar{\mu})g.$$

that is

$$\frac{1 + \sigma}{\mu - \bar{\mu}} < g < \min\left(\frac{1 + \sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar{\mu}}\right) \quad (7.1)$$

For  $s, \bar{s}$ , let  $C_{s, \bar{s}} : [0, \infty) \rightarrow [1, \infty]$  be a family of continuous functions on  $[0, 2]$  such that  $C_{s, \bar{s}}(t) = \infty$  if  $t > 2$ , increasing with respect to  $s, \bar{s} \geq 0$  and let  $\bar{C}_s = \max_{t \leq 2} C_{s, s}(t)$ .

**THEOREM 10.** *There exist  $s_0 > 0$  such that if  $\varepsilon_{p, s}$  is a double sequence satisfying for any  $s, \bar{s}, p \in \mathbb{N}$*

$$\varepsilon_{p+1, s} \leq C_{s, \bar{s}}(1 + \lambda_p^a \varepsilon_{p, 0}) \times (\lambda_p^{a+Ms} \varepsilon_{p, 0}^{1+\sigma_0} + \lambda_p^{a+ms} \varepsilon_{p, s} \varepsilon_{p, 0} + \lambda_p^{a-(\bar{s}-s)\mu} (\varepsilon_{p, \bar{s}} + \lambda_p^{\bar{\mu}\bar{s}} \varepsilon_{p, 0})) \quad (7.2)$$

where  $\lambda_p = L^{(1+\sigma)^p}$ ,  $L = \bar{C}_{s_0}$  and if

$$\varepsilon_{0, 0} \leq (\bar{C}_{s_0})^{-\frac{s_0}{g}}, \quad \varepsilon_{0, s_0} \leq 1$$

then, for any  $s \in \mathbb{N}$   $\varepsilon_{p, s} = O(\lambda_p^{-\infty})$ . Also,  $s_0$  does not depend on the sequence  $(\bar{C}_s)$  and can be taken of the form  $(a+1)\xi(\sigma_0, M, m, \mu, \bar{\mu})$ .

*Proof.* In view of (7.1) we choose  $\kappa$  such that

$$\frac{1 + \sigma + \kappa}{\mu - \bar{\mu}} < g \quad (7.3)$$

The proof consists of several lemmas. We assume that  $(\varepsilon_{p, s})$  is a sequence satisfying (7.2) where  $\lambda_p = L^{(1+\sigma)^p}$  for some  $L > 1$ .

**LEMMA 11.** *Let  $\gamma_0 > 0$ ,  $s_0 = g\gamma_0$ ,  $b = \kappa\gamma_0$ . Then, there exists  $\gamma_0^{\text{ref}}(a, \sigma_0, M, m, \mu, \bar{\mu}, g, \sigma, \kappa)$  such that if  $\gamma_0 > \gamma_0^{\text{ref}}$  and  $\lambda_p = L^{(1+\sigma)^p}$ ,  $L = \bar{C}_{s_0}$  then*

$$\varepsilon_{p, 0} \leq \lambda_p^{-\gamma_0} \quad (7.4)$$

$$\varepsilon_{p, s_0} \leq \lambda_p^b \quad (7.5)$$

for any  $p \in \mathbb{N}$  provided these inequalities hold for  $p = 0$ . Also, there exists  $\Omega^{\text{ref}}(\sigma_0, M, m, \mu, \bar{\mu}, g, \sigma, \kappa) > 0$  such that one can choose  $\gamma_0^{\text{ref}} = (a+1) \times \Omega^{\text{ref}}$ .

*Proof.* In view of (7.2) where we make  $s = 0$ ,  $\bar{s} = s_0$  and  $s = s_0$ ,  $\bar{s} = s_0$  we just have to check that

$$\lambda_p^a \varepsilon_{p, 0} \leq 1$$

and

$$3 \times \bar{C}_{s_0} \lambda_p^a \lambda_p^{-(1+\sigma_0)\gamma_0} \leq \lambda_p^{-\gamma_0(1+\sigma)}, \quad 3 \times \bar{C}_{s_0} \lambda_p^{a+Ms_0} \lambda_p^{-(1+\sigma_0)\gamma_0} \leq \lambda_p^{b(1+\sigma)}, \quad (7.6)$$

$$3 \times \bar{C}_{s_0} \lambda_p^a \lambda_p^{-2\gamma_0} \leq \lambda_p^{-\gamma_0(1+\sigma)}, \quad 3 \times \bar{C}_{s_0} \lambda_p^{a+ms_0} \lambda_p^{-\gamma_0} \lambda_p^b \leq \lambda_p^{b(1+\sigma)}, \quad (7.7)$$

$$3 \times \bar{C}_{s_0} \lambda_p^{a-\mu s_0} (\lambda_p^b + \lambda_p^{\bar{\mu} s_0 - \gamma_0}) \leq \lambda_p^{-\gamma_0(1+\sigma)}, \quad 3 \times \bar{C}_{s_0} \lambda_p^a (\lambda_p^b + \lambda_p^{\bar{\mu} s_0 - \gamma_0}) \leq \lambda_p^{b(1+\sigma)}, \quad (7.8)$$

These inequalities are satisfied if

$$a < \gamma_0 \quad (7.9)$$

and

$$a < \gamma_0(\sigma_0 - \sigma) \quad Ms_0 - (1 + \sigma_0)\gamma_0 < b(1 + \sigma) - a \quad (7.10)$$

$$a < \gamma_0(1 - \sigma) \quad ms_0 - \gamma_0 < b\sigma - a \quad (7.11)$$

$$\begin{cases} a + b < \mu s_0 - \gamma_0(1 + \sigma) \\ a < (\mu - \bar{\mu})s_0 - \sigma\gamma_0 \end{cases} \quad \begin{cases} a < b\sigma \\ \bar{\mu}s_0 - \gamma_0 < b(1 + \sigma) - a \end{cases} \quad (7.12)$$

provided  $L$  is larger than some  $L^{\text{ref}}(a, \sigma_0, M, m, \mu, \bar{\mu}, g, \sigma, \kappa, \gamma_0, \bar{C}_{s_0})$  where in the limit of large  $\gamma_0$  it is possible to take  $L^{\text{ref}} = \bar{C}_{s_0}$ . Also in the limit of large  $\gamma_0$ , the previous conditions (7.9), (7.10)–(7.12) will be satisfied if

$$\kappa < \mu g - (1 + \sigma), \quad Mg - (1 + \sigma_0) < \kappa(1 + \sigma), \quad mg - 1 < \kappa\sigma, \quad \sigma < g(\mu - \bar{\mu}), \quad \bar{\mu}g - 1 < (1 + \sigma)\kappa$$

or equivalently if

$$\max\left(\frac{1 + \sigma + \kappa}{\mu}, \frac{\sigma}{\mu - \bar{\mu}}\right) < g < \min\left(\frac{(1 + \sigma)\kappa + (1 + \sigma_0)}{M}, \frac{1 + \kappa\sigma}{m}, \frac{1 + (1 + \sigma)\kappa}{\bar{\mu}}\right)$$

which follows from (7.1) and (7.3). For further records (see (7.15)), we impose in addition that  $\gamma_0$  is big enough so that

$$1 < \min\left(\frac{\mu g - \kappa - (a/\gamma_0)}{1 + \sigma}, \frac{(\mu - \bar{\mu})g - (a/\gamma_0)}{1 + \sigma}\right) \quad (7.13)$$

and

$$\frac{a}{\sigma\kappa} \leq \gamma_0. \quad (7.14)$$

It is clear that in order for (7.9), (7.10)–(7.12) and (7.13) to be satisfied it is enough to choose  $\gamma_0$  larger than some  $(a + 1) \times \Omega^{\text{ref}}(\sigma_0, M, m, \mu, g, \sigma, \kappa) > 0$ .  $\square$

**LEMMA 12.** *Let  $a > 0$ ,  $0 < \sigma < \sigma_0$  and  $\gamma_0 > \frac{a}{\sigma_0 - \sigma}$ . If  $\gamma_0 < \gamma < \frac{c}{1 + \sigma}$  and if  $u_p = O(\lambda_p^{-\gamma_0})$  satisfies*

$$u_{p+1} \leq C(\lambda_p^a u_p^{1 + \sigma_0} + \lambda_p^{-c})$$

for some constant  $C > 0$ , then  $u_p = O(\lambda_p^{-\gamma})$

*Proof.* We can assume  $0 < u_p < 1$ . Observe that

$$2C\lambda_p^a \lambda_p^{-\gamma(1 + \sigma_0)} \leq \lambda_p^{-\gamma(1 + \sigma)}, \quad 2C\lambda_p^{-c} \leq \lambda_p^{-\gamma(1 + \sigma)}$$

if  $p$  is big enough (since  $\gamma > a/(\sigma_0 - \sigma)$  and  $c/\gamma > 1 + \sigma$ ). Now

i) either for any  $p$  the inequality  $u_p > \lambda_p^{-\gamma}$  is true and then

$$\lambda_p^{-c} < u_p^{c/\gamma}.$$

We have

$$u_{p+1} \leq 2C\lambda_p^a u_p^{\min((c/\gamma), (1 + \sigma_0))}.$$

If we define  $\nu_p$  by  $u_p = \lambda_p^{-\nu_p}$  we see that

$$\nu_{p+1} \geq \frac{\min((c/\gamma), (1 + \sigma_0))}{1 + \sigma} \nu_p - \frac{a}{1 + \sigma} - \frac{\log(2C)}{(1 + \sigma)^{p+1} \log L}.$$

We now introduce  $\sigma < \rho < \sigma_0$  and  $\delta > 0$  such that  $\frac{c}{1+\sigma} > \gamma_0 > \frac{a+\delta}{\rho-\sigma}$ ; it is plain that for  $p \geq p_0$  big enough (remember that by assumption  $\inf_p \nu_p > 0$  and  $1 + \rho < \min((c/\gamma), (1 + \sigma_0))$ ) one has

$$\nu_{p+1} \geq \frac{1+\rho}{1+\sigma} \nu_p - \frac{a+\delta}{1+\sigma},$$

and thus

$$\nu_{p+1} - \frac{a+\delta}{\rho-\sigma} \geq \left( \frac{1+\rho}{1+\sigma} \right)^{p-p_0} \left( \nu_{p_0} - \frac{a+\delta}{\rho-\sigma} \right).$$

Since by assumption  $\liminf \nu_p \geq \gamma_0$ , this implies  $u_p = O(\lambda_p^{-\infty})$ .

ii) or there exists  $p_k \rightarrow \infty$  such that  $u_{p_k} \leq \lambda_{p_k}^{-\gamma}$  and then the induction can be initiated.  $\square$

The next lemma is similar to but easier than Lemma 12 so we leave the proof to the reader.

LEMMA 13. *Let the sequence  $u_p \geq 0$  satisfy*

$$u_{p+1} \leq C(\lambda_p^{-\gamma_1} u_p + \lambda_p^{-\gamma_2})$$

for some  $C > 0$ ,  $\gamma_2 > 0$ ,  $\gamma_1 \in \mathbb{R}$ .

(a) If  $\gamma_1 < 0$  then  $u_p = O(\lambda_p^b)$  for any  $b > |\gamma_1|/\sigma$ .

(b) If  $\gamma_1 > 0$  then  $u_p = O(\lambda_p^{-b})$  for any  $b < \min(|\gamma_1|/\sigma, \gamma_2/(1+\sigma))$ .

The next lemma shows that in fact we can improve the estimates on  $\varepsilon_{p,0}$  significantly without altering the one on  $\varepsilon_{p,s_0}$  but rather extending it to  $\varepsilon_{p,s}$  for any  $s$ .

LEMMA 14. *In view of (7.1) let us choose*

$$0 < \omega < \min\left(\frac{1+\sigma_0}{Mg} - 1, \frac{1}{\bar{\mu}g} - 1, \frac{1}{mg} - 1\right)$$

and define for  $k \geq 1$  sequences  $\gamma_k, s_k$  such that

$$s_k = [(1+\omega)s_{k-1}], \quad \gamma_k = \frac{s_k}{g}.$$

Then, for any  $p \in \mathbb{N}$

$$(P_k) \quad \varepsilon_{p,0} = O(\lambda_p^{-\gamma_k}), \quad \varepsilon_{p,s_k} = O(\lambda_p^b).$$

*Proof.* We prove by induction on  $k$  that  $(P_k)$  holds for  $k$  assuming it is true for  $k-1 \geq 0$  (the case  $k=0$  is the content of Lemma 11). Observe that the sequences  $\gamma_k, s_k$  are increasing. By making  $s = \bar{s} = s_k$  in (7.2)

$$\varepsilon_{p+1,s_k} \leq C_{s_k} (\lambda_p^{a+Ms_k} \varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+Ms_k} \varepsilon_{p,0} \varepsilon_{p,s_k} + \lambda_p^a (\varepsilon_{p,s_k} + \lambda_p^{\bar{\mu}s_k} \varepsilon_{p,0}))$$

and since  $(P_{k-1})$  holds  $\lambda_p^{a+Ms_k} \varepsilon_{p,0}^{1+\sigma_0} = O(\lambda_p^{a+Ms_k-(1+\sigma_0)\gamma_{k-1}})$  with

$$a + Ms_k - (1+\sigma_0)\gamma_{k-1} = a + (Mg(1+\omega) - (1+\sigma_0))\gamma_{k-1} < 0,$$

since  $\gamma_k > \gamma_0$ ,  $Mg(1+\omega) < 1+\sigma_0$  and  $\gamma_0$  was chosen big enough. We have furthermore

$$\bar{\mu}s_k - \gamma_{k-1} = (\bar{\mu}g(1+\omega) - 1)\gamma_{k-1} < 0$$

since  $\bar{\mu}g(1+\omega) < 1$ . The inductive hypothesis  $(P_{k-1})$  also yields that  $\lambda_p^{a+Ms_k} \varepsilon_{p,0} \varepsilon_{p,s_k} = O(\lambda_p^{a+Ms_k-\gamma_{k-1}} \varepsilon_{p,s_k})$  and it follows from  $mg(1+\omega) < 1$  that  $a + Ms_k - \gamma_{k-1} =$

$a + (mg(1 + \omega) - 1)\gamma_{k-1} < a$ . Finally, Lemma 13(a) applies and gives  $\varepsilon_{p,s_k} = O(\lambda_p^b)$  with  $b > a/\sigma$ , hence for  $b = \kappa\gamma_0$  due to (7.14).

Also making  $s = 0, \bar{s} = s_k$  in (7.2)

$$\varepsilon_{p+1,0} \leq C_{s_k}(\lambda_p^a \varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^a \varepsilon_{p,0}^2 + \lambda_p^{a-\mu s_k}(\varepsilon_{p,s_k} + \lambda_p^{\bar{\mu} s_k} \varepsilon_{p,0}))$$

hence

$$\varepsilon_{p+1,0} = O(\lambda_p^a \varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+b-\mu s_k} + \lambda_p^{\bar{\mu} s_k - \gamma_{k-1} + a - \mu s_k}).$$

In order to be able to apply Lemma 12 we have to check that

$$\gamma_0 < \gamma_k < \frac{\mu s_k - (b+a)}{1+\sigma}, \quad \gamma_0 < \gamma_k < \frac{(\mu - \bar{\mu})s_k + \gamma_{k-1} - a}{1+\sigma}$$

or equivalently

$$\gamma_0 < \gamma_k < \min\left(\frac{\mu g \gamma_k - \kappa \gamma_0 - a}{1+\sigma}, \frac{(\mu - \bar{\mu})g \gamma_k + \gamma_{k-1} - a}{1+\sigma}\right). \quad (7.15)$$

But the condition (7.13) ensures that this is the case. Lemma 12 then applies.  $\square$

We now conclude the proof of Theorem 10. Fix  $s > 0$ . Making  $\bar{s} = s_k$  in (7.2) we get

$$\varepsilon_{p+1,s} \leq C_s(\lambda_p^{a+Ms_k} \varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+ms} \varepsilon_{p,0} \varepsilon_{p,s} + \lambda_p^{a-\mu(s_k-s)}(\varepsilon_{p,s_k} + \lambda_p^{\bar{\mu} s_k} \varepsilon_{p,0}))$$

From Lemma 14 we know that for any  $\Gamma > 0$ ,  $\varepsilon_{p,0} = O(\lambda_p^{-\Gamma})$ , hence

$$\varepsilon_{p+1,s} = O\left(\lambda_p^{a+ms-\Gamma} \varepsilon_{p,s} + \lambda_p^{\max(a+Ms_k-\Gamma(1+\sigma_0), a-\mu(s_k-s)+b), a-\mu(s_k-s)+\bar{\mu} s_k-\Gamma}\right).$$

Now, we can choose  $\Gamma_k$  such that (remember  $\lim_{k \rightarrow \infty} s_k = \infty$ )

$$\inf_k \max\left(a-\mu(s_k-s)+b, a+Ms_k-\Gamma_k(1+\sigma_0), a+ms-\Gamma_k, a-(\mu-\bar{\mu})s_k+\mu s-\Gamma_k\right) = -\infty$$

Lemma 13(b) then implies that  $\varepsilon_{p,s} = O(\lambda_p^{-\infty})$  for any  $s \in \mathbb{N}$ .  $\square$

We now assume that  $a > 0$ ,  $0 < \sigma_0 \leq 1$ ,  $M, m > 0$  are fixed and we assume that we are given sequences  $(\mu_p), (\bar{\mu}_p)$  with  $0 < 2\bar{\mu}_p < \mu_p$  taking a finite number of values (to simplify) and such that for any  $p$

$$\frac{1}{\mu_p - \bar{\mu}_p} < \min\left(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar{\mu}_p}\right). \quad (7.16)$$

Let now  $g, \sigma$   $0 < \sigma < \sigma_0$  such that

$$\frac{1}{\mu_p - \bar{\mu}_p} < g < \min\left(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\bar{\mu}_p}\right), \quad 1+\sigma < (\mu_p - \bar{\mu}_p)g.$$

As before, for any  $s, \bar{s}$ , let  $C_{s,\bar{s}} : [0, \infty) \rightarrow [1, \infty]$  be a family of continuous functions on  $[0, 2]$  such that  $C_{s,\bar{s}}(t) = \infty$  if  $t > 2$ , increasing with respect to  $s, \bar{s} \geq 0$  and let  $\tilde{C}_s = \max_{t \leq 2} C_{s,s}(t)$ .

The following extension of Theorem 10 is clear from the previous proof.

**COROLLARY 4.** *There exist  $s_0 > 0$  such that if  $\varepsilon_{p,s}$  is a double sequence satisfying for any  $s, \bar{s}, p \in \mathbb{N}$*

$$\varepsilon_{p+1,s} \leq C_{s,\bar{s}}(1 + \lambda_p^a \varepsilon_{p,0}) \times (\lambda_p^{a+Ms} \varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+ms} \varepsilon_{p,s} \varepsilon_{p,0} + \lambda_p^{a-(\bar{s}-s)\mu_p}(\varepsilon_{p,\bar{s}} + \lambda_p^{\bar{\mu}_p \bar{s}} \varepsilon_{p,0})) \quad (7.17)$$

where  $\lambda_p = L^{(1+\sigma)^p}$ ,  $L = \bar{C}_{s_0}$  and if

$$\varepsilon_{0,0} \leq (\bar{C}_{s_0})^{-\frac{s_0}{g}}, \quad \varepsilon_{0,s_0} \leq 1$$

then, for any  $s \in \mathbb{N}$   $\varepsilon_{p,s} = O(\lambda_p^{-\infty})$ . Also,  $s_0$  does not depend on the sequence  $(\bar{C}_s)_s$  and can be taken of the form  $(a+1)\xi(\sigma_0, M, m, \mu, \bar{\mu})$ .

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