SMOOTH MIXING FLOWS WITH PURELY SINGULAR SPECTRA.

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Abstract. We give a geometric criterion that implies a singular maximal spectral type for a dynamical system on a Riemannian manifold. The criterion, that is based on the existence of fairly rich but localized periodic approximations, is compatible with mixing. Indeed, we check it for an *ad hoc* class of smooth mixing flows on $\mathbb{T}^3$ obtained from linear flows by time change, thus providing natural examples of mixing smooth diffeomorphisms and flows with purely singular spectra.

1. Introduction

1.1. Mixing is one of the principal characteristics of the stochastic behavior of dynamical systems. It is a spectral property and in the great majority of studied cases it is a consequence of much stronger properties of the system, such as the K-property or fast correlation decay, that imply a Lebesgue spectrum for the associated unitary operator.

The only previously known examples where mixing of the system was accompanied by singular spectrum of the associated unitary operator were obtained in an abstract measure theoretical or probabilistic frame, such as Gaussian and related systems which by their nature do not come from differentiable dynamics, or rank one and mixing constructions which do not have yet C^∞ realizations. In this paper, we solve the problem of smooth realizations of mixing with singular spectrum by proving the following

Theorem. *There exist on $\mathbb{T}^d$, $d \geq 3$, volume preserving flows of class C^∞ that are mixing and have purely singular spectra.*

The paper has two parts. In the first one, we introduce an abstract criterion that implies the singularity of the spectrum for a discrete time dynamical system on a Riemannian manifold. The criterion, that is based on the existence of fairly rich families of almost periodic sets, is still compatible with mixing, albeit at a slow rate. We state the criterion in §1.2 below and prove it in Section 2.

In the second part of the paper, Section 3, we rely on this criterion, and on the mechanism of mixing used in [2], to obtain smooth mixing reparametrizations of some Liouvillean linear flows on $\mathbb{T}^3$ that display a purely singular spectrum. It is a general fact that in this case the flow itself must have a purely singular spectrum. Further, as a by-product, we observe by Host's theorem [7] that the latter mixing reparametrizations, because they have a purely singular spectrum, are actually mixing of all orders. Finally, apart from giving a positive answer to the problem of smooth realizations we have posed, our constructions shed some new light to the study of reparametrizations of linear flows on tori of which we give a brief historical account in §1.3.

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1.2. **Periodic approximations and singular spectra.**

We consider dynamical systems (T, M, μ) that are given by a Lebesgue space (M, μ) and an automorphism T on it, that is, a bi-measurable μ-preserving bijection of M. The unitary operator U_T associated to the system (T, M, μ) acts on the Hilbert space $H = L^2(M, \mu, \mathbb{C})$ by $U_T f = f \circ T^{-1}$. Since the operator U_T always has an eigenvalue equal to 1 represented by the contant functions, we usually mean by the spectral properties of T those properties of U_T when restricted on the subspace $H_0 = L_0^2(M, \mu, \mathbb{C})$ of functions with zero integral. This applies in particular to the notion of countable Lebesgue spectrum mentionned in the introduction.

We recall that to every $f \in H$ is associated a (spectral) measure σ_f defined on the circle $\mathbb{S}$ *via* the Fourier transform

$$(U_T^n f, f) = \int_{\mathbb{S}} e^{i2\pi\theta} d_{\sigma_f}(\theta).$$

The maximal spectral type of T is the supremum of the types of all the measures σ_f for all $f \in H$, or all $f \in H_0$ if we want to discard the constant functions as explained above. Regardless to this distinction, T is said to have a purely singular spectrum if its maximal spectral type is singular. Since every type which is absolutely continuous with respect to the maximal spectral type appears as the type of a measure σ_f for some $f \in H$, T has a purely singular spectrum if there is no function with an absolutely continuous spectral measure (with respect to the Lebesgue measure on the circle). Since constant function contribute only with a Dirac mass at 0, it is enough to consider only $f \in H_0$.

A basic property implying the singularity of the spectrum of (T, M, μ) is *rigidity*, i.e. the existence of a sequence of times t_n such that for any measurable set $A \subset M$ it holds that $\mu(T^{t_n} A \triangle A) \to 0$ where the notation $A \triangle B$ stands for the symmetric difference between the sets A and B. For known smooth systems, the latter property is often obtained as a consequence of a stronger one, namely the existence of *good cyclic approximations* in the sense of Katok and Stepin: a system (T, M, μ) is said to have good cyclic approximations if there exist a sequence ξ_{q_n} of partitions of (M, μ) into sets of equal measure C_n^i, $i = 1, \dots, q_n$, and cyclic permutations S_{q_n} of these sets such that

$$\sum_{i=1}^{q_n} \mu(T C_{q_n}^i \triangle S_{q_n} C_{q_n}^i) = o(1/q_n).$$

If T admits good cyclic approximation then T is ergodic and rigid (see for example the original paper [11] or [10] for a definitive account of the general concept of periodic approximations and its applications to the study of various ergodic and spectral properties).

Rigidity of (T, M, μ) is clearly not compatible with mixing. To get a criterion that guarantees a singular spectrum without precluding mixing, we relax the concept of periodic approximations to that of having strongly periodic towers with nice levels (balls) such that on one hand the total measure of the levels in a given tower might tend to zero, but on the other one any measurable set can be approximated by unions of levels from possibly different towers. Such *localized* (as opposed to exhaustive above) periodic approximations are not incompatible with mixing.

DEFINITION (Slowly coalescent periodic approximations). Let T be an ergodic transformation of a Riemannian manifold M preserving a volume μ. We say that the dynamical system (T, M, μ) displays *slowly coalescent periodic approximations*, if there exist a sequence of integers $k_n \in \mathbb{N}^*$, and a sequence ϵ_n of positive numbers with $\sum \epsilon_n < +\infty$, such that for every $n \in \mathbb{N}$ there exists a sequence

$$\mathcal{C}_n = \bigcup_{i \in \mathbb{N}} B_{n,i}$$

where the $B_{n,i}$, $i = 0, \dots,$ are *balls* of M satisfying

(i) $\sup_{i\in\mathbb{N}} r(B_{n,i}) \underset{n\longrightarrow\infty}{\longrightarrow} 0,$

(ii) $\mu\left(T^{k_n} B_{n,i} \triangle B_{n,i}\right) \leq \epsilon_n \mu(B_{n,i}),$

(iii) $\mu\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \mathcal{C}_n\right) = 1.$

In Section 2 we will prove the following theorem

THEOREM (Criterion for the singularity of the spectrum). *A dynamical system* (T, M, μ) *displaying slowly coalescent periodic approximations has a purely singular spectral type.*

REMARK 1. In general, $\mu(\mathcal{C}_n)$ need not converge to zero. For a rotation of the circle, for example, it can be chosen so that it tends on the contrary to 1. For a mixing system (T, M, μ) however, (ii) implies that $\mu(\mathcal{C}_n) \to 0$[1] and this is what we refer to by *coalescent*. The terminology *slowly coalescent* is then used to refer to property (iii) that is the key property in guaranteeing a purely singular spectrum. We will abbreviate slowly coalescent periodic approximations with SCPA.

REMARK 2. The condition $\sum \epsilon_n < +\infty$ can be viewed as a condition on the *speed* of the localized periodic approximations. It is crucial in the proof of the theorem, namely in combining proposition 2.2 and lemma 2.5.

REMARK 3. If the sets $\mathcal{C}_n$ satisfy adequate independence conditions, (iii) will follow from the Borel Cantelli Lemma if $\sum_{n\in\mathbb{N}} \mu(\mathcal{C}_n) = +\infty$.

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1.3. Spectral type of reparametrized linear flows.

The problem of understanding the ergodic and spectral properties of reparametrizations of linear flows on tori was raised by A. N. Kolmogorov in his I.C.M. address of 1954 [15]. Since then and starting with the work of Kolmogorov himself, this problem has been intensively studied and a surprisingly rich variety of behaviors were discovered to be possible for the reparametrized flows. We say surprisingly because at the time when Kolmogorov raised the problem, some strong restrictions on the spectral type of the reparametrized flow were expected to hold, at least in the case of real analytic reparametrizations, Cf. [15] as well as the appendix by Fomin to the Russian version of the book of Halmos on ergodic theory, where absence of mixed spectrum was conjectured for smooth reparametrizations of linear flows.

[1]It is not true in general that for a mixing system and a sequence of measurable sets C_n such that $T^{k_n} C_n \triangle C_n \to 0$, then $\mu(C_n) \to 0$ or $\mu(C_n) \to 1$. But in our case $\mathcal{C}_n$ is a union of balls with radii converging uniformly to 0, hence if $\limsup \mu(\mathcal{C}_n) > \epsilon > 0$ and if we fix $p = [\frac{2}{\epsilon}]$ disjoint open subsets in M, $M_1, \dots, M_p$, of equal measure $1/p$ there will be at least one (say M_1) that must intersect $\mathcal{C}_n$, for infinitely many n, in a set of measure greater than ϵ/p that is almost a union of balls $B_{n,i}$ so as to force $\limsup \mu(T^{k_n} M_1 \cap M_1) \geq \epsilon/p > \mu(M_1)^2$ which contradicts the mixing property.

We denote by R_α^t the linear flow on the torus $\mathbb{T}^n$ given by

$$\frac{dx}{dt} = \alpha,$$

where $x \in \mathbb{T}^n$ and α is a vector of $\mathbb{R}^n$. Given a continuous function $\phi : \mathbb{T}^n \to \mathbb{R}_+^*$ we define the reparameterization flow $T_{\alpha,\phi}^t$ by

$$\frac{dx}{dt} = \frac{\alpha}{\phi(x)}.$$

If the coordinates of α are rationally independent then the linear flow R_α^t is uniquely ergodic and so is $T_{\alpha,\phi}^t$ that preserves the measure with density ϕ. Other properties of the linear flow may change under reparameterization. While the linear flow has a pure discrete spectrum with the group of eigenvalues isomorphic to $\mathbb{Z}^n$, a continuous time change may yield a wide variety of spectral properties. This follows from the theory of monotone (Kakutani) equivalence [9] and the fact that every monotone measurable time change is cohomologous to a continuous one [16]. However, for sufficiently smooth reparameterizations the possibilities are more limited and they depend on the arithmetic properties of the vector α.

If α is Diophantine and the function ϕ is C^∞, then the reparameterized flow is smoothly isomorphic to a linear flow. This was first noticed by A. N. Kolmogorov [15]. Herman found in [6] sharp results of that kind for the finite regularity case. Kolmogorov also showed that for a Liouville vector α a smooth reparametrization could be weak mixing, or equivalently the associated unitary operator to the reparametrized flow could have a continuous spectrum.

M. D. Šklover proved in [17] the existence of real-analytic weak mixing reparametrizations of some Liouvillean linear flows on $\mathbb{T}^2$; his result being optimal in that he showed that for any real-analytic reparametrization ϕ other than a trigonometric polynomial there is α such that $T_{\alpha,\phi}^t$ is weak mixing. In [1], it was shown that for any Liouvillean translation flow R_α^t on the torus $\mathbb{T}^n$, $n \geq 2$, the generic C^∞ reparametrization of R_α^t is weak mixing.

Continuous and discrete spectra are not the only possibilities. In [3] it was proved that for every $\alpha \in \mathbb{R}^2$ with a Liouvillean slope there exists a strictly positive C^∞ function ϕ such that the flow $T_{\alpha,\phi}^t$ on $\mathbb{T}^2$ has a mixed spectrum since it has a discrete part generated by only one eigenvalue. They also construct real-analytic examples for a more restricted class of Liouvillean α. Recently, M. Guenais and F. Parreau [5] achieved real-analytic reparametrizations of linear flows on $\mathbb{T}^2$ that have an arbitrary number of eigenvalues. They also construct an example of a reparametrization of a linear flow on $\mathbb{T}^2$ that is isomorphic to a linear flow on $\mathbb{T}^2$ with "exotic" eigenvalues, i.e. not in the span of the eigenvalues of the original linear flow. Finally, there exist real-analytic functions ϕ that are not trigonometric polynomials, and for which a mixed spectrum is precluded for the flow $T_{\alpha,\phi}^t$ for any choice of α. Indeed, it was proven in [4] that for a class of functions satisfying some regularity conditions on their Fourier coefficients the following dichotomy holds: $T_{\alpha,\phi}^t$ either has a continuous spectrum or is L^2 isomorphic to a constant time suspension.

Reparametrizations and mixing. In [8], Katok showed that any reparametrization of an irrational flow on the two torus with a function of class C^5 has a simple spectrum, a singular maximal spectral type, and thus cannot be mixing. Absence of mixing was extended by A. V. Kočergin to Lipschitz reparametrizations [12]. The argument relies on a Denjoy–Koksma type estimate which was proven to fail in higher dimension by Yoccoz [19]. Based on the latter fact, we showed in [2] that there exist $\alpha \in \mathbb{R}^3$ and a real-analytic strictly positive function ϕ defined on $\mathbb{T}^3$, such that the reparametrized flow $T_{\alpha,\phi}^t$ is mixing. The construction easily extends to higher dimensional tori (cf. [2, Theorem 3]).

The mixing examples obtained by reparametrizations of linear flows belong to a variety of fairly slow mixing systems, also including the mixing flows with singularities constructed on surfaces by Kočergin in the seventies [13], for which the type and the multiplicity of the spectrum remain undetermined.

Modifying the reparametrizations of [2, Theorem 1, Theorem 3], it is possible to maintain mixing while the time one map of the reparametrized flow is forced to satisfy the SCPA criterion stated above. But the singularity of the maximal spectral type of any time map implies that of the flow, thus yielding

THEOREM. *For $d \geq 3$, there exist $\alpha \in \mathbb{R}^d$ and a strictly positive function ϕ over $\mathbb{T}^d$ of class C^∞ such that the reparametrized flow $T^t_{\alpha,\phi}$ is mixing and has a singular maximal spectral type with respect to the Lebesgue measure.*

A dynamical system (T, M, μ) (or flow (T^t, M, μ)) is said to be mixing of order $l \geq 2$ if, for any sequence $(u_n^{(1)}, \cdots, u_n^{(l-1)})_{\{n\in\mathbb{N}\}}$, where for $i = 1, \cdots, l-1$ the $(u_n^{(i)})_{\{n\in\mathbb{N}\}}$ are sequences of integers (or real numbers) such that $\lim_{n\to\infty} u_n^{(i)} = \infty$, and for any l-uple $(A_1, \cdots, A_l)$ of measurable subsets of M, we have

$$\lim_{n\to\infty} \mu\left(T^{-u_n^{(1)}-\cdots-u_n^{(l-1)}} A_l \cap \cdots \cap T^{-u_n^{(1)}} A_2 \cap A_1\right) = \mu(A_{l-1}) \cdots \mu(A_1).$$

The general definition of mixing corresponds to mixing of order 2. A system is said to be mixing of all orders if it is mixing of order l for any $l \geq 2$. Host's theorem [7] asserts that a mixing system with singular spectrum is mixing of all orders, hence we get

COROLLARY. *For $d \geq 3$, there exist $\alpha \in \mathbb{R}^d$ and a strictly positive function ϕ over $\mathbb{T}^d$ of class C^∞ such that the reparametrized flow $T^t_{\alpha,\phi}$ is mixing of all orders.*

Acknowledgments. I am grateful to Jean-Paul Thouvenot for several enlightening discussions and explanations in spectral theory that helped me find the criterion of Theorem 1.2, and to Anatole Katok and the referee for helping me state it in its actual general form. I also thank Arthur Avila, Raphaël Krikorian and François Ledrappier for useful conversations. I am indebted to the referee for many improvements that he suggested.

2. Slowly coalescent periodic approximations

In this section we prove Theorem 1.2.

2.1. We will state now a general criterion that guarantees a singular spectrum for (T, M, μ). Although this will not be the criterion that we will use to prove that systems having SCPA have a singular spectrum it is of interest by itself and is similar, yet more general, to the *ad hoc* one that we will use and that will be stated in the next paragraph.

PROPOSITION. *Let (T, M, μ) be a dynamical system. If for any complex nonzero function $f \in L^2_0(M, \mu)$, i.e. $\int_M f(x)d\mu(x) = 0$, there exists a measurable set $E \subset M$ with $\mu(E) > 0$, and a strictly increasing sequence l_n, such that for every $x \in E$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \left| \sum_{i=0}^{n-1} f(T^{l_i} x) \right| > 0 \tag{2.1}$$

then the maximal spectral type of the unitary operator associated to (T, M, μ) is singular.

Proof. Assume that T has an absolutely continuous component in its spectrum. Then there exists $f \in L^2_0(M, \mu)$ such that the spectral measure corresponding to f on the circle $\mathbb{S}$ can be written as $\sigma_f(dx) = g(x)dx$ where $g \in L^1(\mathbb{S}, \mathbb{R}_+, dx)$ is bounded. With the notation

$$S_n f(x) = \sum_{i=0}^{n-1} f(T^{l_i}(x))$$

we write spectrally

$$\begin{aligned} \left\| \frac{S_n f}{n} \right\|_{L^2}^2 &= \frac{1}{n^2} \int_{\mathbb{S}} \left| \sum_{i=0}^{n-1} z^{l_i} \right|^2 g(z) dz \\ &\leq \frac{\sup_{z\in\mathbb{S}} g(z)}{n^2} \int_{\mathbb{S}} \left| \sum_{i=0}^{n-1} z^{l_i} \right|^2 dz \\ &\leq \frac{\sup_{z\in\mathbb{S}} g(z)}{n}. \end{aligned}$$

From this we deduce that $S_{n^2} f/n^2$ converges to zero for almost every $x \in M$. Furthermore, a similar computation as above gives for $n^2 \leq l \leq (n+1)^2 - 1$ that

$$\| \frac{1}{l}(S_l f - S_{n^2} f) \|_{L^2}^2 \leq \sup_{z\in\mathbb{S}} g(z) \frac{l - n^2}{l^2},$$

hence

$$\sum_{n\geq 0} \sum_{l=n^2}^{(n+1)^2-1} \| \frac{1}{l}(S_l f - S_{n^2} f) \|_{L^2}^2 < +\infty.$$

By Fatou's Lemma we conclude that for almost every $x \in M$ we have $S_n f(x)/n \underset{n\to\infty}{\longrightarrow} 0$, in contradiction with (2.1).[2] □

2.2. Even simpler than Proposition 2.1 and yet more adapted to our purpouse is the following

PROPOSITION. *Let (T, M, μ) be a dynamical system. If for any complex nonzero function $f \in L_0^2(M, \mu)$, there exist a measurable set $E \subset M$ with $\mu(E) > 0$, a sequence $k_n \in \mathbb{N}^*$ and a sequence $\tau_n \in \mathbb{N}^*$ with $\sum(1/\tau_n) < +\infty$, such that for every $x \in E$ we have*

$$\limsup_{n\to\infty} \frac{1}{\tau_n} \left| \sum_{i=0}^{\tau_n - 1} f(T^{ik_n} x) \right| > 0, \tag{2.2}$$

then the maximal spectral type of the unitary operator associated to (T, M, μ) is singular.

Proof. If σ_f has a bounded density, then $\| \sum_{i=0}^{\tau_n-1} f \circ T^{ik_n} \|_{L^2}^2 \leq C\tau_n$ and the fact that $\sum(1/\tau_n) < +\infty$ then gives a contradiciton with (2.2). □

REMARK. Clearly, it is enough to prove the above proposition for all real-valued functions $f \in L_0^2(M, \mu)$.

[2]The proof of this proposition is similar to the proof of the strong law of large numbers in the case of independent random variables with bounded L^2 norms.

2.3. In the sequel we will assume that (T, M, μ) satisfies $(i) - (iii)$ of Definition 1.2. We fix hereafter a sequence τ_n of integers such that $\epsilon_n \tau_n \to 0$ while $\sum (1/\tau_n) < +\infty$ (this is possible since $\sum \epsilon_n < +\infty$). We fix an arbitrary nonzero real-valued function $f \in L_0^2(M, \mu)$. For $\varepsilon > 0$ we define the set

$$D_\varepsilon = \{x \in M \,|\, f(x) \geq 2\varepsilon\}.$$

Since $f \in L_0^2(M, \mu)$ is not null, there exists $\varepsilon_0 > 0$ such that $\mu(D_{\varepsilon_0}) > 0$. From Proposition 2.2, Theorem 1.2 will hold proved if we show that:

PROPOSITION. *For μ a.e. point $x \in D_{\varepsilon_0}$, there exists infinitely many integers n such that*

$$\frac{1}{\tau_n} \sum_{i=0}^{\tau_n - 1} f(T^{ik_n} x) \geq \varepsilon_0. \tag{2.3}$$

2.4. The following proposition states that for every $N > 0$, the set of $x \in D_{\varepsilon_0}$ for which (2.3) fails for all $n \geq N$ has zero measure, which implies Proposition 2.3.

PROPOSITION. *For every $N > 0$ and every measurable set $D \subset D_{\varepsilon_0}$ with $\mu(D) > 0$ there exist $n \geq N$ and $x \in D$ such that (2.3) holds.*

2.5. To prove the proposition, we will need the following Lemma where we let $f_0 = \min(f, 2\varepsilon_0)$:

LEMMA. *There exists N_0 such that if $n \geq N_0$ and B_n is a set satisfying (ii) of Theorem 1.2 and*

$$\int_{B_n} f_0(x) d\mu(x) \geq \frac{3}{2} \varepsilon_0 \mu(B_n)$$

then there exists a set $\overline{B}_n \subset B_n$ with $\mu(\overline{B}_n) \geq \mu(B_n)/5$ such that (2.3) holds with n for every $x \in \overline{B}_n$.

Proof. Let B_n and k_n be as in (ii) of Theorem 1.2. For $x \in M$, we use in this proof the notation

$$S_n f(x) := \sum_{i=0}^{\tau_n - 1} f(T^{ik_n} x).$$

Define

$$\tilde{B}_n = \bigcup_{i=0}^{\tau_n - 1} T^{-ik_n} B_n, \qquad \hat{B}_n = \bigcap_{i=0}^{\tau_n - 1} T^{-ik_n} B_n.$$

Clearly $\hat{B}_n \subset B_n \subset \tilde{B}_n$. Notice that, if $x \in \tilde{B}_n \setminus \hat{B}_n$, then $x \in (T^{-ik_n} B_n) \triangle (T^{-(i+1)k_n} B_n)$ for some i, $0 \leq i < \tau_n$. From (ii) we get

$$\mu(\tilde{B}_n \triangle \hat{B}_n) \leq \tau_n \epsilon_n \mu(B_n) \tag{2.4}$$

but $\tau_n \epsilon_n \to 0$ so that $\mu(\tilde{B}_n \triangle B_n)/\mu(B_n) \to 0$.

Define $\tilde{f}_0 = f_0$ on B_n and equal to zero otherwise. We then have

$$\int_{\tilde{B}_n} \frac{S_n \tilde{f}_0(x)}{\tau_n} d\mu(x) = \int_M \frac{S_n \tilde{f}_0(x)}{\tau_n} d\mu(x) = \int_M \tilde{f}_0(x) d\mu(x) = \int_{B_n} f_0 d\mu(x),$$

hence from our hypothesis

$$\int_{\tilde{B}_n} \frac{S_n \tilde{f}_0(x)}{\tau_n} d\mu(x) \geq \frac{3}{2} \varepsilon_0 \mu(B_n). \tag{2.5}$$

On the other hand, since $\tilde{f}_0 \leq 2\varepsilon_0$ we get

$$\int_{\tilde{B}_n} \frac{S_n \tilde{f}_0(x)}{\tau_n} d\mu(x) \leq \mu(\tilde{B}_n)\varepsilon_0 + 2\mu\left(\left\{x \in \tilde{B}_n \,\middle|\, \frac{S_n \tilde{f}_0(x)}{\tau_n} \geq \varepsilon_0\right\}\right)\varepsilon_0$$

which in light of (2.4) (with n sufficiently large so that $\tau_n \epsilon_n \leq 1/100$) and (2.5) leads to

$$\mu\left(\left\{x \in \tilde{B}_n \,\middle|\, \frac{S_n \tilde{f}_0(x)}{\tau_n} \geq \varepsilon_0\right\}\right) \geq (1/4 - 1/200)\,\mu(B_n),$$

which using (2.4) again yields

$$\mu\left(\left\{x \in \hat{B}_n \,\middle|\, \frac{S_n \tilde{f}_0(x)}{\tau_n} \geq \varepsilon_0\right\}\right) \geq \frac{1}{5}\mu(B_n),$$

which is the desired inequality since $S_n \tilde{f}_0$ and $S_n f_0$ coincide on $\hat{B}_n \subset B_n$. □

2.6. *Proof of Proposition* 2.4. Fix a measurable set $D \subset D_{\varepsilon_0}$ such that $\mu(D) > 0$. Fix $N \in \mathbb{N}$ and let $\overline{N} = \sup(N_0, N)$ where N_0 is as in Lemma 2.5.

By Vitali's Lemma and properties (i) and (iii), there exists a constant $0 < \vartheta < 1$ such that, given any ball B in M, we can find a family of balls $B_{n_i} \subset B$ such that

(P1) The B_{n_i} are disjoint;
(P2) Every B_{n_i} belongs to some $\mathcal{C}_n$ with $n \geq \overline{N}$;
(P3) $\mu(\cup B_{n_i}) \geq \vartheta \mu(B)$.

For $x \in D \subset D_{\varepsilon_0}$, we have $f_0 = 2\varepsilon_0$. Considering a Lebesgue density point we obtain, for any $\epsilon > 0$, a ball $B \subset M$ such that

(B1) $\mu(B \cap D) \geq (1-\epsilon)\mu(B)$;
(B2) $\int_B f_0(x) d\mu(x) \geq (2-\epsilon)\varepsilon_0 \mu(B)$.

We can now choose $\epsilon > 0$ arbitrarily small in (B1), (B2) and then apply (P1)-(P3) to the above ball B. Indeed, as ϵ is made closer to 0, (B1) implies that most of the balls given by (P1)-(P3) must satisfy $\mu(B_n \cap D) \geq (1 - 1/10)\mu(B_n)$. Similarly, (B2) and the fact that $f_0 \leq 2\varepsilon_0$ imply that most of the balls given by (P1)-(P3) must satisfy $\int_{B_n} f_0(x) d\mu(x) \geq (3/2)\varepsilon_0 \mu(B_n)$. We can hence obtain a ball $B_n \in \mathcal{C}_n$ with $n \geq \overline{N}$ satisfying both $\mu(B_n \cap D) \geq (1 - 1/10)\mu(B_n)$ and $\int_{B_n} f_0(x) d\mu(x) \geq (3/2)\varepsilon_0 \mu(B_n)$. We then conclude the proof using Lemma 2.5. □

3. Application: Slow mixing and singular spectrum

This section is devoted to the proof of Theorem 1.3.

3.1. **Reduction to special flows.**

Definition. (Special flows) Given a Lebesgue space L, a measure preserving transformation T on L and an integrable strictly positive real function defined on L we define the special flow over T and under the *ceiling function* φ by inducing on $M(L, T, \varphi) = L \times \mathbb{R}/\sim$, where $\sim$ is the identification $(x, s + \varphi(x)) \sim (T(x), s)$,the action of

$$\begin{array}{rcl} L \times \mathbb{R} & \rightarrow & L \times \mathbb{R} \\ (x, s) & \rightarrow & (x, s+t). \end{array}$$

If T preserves a unique probability measure λ, then the special flow will preserve a unique probability measure μ that is the normalized product measure of λ on the base and the Lebesgue measure on the fibers.

We will be interested in special flows above minimal translations $R_{\alpha,\alpha'}$ of the two torus and under smooth functions $\varphi(x,y) \in C^\infty(\mathbb{T}^2, \mathbb{R}_+^*)$ that we will denote by $T^t_{\alpha,\alpha',\varphi}$. For $r \in \mathbb{N} \bigcup \{+\infty\}$, we denote by $C^r(\mathbb{T}^2, \mathbb{R})$ the set of real functions on $\mathbb{R}^2$ of class C^r and $\mathbb{Z}^2$-periodic. We denote by $C^r(\mathbb{T}^2, \mathbb{R}_+^*)$ the set of strictly positive functions in $C^r(\mathbb{T}^2, \mathbb{R})$.

In all the sequel we will use the following notation: for $m \in \mathbb{N}$,

$$S_m\varphi(x,y) = \sum_{l=0}^{m-1} \varphi(x + l\alpha, y + l\alpha'). \tag{3.1}$$

With this notation, given $t \in \mathbb{R}_+$ we have for $z \in \mathbb{T}^2$

$$T^t(z,0) = \left(R^{N(t,z)}_{\alpha,\alpha'}(z), t - S_{N(t,z)}\varphi(z)\right)$$

where $N(t,z)$ is the largest integer m such that $t - S_m\varphi(x) \geq 0$, that is the number of fibers covered by z during its motion under the action of the flow until time t.

By the equivalence between special flows and reparametrizations Theorem 1.3, in the case of the three torus, follows if we prove

THEOREM. *There exist a vector $(\alpha, \alpha') \in \mathbb{R}^2$ and $\varphi \in C^\infty(\mathbb{T}^2, \mathbb{R}_+^*)$ such that the special flow $T^t_{\alpha,\alpha',\varphi}$ is mixing and satisfies $(i)-(iii)$ of Theorem 1.2, which implies that the spectral type of the flow is purely singular.*

The equivalence between the above theorem and Theorem 1.3 is standard and can be found in [2], Section 4. The generalization to any dimension $d \geq 3$ of the construction that will be described on the three torus is straightforward.

In the construction of the special flow $T^t_{\alpha,\alpha',\varphi}$, we will first choose a special translation vector on $\mathbb{T}^2$, then we will give two criteria on the Birkhoff sums of the special function φ above $R_{\alpha,\alpha'}$ that will guarantee mixing and SCPA. Finally, we will build a smooth function φ satisfying these criteria.

3.2. **Choice of the translation on $\mathbb{T}^2$.** Given a real number u, we will use the following notations: $[u]$ to indicate the integer part of u, $\{u\}$ its fractional part and $|||u|||$ its closest distance to integers. Let α be an irrational real number, then there exists a sequence of rationals $\{\frac{p_n}{q_n}\}_{n\in\mathbb{N}}$, called the best rational approximations of α, that satisfy

$$|||q_{n-1}\alpha||| \leq |||k\alpha|||, \quad \forall k < q_n \tag{3.2}$$

and for any $n \in \mathbb{N}$

$$\frac{1}{q_n(q_n + q_{n+1})} \leq (-1)^n(\alpha - \frac{p_n}{q_n}) \leq \frac{1}{q_n q_{n+1}}. \tag{3.3}$$

The numbers q_n are called the approximation denominators of α. We recall also that any irrational number $\alpha \in \mathbb{R} - \mathbb{Q}$ can be written as a continued fraction where $\{a_i\}_{i\geq 1}$ is a sequence of integers ≥ 1, $a_0 = [\alpha]$. Conversely, any sequence $\{a_i\}_{i\in\mathbb{N}}$ corresponds to a unique number α. The best approximations of α are given by the a_i in the following way:

$$p_n = a_n p_{n-1} + p_{n-2} \text{ for } n \geq 2, \qquad p_0 = a_0, \quad p_1 = a_0 a_1 + 1,$$
$$q_n = a_n q_{n-1} + q_{n-2} \text{ for } n \geq 2, \qquad q_0 = 1, \quad q_1 = a_1.$$

Following [19] and as in [2], we take α and α' with there approximation denominators q_n and q'_n satisfying for all $n \geq 1$

$$q'_n \geq e^{3q_n}, \tag{3.4}$$

$$q_{n+1} \geq e^{3q'_n}. \tag{3.5}$$

Vectors $(\alpha, \alpha') \in \mathbb{R}^2$ satisfying (3.4) and (3.5) are obtained by a suitable inductive choice of the sequences a_n and a'_n in their continued fraction expansions respectively. Moreover, it is easy to see that the set of vectors $(\alpha, \alpha') \in \mathbb{R}^2$ satisfying (3.4) and (3.5) is a continuum (Cf. [19], Appendix 1).

3.3. **Mixing criterion.** We will consider special flows $T^t_{\alpha,\alpha',\varphi}$ above $R_{\alpha,\alpha'}$ and use the same criterion implying mixing that we used in [2]. It is based on the uniform stretch of the Birkhoff sums $S_m\varphi$ (given by (3.1) of the ceiling function above the x or the y direction alternatively depending on whether m is far from q_n or from q'_n. In [2, Proposition 3], the key property underlying mixing, namely the alternated uniform stretch of $S_m\varphi$, is expressed in terms of properties of the first two derivatives of $S_m\varphi$, properties that are in their turn derived from simpler ones on the first derivatives [2, Proposition 4] and on the second derivatives [2, Proposition 5]. The property required in the latter is simply a linear bound $|D_{xx}S_m\varphi(x,y)| \leq Cm$ (as well as $|D_{yy}S_m\varphi(x,y)| \leq Cm$) which trivially follows from the fact that φ is of class C^2. It remains to state Proposition 4 of [2] and later check it for the ceiling function that we will consider:

PROPOSITION (Mixing Criterion). *Let (α, α') be as in (3.4) and (3.5) and $\varphi \in C^2(\mathbb{T}^2, \mathbb{R}^*_+)$. If for every $n \in \mathbb{N}$ sufficiently large, we have two sets I_n and I'_n, each one being equal to the circle minus two intervals whose lengths converge to zero, and if*

- *For any $y \in \mathbb{T}$ and any x such that $\{q_n x\} \in I_n$, for any $m \in \left[e^{2q_n}/2, 2e^{2q'_n}\right]$, we have*
$$|D_x S_m\varphi(x,y)| \geq \frac{m}{e^{q_n}}\frac{q_n}{n},$$
- *For any $x \in \mathbb{T}$ and any x such that $\{q'_n y\} \in I'_n$, for any $m \in \left[e^{2q'_n}/2, 2e^{2q_{n+1}}\right]$, we have*
$$|D_y S_m\varphi(x,y)| \geq \frac{m}{e^{q'_n}}\frac{q'_n}{n},$$

then the special flow $T^t_{\alpha,\alpha',\varphi}$ is mixing.

REMARK 4. In the proof of mixing given in [2], the factor q_n/n appears in the lower bound of $|D_x S_m\varphi|$ due to the specific form of the function φ considered there (see §3.5 below), but is not used in the proof of mixing. The same is true for its counterpart in the y direction. However, we keep them here since this will not require any additional difficulty in the construction of φ.

3.4. **Criterion for the existence of slowly coalescent periodic approximations.** We give now a condition on the Birkhoff sums of φ above $R_{\alpha,\alpha'}$ that is sufficient to ensure SCPA for the time one map of the flow $T^t_{\alpha,\alpha',\varphi}$ on $M = M(\mathbb{T}^2, R_{\alpha,\alpha'}, \varphi)$.

PROPOSITION. *If for n sufficiently large, we have for any x such that $1/2 - 1/(2n) + 1/n^2 \leq \{q_n x\} \leq 1/2 + 1/(2n) - 1/n^2$ and for any $y \in \mathbb{T}$*

$$\left|S_{q_n q'_n}\varphi(x,y) - q_n q'_n\right| \leq \frac{1}{e^{q_n}}, \tag{3.6}$$

then the time one map of the special flow $T^t_{\alpha,\alpha',\varphi}$ has slowly coalescent periodic approximations as in Definition 1.2.

Proof. Let C_n be the set of points $(x, y, s) \in M$ satisfying $1/2 - 1/(2n) + 2/n^2 \leq \{q_n x\} \leq 1/2 + 1/(2n) - 2/n^2$. It follows from the definition of special flows and (3.6) that for $(x, y, s) \in M$ such that $1/2 - 1/(2n) + 1/n^2 \leq \{q_n x\} \leq 1/2 + 1/(2n) - 1/n^2$ we have

$$T^{q_n q_n'}(x, y, s) = \left(x + q_n q_n' \alpha, y + q_n q_n' \alpha', s + S_{q_n q_n'} \varphi(x, y) - q_n q_n'\right)$$

but from (3.3) we have that $|||q_n q_n' \alpha||| \leq q_n'/q_{n+1} = o(e^{-q_n})$ as well as $|||q_n q_n' \alpha'||| \leq q_n/q_{n+1}' = o(e^{-q_n})$. Therefore (3.6) implies that $d(T^{q_n q_n'}(x, y, s), (x, y, s)) \leq 2e^{-q_n}$. It is therefore possible to cover C_n with a collection of balls $\mathcal{C}_n$ such that each ball $B \in \mathcal{C}_n$ has radius less than $1/(nq_n)$ and satisfies $\mu\left(T^{q_n q_n'} B \triangle B\right) \leq e^{-n}\mu(B)$ which yields conditions (i) and (ii) of Definition 1.2.

The key fact in the statement of the criterion is that the sets C_n are not too small, indeed

$$\mu(C_n) \geq \frac{1}{2n} \inf_{(x,y) \in \mathbb{T}^2} \varphi(x, y).$$

Next, due to the difference of scale between the successive terms of the sequence q_n, it is easy to see that for any $k \in \mathbb{N}$, we have for n sufficiently large

$$\mu(C_n \cap \bigcap_{j=k}^{n-1} C_j^c) \geq \frac{1}{2}\mu(C_n)\mu(\bigcap_{j=k}^{n-1} C_j^c)$$

which implies due to a variant of the Borel-Cantelli lemma (see [18], Prop. IV-4.4) that $\mu(\limsup C_n) = 1$ and condition (iii) holds thus satisfied. □

3.5. **Choice of the ceiling function φ.** Let $(\alpha, \alpha') \in \mathbb{R}^2$ be as in §3.2 and define

$$f(x, y) = 1 + \sum_{n \geq 2} X_n(x) + Y_n(y)$$

where

$$X_n(x) = \frac{1}{e^{q_n}} \cos(2\pi q_n x) \tag{3.7}$$

$$Y_n(y) = \frac{1}{e^{q_n'}} \cos(2\pi q_n' y). \tag{3.8}$$

Relying on the Proposition-Criterion 3.3 stated above, it is possible to prove as in [2] that the flow $T^t_{\alpha,\alpha',f}$ is mixing. The fact that f satisfies the Proposition-Criterion 3.3 is explained in the beginning of the proof of Theorem 3.1 in §3.6 below. In order to keep the mixing criterion valid but have in addition the conditions of the SCPA Criterion 3.4 satisfied we modify the ceiling function in the following way:

- We keep $Y_n(y)$ unchanged.
- We replace $X_n(x)$ by a trigonometric polynomial $\tilde{X}_n$ with integral zero, that is essentially equal to 0 for $\{q_n x\} \in [1/2 - 1/(2n), 1/2 + 1/(2n)]$ and whose derivative has its absolute value bounded from below by e^{-q_n} for $\{q_n x\} \in [0, 1/2 - 1/n] \cup [1/2 + 2/n, 1]$. The first listed properties of $\tilde{X}_n$ will yield Criterion 3.4 while the lower bound on the absolute value of its derivative will ensure Criterion 3.3.

More precisely, the following Proposition enumerates some properties that we will require on $\tilde{X}_n$ and its Birkhoff sums that will be sufficient for our purposes, and that we will realize with a specific construction at the end of the section.

Proposition. *Let (α, α') be as in Section 3.2. There exists a sequence of trigonometric polynomials $\tilde{X}_n(x)$ satisfying*

(1) $\displaystyle\int_{\mathbb{T}} \tilde{X}_n(x)dx = 0;$

(2) *For any $r \in \mathbb{N}$, there exists $N(r) \in \mathbb{N}$ such that for every $n \geq N(r)$,* $\| \tilde{X}_n \|_{C^r} \leq \frac{1}{e^{\frac{q_n}{2}}}$;

(3) *For* $\{q_n x\} \in [1/2 - 1/(2n), 1/2 + 1/(2n)]$, $|\tilde{X}_n(x)| \leq \frac{1}{{q'_n}^2}$;

(4) *For* $\{q_n x\} \in [0, 1/2 - 2/n] \cup [1/2 + 2/n, 1]$, $\tilde{X}'_n(x) \geq \frac{q_n^2}{e^{q_n}}$;

(5) *For $n \in \mathbb{N}$ sufficiently large,* $\left\| S_{q_n} \sum_{l \leq n-1} \tilde{X}_l \right\| \leq \frac{1}{{q'_n}^2}$;

(6) *For $n \in \mathbb{N}$ sufficiently large, we have for any $m \in \mathbb{N}$,* $\left\| S_m \sum_{l \leq n-1} \tilde{X}'_l \right\| \leq q_n$.

Before we prove this proposition, let us show how it allows us to produce the example of Theorem 3.1.

3.6. **Proof of Theorem 3.1.**

Define for some $n_0 \in \mathbb{N}$

$$\varphi(x, y) = 1 + \sum_{n=n_0}^{\infty} \tilde{X}_n(x) + Y_n(y) \tag{3.9}$$

where Y_n is as in (3.8) and $\tilde{X}_n$ is as in the proposition above. From (3.8) and Property (2) of $\tilde{X}_n$, we have that $\varphi \in C^\infty(\mathbb{T}, \mathbb{R})$. Also from (3.8) and (2), we can choose n_0 sufficiently large so that φ be strictly positive. We then have

THEOREM. *Let $(\alpha, \alpha') \in \mathbb{R}^2$ be as in Section 3.2 and φ be given by (3.9). Then the special flow $T^t_{\alpha,\alpha',\varphi}$ satisfies the conditions of Propositions 3.3 and 3.4 and is therefore mixing with a singular maximal spectral type.*

Proof. The second part of Proposition 3.3 is valid exactly as in [2] since Y_n has not been modified. Briefly, the reason is that due to (3.3) and (3.4)-(3.5) we have $Y'_n(y + l\alpha') \sim Y'_n(y)$ for every $l \leq m \ll q'_{n+1}$ so that $|S_m Y'_n|$ is large as required for $m \in [e^{2q'_n}/2, 2e^{q_{n+1}}]$ and $\{q'_n y\} \in [1/n, 1/2 - 1/n] \cup [1/2 + 1/n, 1 - 1/n]$. Meanwhile, $S_m \sum_{k<n} Y'_k$ is much smaller because these lower frequencies behave as controlled coboundaries for this range of m (we can write $Y_k(y) = h_k(y + \alpha') - h_k(y)$ with $\| h_k \|_{C^r} = o(q'_{k+1})$). As for $S_m \sum_{k>n} Y'_k$, it is still very small since $m \ll e^{q'_{n+1}}$. These phenomena will be further explicated as similar ones are used in the sequel.

Let $m \in [e^{2q_n}/2, 2e^{2q'_n}]$ and define $I_n := [1/n, 1/2 - 3/n] \cup [1/2 + 3/n, 1 - 1/n]$. It follows from (3.3) that for x such that $\{q_n x\} \in [1/n, 1/2 - 3/n]$ and for any $l \leq m$, $0 \leq \{q_n(x + l\alpha)\} \leq 1/2 - 2/n$. Hence, by Property (4) of $\tilde{X}_n$

$$S_m \tilde{X}'_n(x) \geq \frac{m q_n^2}{e^{q_n}}.$$

On the other hand, Properties (2) and (6) imply that

$$\begin{aligned}\left\|D_x S_m \varphi - S_m \tilde{X}'_n\right\| &\leq \left\|S_m \sum_{l<n} \tilde{X}'_l\right\| + \left\|S_m \sum_{l>n} \tilde{X}'_l\right\| \\ &\leq q_n + m \sum_{l\geq n+1} \frac{1}{e^{\frac{q_l}{2}}} \\ &\leq q_n + \frac{2m}{e^{\frac{q_{n+1}}{2}}} \\ &= o(\frac{m}{e^{q_n}})\end{aligned}$$

for the current range of m. With an exactly similar computation for the other part of I_n, the criterion of Proposition 3.3 holds proved.

Let now x be as in Proposition 3.4, that is $1/2 - 1/(2n) + 1/n^2 \leq \{q_n x\} \leq 1/2 + 1/(2n) - 1/n^2$. From (3.2) we have for any $l \leq q_n q'_n$ that $1/2 - 1/(2n) \leq \{q_n(x + l\alpha)\} \leq 1/2 + 1/(2n)$, hence Property (3) implies

$$|S_{q_n q'_n} \tilde{X}_n(x)| \leq \frac{q_n}{q'_n} \tag{3.10}$$

the latter being very small compared to $1/e^{q_n}$ since $q'_n \geq e^{3q_n}$. From Properties (5) and (2) we get for n sufficiently large

$$\begin{aligned}\left\|S_{q_n q'_n} \sum_{l\neq n} \tilde{X}_l\right\| &\leq \frac{1}{q'_n} + q_n q'_n \sum_{l\geq n+1} \frac{1}{e^{\frac{q_l}{2}}} \\ &\leq \frac{2}{q'_n}. \end{aligned} \tag{3.11}$$

On the other hand, it follows from (3.2) and (3.3) that for any $y \in \mathbb{T}$, for any $|l| < q'_n$, we have

$$\begin{aligned}|S_{q'_n} e^{i2\pi l y}| &= \left|\frac{\sin(\pi l q'_n \alpha')}{\sin(\pi l \alpha')}\right| \\ &\leq \frac{\pi l q'_n}{q'_{n+1}}, \end{aligned} \tag{3.12}$$

which yields for Y_l as in (3.8)

$$\left\|S_{q'_n} \sum_{l<n} Y_l\right\| = o(\frac{1}{e^{q'_n}}) \tag{3.13}$$

while clearly

$$\left\|S_{q'_n} \sum_{l\geq n} Y_l\right\| = o(\frac{1}{e^{\frac{q'_n}{2}}}). \tag{3.14}$$

In conclusion, (3.6) follows from the definition (3.9) of φ and (3.10), (3.11), (3.13), (3.14). □

It remains to construct $\tilde{X}_n$ satisfying (1)-(6).

3.7. **Proof of Proposition 3.5.**

3.7.1. For $n \in \mathbb{N}$, we define for $x \in \mathbb{R}$ the function ξ_n equal to $2q_n^2 e^{-q_n} x$ on $(-\frac{1}{2q_n} + \frac{1}{nq_n}, \frac{1}{2q_n} - \frac{1}{nq_n})$ and identically zero outside this interval.

Consider on $\mathbb{R}$ a C^∞ positive even function K equal to zero outside the interval $(-1,1)$ and such that $\int_{\mathbb{R}} K(x)dx = 1$. Define $K_n(x) = n^2 q_n K(n^2 q_n x)$.

We then consider the (odd) function $\hat{\xi}_n = \xi_n \star K_n$ that satisfies:

- $\displaystyle\int_{\mathbb{R}} \hat{\xi}_n(x)dx = 0$;
- For any $r \in \mathbb{N}$, we have for n sufficiently large $\left\|\hat{\xi}_n\right\|_{C^r} \leq \frac{1}{e^{\frac{3q_n}{4}}}$;
- $\hat{\xi}_n(x) = 0$ for $x \in (-\infty, -\frac{1}{2q_n} + \frac{1}{2nq_n}) \cup (\frac{1}{2q_n} - \frac{1}{2nq_n}, +\infty)$;
- $\hat{\xi}'_n(x) = \frac{2q_n^2}{e^{q_n}}$ for $x \in [-\frac{1}{2q_n} + \frac{2}{nq_n}, \frac{1}{2q_n} - \frac{2}{nq_n}]$.

Clearly, we can restrict ξ_n to the interval $(-\frac{1}{2q_n}, \frac{1}{2q_n})$ and then extend it to $\mathbb{R}$ as a smooth periodic function with period $1/q_n$, and finally consider the resulting function as a function $\hat{X}_n$ defined on the torus. As a consequence of the properties proved for $\hat{\xi}_n$, $\hat{X}_n$ satisfies the properties (1) to (4) required in proposition 3.5. To obtain the other two properties we need to replace $\hat{X}_n$ by a trigonometric polynomial.

3.7.2. We consider the Fourier series of $\hat{X}_n(x) = \sum_{k\in\mathbb{Z}} \hat{X}_{n,k} e^{i2\pi kx}$ and let

$$\tilde{X}_n := \sum_{k=-q_{n+1}+1}^{q_{n+1}-1} \hat{X}_{n,k} e^{i2\pi kx}.$$

The Fourier coefficients f_k of a function $f \in C^\infty(\mathbb{T},\mathbb{R})$ satisfy for any $k \in \mathbb{Z}$

$$(2\pi)^{r-1}|k|^r|f_k| \leq \|f\|_{C^r} \leq \sup_{k\in\mathbb{N}} (2\pi|k|)^{r+2}|f_k|. \tag{3.15}$$

Hence, we have for any $r \in \mathbb{N}$

$$\begin{aligned} \left\|\tilde{X}_n - \hat{X}_n\right\|_{C^r} &\leq \sum_{|k|\geq q_{n+1}} (2\pi k)^r |\hat{X}_{n,k}| \\ &\leq \frac{1}{2\pi}\|\hat{X}_n\|_{C^{r+2}} \sum_{|k|\geq q_{n+1}} \frac{1}{k^2} \\ &= o(\frac{1}{{q'_n}^2}) \end{aligned}$$

which allows to check (1), (2), (3) and (4) for $\tilde{X}_n$ from the properties of $\hat{X}_n$.

Proof of Properties (5) and (6). We have due to our truncation

$$\tilde{X}_n(x) = \psi_n(x+\alpha) - \psi_n(x) \tag{3.16}$$

where

$$\psi_n(x) = \sum_{k=-q_{n+1}+1}^{q_{n+1}-1} \psi_{n,k} e^{i2\pi kx}$$

with

$$\psi_{n,0} = 0 \qquad \text{and for k} \neq 0,\ \ \psi_{n,k} = \frac{\hat{X}_{n,k}}{e^{i2\pi k\alpha} - 1}.$$

Since $|k| < q_{n+1}$, it follows from (3.2) that

$$|\psi_{n,k}| \leq q_{n+1}|\hat{X}_{n,k}|$$

which with (3.15) implies

$$\begin{aligned} \|\psi_n\|_{C^r} &\leq 2\pi q_{n+1} \|\hat{X}_n\|_{C^{r+2}} \\ &\leq 2\pi \frac{q_{n+1}}{e^{\frac{3q_n}{4}}} \end{aligned}$$

for sufficiently large n. Hence, from (3.16) and (3.3) we get

$$\begin{aligned} \left\| S_{q_n} \sum_{l \leq n-1} \tilde{X}_l \right\| &\leq \frac{1}{q_{n+1}} \sum_{l \leq n-1} \|\psi_l\|_{C^1} \\ &\leq \frac{1}{q_{n+1}} \sum_{l \leq n-1} \frac{q_{l+1}}{e^{\frac{3q_l}{4}}} \\ &\leq \frac{q_n}{q_{n+1}} \end{aligned}$$

so that property (5) of Proposition 3.5 follows. Similarly, property (6) holds true since we have for sufficiently large n

$$\begin{aligned} \left\| S_m \sum_{l \leq n-1} \tilde{X}'_l \right\| &\leq 2 \sum_{l \leq n-1} \|\psi_l\|_{C^1} \\ &\leq q_n. \end{aligned}$$

□

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