ON THE ERGODICITY OF THE WEYL SUMS COCYCLE.

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ABSTRACT. For $\theta \in [0,1]$, we consider the map $T_\theta : \mathbb{T}^2 \to \mathbb{T}^2$ given by $T_\theta(x,y) = (x+\theta, y+2x+\theta)$. The skew product $f_\theta : \mathbb{T}^2 \times \mathbb{C} \to \mathbb{T}^2 \times \mathbb{C}$ given by $f_\theta(x,y,z) = (T_\theta(x,y), z + e^{2\pi i y})$ generates the so called Weyl sums cocycle $a_\theta(x,n) = \sum_{k=0}^{n-1} e^{2\pi i (k^2\theta + kx)}$ since the n^{th} iterate of f_θ writes as $f_\theta^n(x,y,z) = (T_\theta^n(x,y), z + e^{2\pi i y} a_\theta(2x,n))$.

In this note, we improve the study developed by Forrest in [5, 6] around the density for $x \in \mathbb{T}$ of the complex sequence $\{a_\theta(x,n)\}_{n\in\mathbb{N}}$, by proving the ergodicity of f_θ for a class of numbers θ that contains a residual set of positive Hausdorff dimension in $[0,1]$. The ergodicity of f_θ implies the existence of a residual set of full Haar measure of $x \in \mathbb{T}$ for which the sequence $\{a_\theta(x,n)\}_{n\in\mathbb{N}}$ is dense.

1. Let $\mathbb{T}^2$ denote the torus $\mathbb{R}^2/\mathbb{Z}^2$. For $\theta \in [0,1]$ define the map (*skew shift*) T_θ:

$$\begin{array}{ccc} \mathbb{T}^2 & \to & \mathbb{T}^2 \\ (x,y) & \mapsto & (x+\theta, y+2x+\theta) \end{array}$$

and the *skew product* f_θ:

$$\begin{array}{ccc} \mathbb{T}^2 \times \mathbb{C} & \to & \mathbb{T}^2 \times \mathbb{C} \\ (x,y,z) & \mapsto & (x+\theta, y+2x+\theta, z+e(y)) \end{array}$$

where $e(y)$ is the usual notation for $e^{2\pi i y}$. The diffeomorphism f_θ preserves the product measure $\mu = m \times \nu$ where m denotes the Haar measure on $\mathbb{T}^2$ and ν denotes the Lebesgue measure on $\mathbb{C}$. We say that the map f_θ is *ergodic* if and

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only if for every μ-measurable set $A \subset \mathbb{T}^2 \times \mathbb{C}$ such that $f_\theta(A) = A$ we have $\mu(A) = 0$ or $\mu(A^c) = 0$.

Definition 1. We define $\mathcal{F}$ to be the set of numbers $\theta \in [0,1] \setminus \mathbb{Q}$ having a continued fraction representation

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{\cdots}}},$$

such that $\sum_n 1/a_n < \infty$, and such that $\liminf_{q \geq 1} q^{3+\varepsilon} \|q\theta\| = 0$ for some $\varepsilon > 0$. Here and in all the text $\|\cdot\|$ stands for the closest distance of a real number to the integers. Let $p_l/q_l = [a_1, \ldots, a_l] = 1/(a_1 + 1/(a_2 + \ldots + (1 + 1/a_l) \ldots))$, p_l and q_l relatively prime. The sequence p_l/q_l is called the sequence of the best rational approximations of θ since we have $\|k\theta\| \leq q_{l-1}\theta$ for every $k < q_l$. The sequence q_l is simply called the sequence of *approximation denominators* of θ.

We will elaborate on the paper by Forrest [6] to obtain the following result:

Theorem 1. *Let $\theta \in \mathcal{F}$. Then f_θ is ergodic.*

The set $\mathcal{F}$ has positive Hausdorff dimension but the condition $\sum 1/a_n < \infty$ on θ is actually very restrictive since it involves *all* the convergents of θ. For instance $\mathcal{F}$ has zero measure and is contained in the complementary of a residual set (this can be checked by the ergodicity of the Gauss transformation $\theta \mapsto \{1/\theta\}$). But we can show using a classical general argument of Halmos, exposed in his introductory book to ergodic theory [3, proof of the second category theorem], that the set of θ such that f_θ is ergodic is a G_δ set, call it $\tilde{\mathcal{F}}$. Since $\mathcal{F}$ is dense and $\mathcal{F} \subset \tilde{\mathcal{F}}$, we have

Corollary 1. *The set $\tilde{\mathcal{F}} \subset [0,1]$ of θ such that f_θ is ergodic is a residual set of positive Hausdorff dimension.*

2. Theorem 1 and its corollary are a strengthening of the main result of [6] where the density in $\mathbb{C}$ of the Weyl sums

$$\sum_{k=0}^{n-1} e(k^2\theta + kx), \quad n = 1, 2, \cdots \tag{1}$$

was proved, if $\theta \in \mathcal{F}$, for almost every $x \in [0, 1]$. Indeed, we have

Corollary 2. *Let $\theta \in \tilde{\mathcal{F}}$. Then the set*

$$B(\theta) = \{x \in [0,1] : \sum_{k=0}^{n-1} e(k^2\theta + kx),\ n = 1, 2, \cdots, \textit{ is dense in } \mathbb{C}\}$$

is a G_δ dense set of full Lebesgue measure in $[0, 1]$.

Proof. If f_θ is ergodic then for μ-a.e. $u = (x, y, z)$ we have that the sequence $u, f_\theta(u)$,

$f_\theta^2(u), \cdots$, is dense in $\mathbb{T}^2 \times \mathbb{C}$. This is indeed a genral fact that can be proven considering a countable base $\{O_j\}_{j\in\mathbb{N}}$ of open balls of $\mathbb{T}^2 \times \mathbb{C}$ and observing that the complementary of the invariant set $\cup_{n\in\mathbb{Z}} f_\theta^n(O_j)$ has zero measure from which it follows that the complemetary of the set $\mathcal{D} = \cap_{j\in\mathbb{N}} \cup_{n\in\mathbb{Z}} f_\theta^n(O_j)$. But by definition a point $x \in \mathcal{D}$ has a dense orbit under f_θ. Now

$$f_\theta^n(x, y, z) = (T_\theta^n(x, y), z + \sum_{k=0}^{n-1} e(k^2\theta + 2kx + y)), \tag{2}$$

so that for μ-a.e. (x, y, z) we have that the sequence $z + \sum_{k=0}^{n-1} e(k^2\theta + 2kx + y)$, $n = 1, 2, \cdots$, is dense in $\mathbb{C}$. The density of the latter sequence clearly does not depend on y and z and the measurable statement of the corollary follows. Further, $\mathcal{D}$ is a G_δ set and since its complementary has zero measure it follows that it is a dense G_δ. For the same reason as above this means that $B(\theta)$ is a G_δ-dense set. □

We will see that in proving the density of the Weyl sums (1) for almost every x when $\theta \in \mathcal{F}$, Forrest actually went a long way towards proving the ergodicity of f_θ. Yet, he left this question unsolved and put it as an open problem even for a single value of θ. In a sense, we will finish here his work.

Finally, we recall that prior to [6], Forrest had already proved in [5] the transitivity of f_θ under the sole hypothesis $\liminf_{q\geq 1} q^{3/2}\|q\theta\| < \infty$. From the transitivity of f_θ, the density of the Weyl sums follows for a dense G_δ set of $x \in [0,1]$. Although T_θ is uniquely ergodic, the cocycles $\sum_{k=0}^{n-1} e(k^2\theta + 2kx + y))$ behave differently for different points $(x,y) \in \mathbb{T}^2$ as shown by the following remark:

Remark 1. While it is not clear whether 0 could be in $B(\theta)$ for some choice of θ[1], it does follow from an argument by Besicovitch [2] that for any θ there exists always an x such that $x \notin B(\theta)$.

3. The question of knowing whether the set $\mathcal{F}$ of θ for which f_θ is ergodic (or even transitive) has full measure (or contains all irrationals!) is still open and we have not much to say about this as explained in the following list of remarks:

Remark 2. It does not seem to be known whether there exists a class of irrational numbers θ for which the Weyl sums could fail to be dense for every x. In [6] it is claimed erroneously[2] that the estimate $|\sum_{k=0}^{n-1} e(k^2\theta + kx)| \geq c_\theta\sqrt{n}$ (uniformly in $x \in [0,1]$) was proved in [4] for constant type numbers θ (numbers with bounded partial quotients, or equivalently numbers that satisfy $\liminf_{q\geq 1} q\|q\theta\| > 0$). If this however turns out to be true, it would obviously preclude, if θ is of constant type, the density of the Weyl sums for any choice of x.

Remarkably, if true, the latter estimate turns out to be paradoxically helpful in showing ergodicity of the Weyl sums without the restrictive hypothesis $\sum 1/a_n < \infty$. Indeed, an elegant proof of ergodicity of f_θ for some class of θ (included in

[1]The claim made by Forrest that it follows from [4] that $0 \notin B(\theta)$ for any irrational θ probably stems from his missinterpretation of the formula $a(0,n) = \Omega(\sqrt{n})$ which is used in [4] (cf. §4 below) as the negation of $a(0,n) = o(\sqrt{n})$ and not as $\sqrt{n} = O(|a(0,n)|)$ like Forrest might have understood it. It is clear from the formulae of $a(0,n)$ in the case of θ rational that one can construct an irrational θ for which there exists a seqeunce $q_n \to \infty$ such that $a(0,q_n) \to 0$.

[2]For the same reason as in the precedent footnote.

those satisfying $\liminf_{q\geq 1} q^5\|q\theta\| = 0$) was given in [9], that is based on the alleged uniform lower bound on the Weyl sums for constant type numbers θ.

Remark 3. While a property on the rational approximations of θ, at least like the one used in [5], namely $\liminf q^{3/2}\|q\theta\| = 0$, seems necessary to study the density of the Weyl sums using the dynamics of f_θ, the condition $\sum_{n\geq 1} 1/a_n < +\infty$ could be removed as in [9] from the proof if some upper bounds on the measure of the sets where $|\sum_{k=0}^{n-1} e(k^2\theta + kx)|$ are not large were known. It would be helpful for example if one knows that for any constant $C > 0$,

$$\lim_{q\to\infty} \sup_{1\leq p\leq q-1} \lambda\{x : |\sum_{k=0}^{q-1} e(k^2 p/q + kx)| \leq C\} = 0.$$

Remark 4. If we denote for $l \geq 1$ by $f_\theta^{(l)}$ the skew product $f_\theta^{(l)}(x, y, z) = (x + \theta, y + 2x + \theta, z + e(ly))$, then the same proof of ergodicity for $\theta \in \mathcal{F}$ of $f_\theta^{(1)}$ implies the ergodicity of every $f_\theta^{(l)}$. But the set of $\theta \in [0, 1]$ with the latter property is invariant by multiplication by l on the circle so has measure either 0 or 1.

To compare with our problem, note that twist maps of the type $\mathbb{T}^d \times \mathbb{R}^k \to \mathbb{T}^d \times \mathbb{R}^k$, $(x, z) \to (x + \alpha, z + \varphi(x))$ with a smooth function φ having zero average and that is not a trigonometric polynomial are always ergodic for a G_δ-dense set of $\alpha \in \mathbb{T}^d$ (of zero Hausdorff dimension however) and not ergodic for a set of α of full measure which consists of the Diophantine vectors, that is vectors for which there exists N such that $\liminf_{q\geq 1} q^N\|q\alpha\| > 0$.

4. In [4], Hardy and Littlewood studied the growth of $|\sum_{k=0}^{n-1} e(k^2\theta + kx)|$ for different values of $\theta \in [0, 1]$. The principal bounds they obtained were

Theorem [4, Theorems 2.14, 2.141, 2.18, 2.181, 2.22, 2.221] *For any irrational* $\theta \in [0, 1]$,

$$\left|\sum_{k=0}^{n-1} e(k^2\theta + kx)\right| = o(n), \quad \text{uniformly for all values of } x.$$

If the partial quotients a_n in the continued fraction expansion of θ are bounded then

$$\left|\sum_{k=0}^{n-1} e(k^2\theta + kx)\right| = O(\sqrt{n}), \quad \text{uniformly for all values of } x.$$

These are optimal bounds. Indeed, for any irrational $\theta \in [0,1]$ we have

$$\left|\sum_{k=0}^{n-1} e(k^2\theta)\right| = \Omega(\sqrt{n}),$$

and for every sequence $\varphi_n > 0$ tending to 0 as $n \to \infty$, it is possible to find irrationals θ such that

$$\left|\sum_{k=0}^{n-1} e(k^2\theta)\right| = \Omega(n\varphi_n).$$

Here the notation $u_n = \Omega(v_n)$ for positive sequences u_n and v_n stands for the negation of $u_n = o(v_n)$.

With the dynamical approach adopted in this paper, the first one of these equations follows immediately from two classical and elementary facts in ergodic theory, see e.g. [7]: first, that T_θ is uniquely ergodic as soon as θ is irrational; and second, that this implies that the function $\Phi(x,y) = e(y)$, of zero average, has its Birkhoff means $1/n \sum_{k=0}^{n-1} e(k^2\theta + 2kx + y)$ converging uniformly to zero.

It would be nice if a an additional qualitative ergodic property of T_θ could be displayed in the case of irrationals θ with bounded partial quotient that would explain the second bound in the above theorem of Hardy and Littlewood.

5. We now procede to the proof of theorem 1. In all the sequel, θ will be a fixed irrational number in $\mathcal{F}$. For every $n, m \in \mathbb{N}$ and $(x,y) \in \mathbb{T}^2$, let

$$a(x,y,n) = \sum_{k=0}^{n-1} e(k^2\theta + 2kx + y),$$

and

$$b(x,m) = \sum_{k=0}^{m-1} e(kx).$$

Definition 2. [Essential value] We say that $l \in \mathbb{C}$ is an *essential value* for the cocycle a above T_θ if for any measurable set $E \subset \mathbb{T}^2$ such that $m(E) > 0$ and for any $\nu > 0$, there exists $n \in \mathbb{N}$ such that

$$m\left(E \cap T_\theta^{-n} E \cap \{(x,y) \ / \ |a(x,y,n) - l| \leq \nu\}\right) > 0.$$

We say that $l \geq 0$ is an *essential value for the modulus of* a if for any measurable set $E \subset \mathbb{T}^2$ such that $m(E) > 0$ and for any $\nu > 0$, there exists $n \in \mathbb{N}$ such that

$$m\left(E \cap T_\theta^{-n} E \cap \{(x,y) \ / \ ||a(x,y,n)| - l| \leq \nu\}\right) > 0.$$

Since $|a(x,y,n)|$ does not depend on y we write it simply $|a(x,n)|$.

A very useful general criterion for ergodicity established by K. Schmidt in [8]) states that f_θ is ergodic if and only if any $l \in \mathbb{C}$ is an essential value for a (above T_θ), but due to the symmetries of the system we have the following sufficient criterion for ergodicity that we took from [9]:

Lemma 1. *If* $1/2$ *(or any other strictly positive number) is an essential value for the modulus of* a *then* f_θ *is ergodic.*

Proof. The proof contains two parts. First, it is shown that a has a nonzero essential value. Indeed, if this was not true, then by lemma [8, Lemma 3.8] (the proof of this lemma can also be found in [1, Lemma 8.4.3]), we have that for any compact set $K \subset \mathbb{C}$ that does not contain 0, there exists a measurable set $B \subset \mathbb{T}^2$ such that for every $n \in \mathbb{N}$,

$$B \cap T_\theta^{-n} B \cap \{(x,y) \ / \ a(x,y,n) \in K\} = \emptyset$$

which clearly contradicts the assumption of the lemma.

Next, assume that $l \neq 0$ is an essential value for a. For $y_0 \in \mathbb{T}$ denote by S_{y_0} the map of $\mathbb{T}^2$ on itself $S_{y_0}(x,y) = (x, y + y_0)$. Then, the fact that for a measurable set B with $m(B) > 0$ we have an $n \in \mathbb{N}$ such that

$$m\left(S_{y_0} B \cap T_\theta^{-n}(S_{y_0} B) \cap \{(x,y) \ / \ |a(x,y,n) - l| \leq \nu\}\right) > 0,$$

implies for the same n that

$$m\left(B\cap T_\theta^{-n}B\cap\{(x,y)\ /\ |a(x,y,n)-le(-y_0)|\le\nu\}\right)>0,$$

which implies that all the circle of modulus $|l|$ is included in the set of essentail values of a. Since the set of essential values of a complex cocycle above an ergodic map is a closed subgroup of $\mathbb{C}$ (cf. [8, Lemma 3.3]), it follows that for the cocycle a it is equal to $\mathbb{C}$ and f_θ is hence ergodic. □

6. The general strategy in controlling $|a(x,n)| = |\sum_{k=0}^{n-1} e(k^2\theta+2kx)|$ starts by showing that for a typical value of x, $|a(x,q_n)|\to\infty$ along a subsequence of any infinite set of the approximation denominators of θ, thus in particular along a subsequence that satisfies the hypothesis $q_n^{3+\varepsilon}\|q_n\theta\|\to 0$. The latter condition implies an approximation formula for $|a(x,mq_n)|$, when m is not too large, by $|a(x,q_n)||b(2q_nx,m)|$ and m is then chosen to bring this product close to $1/2$. Typically, when $2q_nx$ behaves like a badly approximated number, $|b(2q_nx,l)|$, $l=1,\cdots,m$ contains a $O(1/m^{1-\epsilon})$-dense set in $[0,1]$ (here $\epsilon>0$ is an arbitrarilly small number). If we prove that $|a(x,q_n)|$ is tipically bounded by $q_n^{1/2+\epsilon}$ then the m_n we need to modulate the product $|a(x,q_n)||b(2q_nx,m)|$ is not larger than $q_n^{1/2+2\epsilon}$ and the condition $q_n^{3+\varepsilon}\|q_n\theta\|\to 0$ appears then to be the exact condition that allows the approximation formula to hold up to this value of m.

Finally, to show that $1/2$ is actually an essential value for the modulus of a we compute a bound on the derivative with respect to x of the product $|a(x,q_n)||b(2q_nx,m_n)|$ and show that, under the same assumption $q_n^{3+\varepsilon}\|q_n\theta\|\to 0$, the interval I_n containing x where the product is close to $1/2$ is sufficiently large so that $R_\theta^{m_nq_n}(I_n)$ is almost equal to I_n. This and the fact that $|a(x,y,l)|$ does not depend on y will allow us to conclude.

In this scheme, the first step is the most delicate. It was proved by Forrest in [6] who based his proof on the following approximate functional equation, established by Hardy and Littlewood in [4, Theorem 2.128, Theorem 2.17]: for

$0 < \theta,\, x < 1$ and $k \geq 1$

$$\sqrt{\theta}|a(\theta/2, x/2, k)| = |a(\{1/\theta\}/2, \{-x/\theta\}/2, [k\theta])| + O(1) \qquad (3)$$

where $\{\cdot\}$ and $[\cdot]$ denote the fractional and the integer part of a number and where the constant involved in the Landau's error notation is absolute. Under an additional assumption on θ it is possible to apply a dynamical approach where θ is viewed as a parameter and obtain by induction from the above functional equation a lower estimate on the Weyl sums. The upshot of this approach is the following key ingredient of [6] as well as for us here:

Proposition 1. [6, Proposition 4.3] *Suppose $\theta \in [0,1]\backslash\mathbb{Q}$ has a continued fraction representation $[a_1, a_2, \cdots]$ such that $\sum_n 1/a_n < \infty$. Then, given any $\delta > 0$ and any infinite subset Q of the set of approximation denominators of θ we have that for Lebesgue almost every $x \in [0,1]$, there exists a sequence $q_n \in Q$ such that $\delta/2 \leq \|2q_n x\| \leq \delta$ and $\lim_{n\to\infty} |a(x, q_n)| = \infty$.*

For the commoditiy of the reader and to keep this note as much self contained as possible (modulo the functional equation (3) that is admitted), we include in an appendix the scheme of the proof given in [6] of the above proposition.

7. To proceed we need the following construction similar to the one made in [6]. Suppose $\theta \in \mathcal{F}$, then there exists a sequence q_n of approximation denominators of θ such that:

7.a. $q_n^{3+\varepsilon}\|q_n\theta\| \longrightarrow 0$.

7.b. For almost every $x \in [0,1]$ there is a sequence $U_n \to \infty$ and infinitly many n such that $\delta/2 \leq \|2q_n x\| \leq \delta$ and $|a(x, q_n)| \geq U_n$ (this is exactly proposition 1).

7.c. For almost every $x \in [0,1]$, there is an n_1 such that for $n \geq n_1$, we have $|a(x, q_n)| \leq q_n^{1/2+\varepsilon/10}$.

This is because the fact that $\int_0^1 |a(x,q_n)|^2 dx = q_n$ implies $\lambda\{x : |a(x,q_n)| \geq q_n^{1/2+\varepsilon/10}\} \leq 1/q_n^{\varepsilon/5}$; but 7.a implies that $q_{n+1} \geq q_n^3$, hence $\sum 1/q_n^{\varepsilon/5} < \infty$ and 3 follows by the Borel Cantelli lemma.

7.d. For almost every $x \in [0,1]$, there is an n_2 such that for $n \geq n_2$, the set $\{|b(2q_n x, m)| : 0 \leq m \leq q_n^{1/2+\varepsilon/4}\}$ is $1/(q_n^{1/2+\varepsilon/8}\|2q_n x\|)$-dense in $[0,1]$.

To prove this we define $H_n := q_n^{1/2+\varepsilon/4}$. We let $A_k^\varepsilon \subset [0,1]$ be the subset of irrationals such that for each $\alpha \in A_k^\varepsilon$, and for $m \geq k$, there exists a continued fraction approximation p/q for α such that $q \in [m^{1-\varepsilon/10}, m]$. We clearly have $\lambda(\cup_k A_k^\varepsilon) = 1$ and we pose $\lambda(A_k^\varepsilon) = 1 - \upsilon(k)$. In our choice of the sequence q_n in 7.a we can assume up to extracting that $\sum_n \upsilon(H_n) < \infty$. Since $\lambda\{x \in [0,1] : 2q_n x \mod [1] \in A_{H_n}^\varepsilon\} = \lambda(A_{H_n}^\varepsilon) = 1 - \upsilon(H_n)$ we deduce that for almost every $x \in [0,1]$, there exists n_2 such that for $n \geq n_2$, then $2q_n x \mod [1] \in A_{H_n}^\varepsilon$ from which 7.d follows easily.

8. Note that a simple computation (see [6, Lemma A.4]) gives that for some constant C and for any $x \in [0,1]$, $l, m \in \mathbb{N}$, we have

$$|a(x,ml) - a(x,l)b(2lx,m)| \leq C|a(x,l)|m^3 l\|l\theta\|,$$

which in the case of q_n satisfying 7.a and $m \leq q_n^{1/2+\varepsilon/4}$ yields

$$|a(x,mq_n) - a(x,q_n)b(2q_n x,m)| \leq C|a(x,q_n)|q_n^{-1/2-\varepsilon/4},$$

and finally, if in addition $|a(x,q_n)| \leq 2q_n^{1/2+\varepsilon/10}$, then

$$|a(x,mq_n) - a(x,q_n)b(2q_n x,m)| \leq Cq_n^{-\varepsilon/8}. \tag{4}$$

It is in the above equations that the restrictive assumption $\liminf q^{3+\varepsilon}\|q\theta\| = 0$ is really crucial.

On another hand, we have $b(2q_n x, m) = e^{i2\pi(m-1)q_n x}\sin(2\pi m q_n x)/\sin(2\pi q_n x)$. Hence for $\delta/4 \leq \|2q_n x\| \leq 2\delta$ we have $|b(2q_n x,m)| \leq 1/\delta$ and $|D_x(b(2q_n x,m))| \leq 4\pi m q_n/\delta$. Also, we clearly have $|a(x,q_n)| \leq q_n$ and $|D_x(a(x,q_n)| \leq 2\pi q_n^2$. From these observations we conclude that for n sufficently large, for any $m \leq q_n^{1/2+\varepsilon/4}$ and $\delta/4 \leq \|2q_n x\| \leq 2\delta$, we have

$$|D_x\left[a(x,q_n)b(2q_n x,m)\right]| \leq \frac{5\pi}{\delta} q_n^{2+1/2+\varepsilon/4}. \tag{5}$$

We deduce from 7.a to 7.d the following:

Proposition 2. *Let $\theta \in \mathcal{F}$. For almost every $x \in [0,1]$ there exists an infinite sequence of integers M_n and a sequence $\epsilon_n \to 0$ such that*

(i) $\|M_n\theta\| \leq q_n^{-(2+1/2+3\varepsilon/4)}$;

(ii) *For every $\tilde{x} \in [x - q_n^{-(2+1/2+\varepsilon/2)}, x + q_n^{-(2+1/2+\varepsilon/2)}]$, we have* $||a(\tilde{x}, M_n)| - 1/2| \leq \epsilon_n$;

(iii) $\|M_n^2\theta + 2M_n x\| \leq \epsilon_n$;

Proof. Take a sequence q_n satisfying 7.a. Take an x that satisfies 7.b, 7.c and 7.d. Up to extracting from q_n we have that $\delta/2 \leq \|2q_n x\| \leq \delta$ and $|a(x,q_n)| \to \infty$. From 7.c and 7.d, we find $m_n \leq q_n^{1/2+\varepsilon/4}$ such that $|a(x,q_n)b(2q_n x, m_n)| \to 1/2$. Since the conditions of (4) are satisfied by x and m_n, (ii) follows for the particular value $\tilde{x} = x$ if we take $M_n := m_n q_n$.

For $|\tilde{x} - x| \leq q_n^{-(2+1/2+\varepsilon/2)}$ we have that $\delta/4 \leq \|2q_n\tilde{x}\| \leq 2\delta$, and since $|D_x(a(\tilde{x},q_n))| \leq 2\pi q_n^2$, we have from 7.c that $|a(\tilde{x},q_n)| \leq 2q_n^{1/2+\varepsilon/10}$, hence (4) holds for $\tilde{x}$ and for the same m_n considered above. At last, (ii) then follows from (5).

From 7.a we get (i) and the fact that $\|M_n^2\theta\| \to 0$. Finally the combination of $|a(x,q_n)| \to \infty$ and $|a(x,q_n)b(2q_n x, m_n)| \to 1/2$ forces $\|2M_n x\| \to 0$ and (iii) is proved. $\square$

Remark 5. It would be possible to insure that $|a(x, q_n)b(2q_n x, m_n)|$ stays close to 1/2 on larger intervals than in (ii) which would allow to relax the requirement (i) and from there relax the arithmetic condition 7.a on θ. But this condition, as we saw, is optimal if we want to insure (4) without which the product $|a(x, q_n)b(2q_n x, m_n)|$ stops being interesting to our end.

9. To see how the latter proposition implies that 1/2 is an essential value for the modulus of a we need an elementary measure theoretical lemma for which we introduce the following notations: For $(x, y) \in \mathbb{T}^2$, $r > 0$ and $\gamma > 0$, define

$$B_v^\gamma((x, y), r) = \{(\tilde{x}, \tilde{y}) \ / \ |\tilde{x} - x| \leq r, \ |\tilde{y} - y| \leq \gamma\}$$

that we call vertical bands since we will essentially use them with r small compared to γ. We note however that $B_v^\gamma((x, y), \gamma)$ is the square of size 2γ centered at (x, y). We have $m(B_v^\gamma((x, y), r)) = 4r\gamma$.

Lemma 2. *Let $E \subset \mathbb{T}^2$, $m(E) > 0$ and take $(x, y) \in E$ a Lebesgue density point. For $\gamma > 0$ we define the set $\mathcal{E}_\gamma \subset [x - \gamma/2, x + \gamma/2]$ of the points $\tilde{x}$ such that for every $r \leq \gamma/2$ we have $m(B_v^\gamma((\tilde{x}, y), r) \cap E) \geq \frac{1}{10} m(B_v^\gamma((\tilde{x}, y), r))$. Then there exists $\gamma_0 > 0$ such that for $\gamma \leq \gamma_0$, we have $Leb(\mathcal{E}_\gamma) > 0$.*

Proof. By contradiction. Take γ_0 such that for any $\gamma \leq \gamma_0$, $m(B_v^\gamma((x, y), \gamma) \cap E) \geq 99/100 m(B_v^\gamma((x, y), \gamma))$ (this is possible since $(x, y) \in E$ is a density point) and assume that for some $\gamma \leq \gamma_0$, $m(\mathcal{E}_\gamma) = 0$. For almost every $\tilde{x} \in [x - \gamma/2, x + \gamma/2]$ take $r(\tilde{x}) \leq \gamma/2$ such that $m(B_v^\gamma((\tilde{x}, y), r(\tilde{x})) \cap E) < \frac{1}{10} m(B_v^\gamma((\tilde{x}, y), r(\tilde{x})))$. The union $\cup_{\tilde{x}} B_v^\gamma((\tilde{x}, y), r(\tilde{x}))$ covers in measure all of $B_v^\gamma((x, y), \gamma/2)$ and we can extract from it a disjoint union of that covers more than 1/4 of it and get therefore a contradiction with the fact that $m(B_v^\gamma((x, y), \gamma) \cap E) \geq 99/100 m(B_v^\gamma((x, y), \gamma))$. □

Proof of theorem 1. Let $\xi > 0$ and take $(x, y) \in E$ and γ_0 as in the lemma. Take $\overline{x}$ in $\mathcal{E}_{\gamma_0}$ for which the result of proposition 2 holds. Recall that for $(\tilde{x}, \tilde{y}) \in \mathbb{T}^2$ we write indifferently $|a(\tilde{x}, \tilde{y}, l)|$ or $|a(\tilde{x}, l)|$ since the modulus of a does not depend on y. If we denote by $B_v^{(n)}$ the band $B_v^{\gamma_0}((\overline{x}, y), q_n^{-(2+1/2+\varepsilon/2)})$ we have from (ii) that

$$||a(\tilde{x}, \tilde{y}, M_n)| - 1/2| \leq \epsilon_n, \quad \forall\ (\tilde{x}, \tilde{y}) \in B_v^{(n)}. \tag{6}$$

But (i) and (iii) imply that

$$\lim_{n\to\infty} \frac{m\left(T_\theta^{-M_n} B_v^{(n)} \triangle B_v^{(n)}\right)}{m(B_v^{(n)})} = 0, \tag{7}$$

where $A \triangle B$ stands for the symetic difference between A and B.

On the other hand, by definition of $\overline{x} \in \mathcal{E}_{\gamma_0}$ we have that

$$m(B_v^{(n)} \cap E) \geq \frac{1}{10} m(B_v^{(n)}). \tag{8}$$

It immediately follows from (7) and (8) that for n sufficiently large

$$m(T_\theta^{-M_n} E \cap E \cap B_v^{(n)}) > 0,$$

and (6) then implies that $1/2$ is an essential value for the modulus of a which ends the proof of theorem 1 due to lemma 1. $\square$

Appendix: *Proof of proposition 1.*

We sketch here the proof given in [6] of proposition 1. For the bound on $\|2q_n x\|$ note that for any strictly increasing sequence of integers l_n, the set of x such that the sequence $(l_n x)_{n\in\mathbb{N}}$ is dense has full Lebesgue measure. Hence we just have to show that for any infinite subset Q of the set of approximation denominators of θ we have that for Lebesgue almost every $x \in [0,1]$, there exists a sequence $q_n \in Q$ such that $\lim |a(x, q_n)| = \infty$. First, it is easy to see that the set of $x \in [0,1]$ satisfying the above condition is invariant by translation by θ, but $x \mapsto x + \theta$ is ergodic, hence it is enough to prove that the set in question has positive measure. Next, by a simple computation we obtain for a given $k \in \mathbb{N}$ and any sequence q_n such that $q_n\|q_n\theta\| \to 0$, $2\max\{|a(x+k\theta, q_n)|, |a(x, q_n)|\} \geq \|2q_n x\||a(x,k)| + C_k u_n$

where $u_n \to 0$ as $n \to \infty$ (cf. [6, Corollary A.5]). Hence the proof of the proposition is reduced to the following

Proposition 3. [6, Proposition 3.13] *Suppose $\theta \in [0,1] \setminus \mathbb{Q}$ has a continued fraction representation $[a_1, a_2, \cdots]$ such that $\sum_n 1/a_n < \infty$. Then there is a $\rho > 0$ such that for all $C > 0$, there is a k such that $\lambda\{x : |a(x,k)| \geq C\} \geq \rho$.*

To prove proposition 3, it is convenient to define first the following function similar to the modulus of the Weyl sums

$$\psi(\theta, x, k) := \left|\sum_{j=0}^{k-1} e(j^2\theta/2 + jx)\right|$$

that satisfies for $0 < \theta$ and $x < 1$

$$\sqrt{\theta}\psi(\theta, x, k) = \psi(\{1/\theta\}, \{-x/\theta\}, [k\theta]) + O(1) \tag{9}$$

where $\{\cdot\}$ and $[\cdot]$ denote the fractional and the integer part of a number and where the constant involved in the Landau's error notation is absolute. Equation (9) is the only "hard analysis" esitmate that is needed in [6], but is really crucial since it is at the center of the proof of proposition 3. It was obtained by Hardy and Littelwood [4, 2.128, 2.17] as a generalisation of a formula of Lindelöf in the case of θ rational and its proof is based on the calculus of residues.

We explain now how (9) is used to prove proposition 3. Given $k \in \mathbb{N}$, let $S\theta = \{1/\theta\}$ and define $\tilde{S}(\theta, x) = (S\theta, \{-x/\theta\})$ and write $(S^m\theta, U_\theta^{(m)}x) = \tilde{S}^m(\theta, x)$. Let $\sigma_m(\theta) = \sqrt{S^{m-1}\theta}\sigma_{m-1}(\theta)$ with $\sigma_0(\theta) = 1$, and $k(m) = [k(m-1)S^{m-1}(\theta)]$ with $k(0) = k$. We have by induction from (9)

$$\sigma_m(\theta)\psi(\theta, x, k) = \psi(S^m\theta, U_\theta^{(m)}x, k(m)) + O(1) \tag{10}$$

(the constant in $O(1)$ is absolute and comes from the fact that $O(\sum_{l=1}^m \sigma_{m-l}(S^l\theta)) = O(\sum_{l=1}^m 2^{-l/2})$ since the hypothesis $\sum 1/a_n < \infty$ implies $\limsup a_n \geq 2$ which in its trun implies that $\sigma_j(S^p\theta) \leq C(\theta)2^{-j/2}$ for any p and j).

Recall the notation $b(x,k)=\sum_{j=0}^{m-1} e(jx)$. Since $\psi(\theta,x,k)=|b(x,k)|+O(k^3\|\theta\|)$ with an absolute constant in the error term, we have from (10)

$$\sigma_m(\theta)\psi(\theta,x,k) \geq |b(U_\theta^{(m)}x, k(m))| - C(k(m)^3\|S^m\theta\|+1), \tag{11}$$

for some absolute constant C. On another hand, the condition $\sum 1/a_n < \infty$ is crucial (see [6, Corollary 3.6]) in checking that for all $0<\eta<1/2$ and for all $m \geq 1$

$$\lambda\{x : \|U_\theta^{(m)}x\| < \eta\} \geq \tilde{C}\eta, \quad \text{for some absolute constant } \tilde{C}$$

which by an elementary computation implies that for any $C_0 \geq 1$

$$\lambda\{x : |b(U_\theta^{(m)}x, [2\pi C_0]+1)| \geq C_0\} \quad \geq \quad \tilde{C}/(2[2\pi C_0]+2). \tag{12}$$

Fix now $C_0 \geq 3C$ where C is the constant of (11). Given any $C' > 0$ pick m sufficently large so that $C_0/(3\sigma_m(\theta)) \geq C'$ and $([2\pi C_0]+1)^3\|S^m\theta\| \leq 1$ (possible due to the arithmetical condition on θ). Let $k=k(0)$ be such that $k(m) = [2\pi C_0]+1$. We have from (11) that

$$\{x : \psi(\theta,x,k) \geq C'\} \subset \{x : \sigma_m(\theta)\psi(\theta,x,k) \geq C_0/3\} \subset \{x : |b(U_\theta^{(m)}x, [2\pi C_0]+1)| \geq C_0\}$$

and the latter set has by (12) a measure greater than the constant $\rho = \tilde{C}/(2[2\pi C_0]+2)$.

□

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