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A highly efficient solver for density-dependent shallow water flows

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Abstract. *A highly efficient solver is proposed for the numerical solution of density-dependent shallow water flows. The governing equations consist on coupling the multi-layer shallow water equations for the hydraulic variables with a suspended sediment transport equation for the concentration variable. The layers can be formed in the shallow water model based on the variation of vertical and horizontal density which depend on the water temperature and salinity. At each time step, the method consists of two stages to update the numerical solution. In the first stage, the multi-layer shallow water equations are rewritten in a non-conservative form and the intermediate solutions are calculated using the modified method of characteristics. In the second stage, the numerical fluxes are reconstructed from the intermediate solutions in the first stage and used in the conservative form of the multi-layer shallow water equations. The proposed method avoids Riemann problem solvers and suitable for multi-layer shallow water equations on non-flat topography.*

1 The Model

In the current study we are interested on hydraulic flows occurring on the water free-surface where assumptions of shallow water flows applied. We consider the one-dimensional multi-layer shallow water equations written in a conservative form as

$$\begin{aligned} \partial_t\left(\rho_j h_j\right) + \partial_x\left(\rho_j h_j u_j\right) &= 0, \\ \partial_t\left(\rho_j h_j u_j\right) + \partial_x\left(\rho_j h_j u_j^2 + \frac{1}{2} g \rho_j h_j^2\right) &= -g\rho_j h_j \partial_x Z - \\ g\rho_j h_j \sum_{k=1}^{j-1} \partial_x h_k &- g h_j \sum_{k=j+1}^{M} \partial_x\left(\rho_k h_k\right), \end{aligned} \tag{1}$$

where $j = 1, \ldots, M$ with M is the total number of layers, ρ_j is the water density of the jth layer, $h_j(t,x)$ is the water height of the jth layer, $u_j(t,x)$ is the local water velocity for the jth layer, $Z(x)$ is the bottom topography and g the gravitational acceleration. For two layers with constant density ρ_1 and ρ_2, the equations (1) reduce to the standard two-layer shallow water equations studied for example in [3] among others. In the current work, we assume that a sediment transport takes place such that the density depends on space and time variables, *i.e.*, $\rho_j = \rho_j(t,x)$. This requires additional equations for its evolution. Here, the equations used to close the system are given by

$$\rho_j = \rho_w + \left(\rho_{s_j} - \rho_w\right) c_j, \qquad j = 1, \ldots, M, \tag{2}$$

where ρ_{s_j} is the sediment density with $\rho_{s_j} > \rho_w$, and c_j is the depth-averaged concentration of the suspended sediment for the jth layer. The equation for mass conservation of species is modeled by

$$\partial_t\left(\rho_{s_j} h_j c_j\right) + \partial_x\left(\rho_{s_j} h_j u_j c_j\right) = 0, \qquad j = 1, \ldots, M. \tag{3}$$

For simplicity in presentation we rewrite the equations (1) and (3) in a compact conservative form as

$$\partial_t \mathbf{W} + \partial_x \mathbf{F}(\mathbf{W}) = \mathbf{Q}(\mathbf{W}), \tag{4}$$

where $\mathbf{W}$ is the vector of conserved variables, $\mathbf{F}$ the vector of flux functions, $\mathbf{Q}$ is the vector of source terms. An equivalent system to the water flow equations (1) and the suspended sediment equations (3) can be obtained by using the

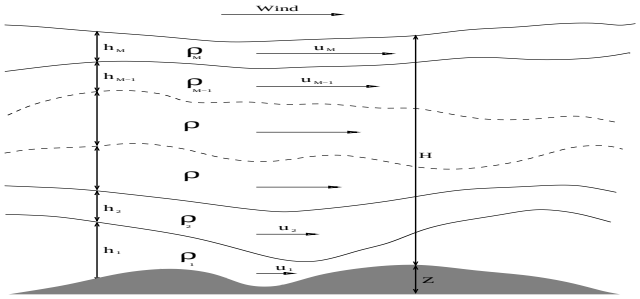

Figure 1: Schematic of a multi-layer shallow water equations.

physical variables as

$$\begin{aligned} D_t^{(j)}\left(\rho_j h_j\right) + \rho_j h_j \partial_x u_j &= 0, \\ D_t^{(j)} u_j + g\partial_x\left(Z + \frac{1}{2}h_j + \sum_{k=j+1}^{M} h_k\right) &= -\frac{g}{\rho_j}\partial_x\left(\frac{1}{2}\rho_j h_j + \sum_{k=1}^{j-1}\left(\rho_k h_k\right)\right), \\ D_t^{(j)}\rho_j &= 0, \qquad j = 1,\ldots,M, \end{aligned}$$

where $D_t^{(j)}$ denotes the total derivative defined as

$$D_t^{(j)}\omega = \partial_t\omega + u_j\partial_x\omega, \qquad j = 1,\ldots,M. \tag{5}$$

2 The Method

Let us discretize the spatial domain into control volumes $[x_{i-1/2}, x_{i+1/2}]$ with uniform size $\Delta x = x_{i+1/2} - x_{i-1/2}$ and divide the temporal domain into subintervals $[t_n, t_{n+1}]$ with stepsize Δt. Here, $t_n = n\Delta t$, $x_{i-1/2} = i\Delta x$ and $x_i = (i+1/2)\Delta x$ is the center of the control volume. Integrating the equation (4) with respect to space over the control volume $[x_{i-1/2}, x_{i+1/2}]$ and in time we obtain the following fully-discrete equations

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \Delta t\frac{\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n}{\Delta x} + \Delta t\mathcal{Q}_i^n, \tag{6}$$

where $\mathbf{W}_i^n$ is the space average of the solution $\mathbf{W}$ in the control volume $[x_{i-1/2}, x_{i+1/2}]$ at time t_n and $\mathcal{F}_{i\pm1/2}^n = \mathbf{F}(\mathbf{W}_{i\pm1/2}^n)$ are the numerical fluxes at $x = x_{i\pm1/2}$ and time t_n. To reconstruct the numerical fluxes $\mathcal{F}_{i\pm1/2}^n$ in (6), we consider the method of characteristics applied to the advective version of the system (4). In general, the advective form of the multi-layer system (4)

is built such that the non-conservative variables are transported with the same velocity field associated with each layer. Thus, the characteristic curves associated with the equation (5) are solutions of the initial-value problems

$$\begin{aligned} \frac{dX_{j,i+1/2}(\tau)}{d\tau} &= u_{j,i+1/2}\Big(\tau, X_{j,i+1/2}(\tau)\Big), \quad \tau \in [t_n, t_n + \Delta t/2], \\ X_{j,i+1/2}(t_n + \Delta t/2) &= x_{i+1/2}, \qquad j = 1, \ldots, M. \end{aligned} \tag{7}$$

Note that $X_{j,i+1/2}(\tau)$ is the departure point at time τ of a particle that will arrive at point $x_{i+1/2}$ in time $t_n + \Delta t/2$. The method of characteristics does not follow the flow particles forward in time, as the Lagrangian schemes do, instead it traces backward the position at time t_n of particles that will reach the points of a fixed mesh at time $t_n + \Delta t/2$. By doing so, the method avoids the grid distortion difficulties that the conventional Lagrangian schemes have.

Once the characteristics curves $X_{j,i+1/2}(t_n)$ are known, a solution at the cell interface $x_{i+1/2}$ is reconstructed as

$$\mathbf{U}^n_{j,i+1/2} = \mathbf{U}_j\left(t_n + \Delta t/2, x_{i+1/2}\right) = \tilde{\mathbf{U}}_j\left(t_n, X_{j,i+1/2}(t_n)\right), \tag{8}$$

where $\tilde{\mathbf{U}}_j\left(t_n, X_{j,i+1/2}(t_n)\right)$ is the solution at the characteristic foot computed by interpolation from the gridpoints of the control volume where the departure point resides *i.e.*

$$\tilde{\mathbf{U}}_j\left(t_n, X_{j,i+1/2}(t_n)\right) = \mathcal{P}\Big(\mathbf{U}_j\left(t_n, X_{j,i+1/2}(t_n)\right)\Big), \tag{9}$$

where $\mathcal{P}$ represents the interpolating polynomial.

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