Numerical Simulation of Complex and Multiphase Flows.
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Non Homogenous Riemann Solver to simulate two-phase flows.

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1 Introduction :

– Presentation of VF scheme for non hongeneous systems, using flux values instead of eigenvectors.
– Scheme analysis.
– Numerical results in 1D and 2D.

2 Equations : balance laws

$$\frac{\partial U(x,t)}{\partial t} + \sum_{j=1}^{d} \frac{\partial F_j(U(x,t))}{\partial x_j} = Q(x,t,U), \tag{1}$$

$$x = (x_1, x_2, ..., x_d) \in D \subset \mathbb{R}^d, t > 0,$$

$U : D \times \mathbb{R}^+ \longrightarrow \Omega$: physical values (p components),
Ω open bounded in $\mathbb{R}^p$,
F_j $(1 \leq j \leq d)$: flux functions.
Non homogeneous term $Q(x,t,U)$ (source terms or non conservative terms).
$U(x,0) = U_0(x)$: initial condition + boundary conditions.

$\Rightarrow$ Modelisation of shallow water or multiphase flows.

1D uniform case : $\dfrac{\partial U}{\partial t} + \dfrac{\partial F(U)}{\partial x} = Q(x, U), \qquad 0 < t < T,$

$$U(x, 0) = U_0(x), \quad \forall x \in \Omega \subset \mathbb{R}.$$

SRNHR scheme

$$\begin{cases} U^n_{j+\frac{1}{2}} = \frac{1}{2}(U^n_j + U^n_{j+1}) - \frac{\alpha^n_{j+\frac{1}{2}}}{2S^n_{j+\frac{1}{2}}}\left[F(U^n_{j+1}) - F(U^n_j)\right] + \frac{\alpha^n_{j+\frac{1}{2}}}{2}\frac{\Delta x}{S^n_{j+\frac{1}{2}}}\widehat{Q}^n_{j+\frac{1}{2}} \\ U^{n+1}_j = U^n_j - r^n\left[F(U^n_{j+\frac{1}{2}}) - F(U^n_{j-\frac{1}{2}})\right] + \tau^n Q^n_j, \end{cases}$$

where $S^n_{j+\frac{1}{2}} = \max\limits_{p=1,\ldots,m}\left(\left|\lambda^n_{p,i}\right|, \left|\lambda^n_{p,i+1}\right|\right)$: local Rusanov velocity,
$\alpha^n_{j+\frac{1}{2}}$ real and positive parameter, $r^n = \frac{\tau^n}{\Delta x}$.

Remark : $\alpha^n_{j+\frac{1}{2}}$ is a parameter which aims to control numerical diffusion of scheme.

For example, for linear scalar equation on uniform mesh, $\alpha^n_{j+\frac{1}{2}} = 1$ corresponds to Lax-Wendroff scheme.

3 Scheme analysis :

Hypothesis : (scalar case)

H1) f' and f" have a constant sign and does not vanish.

H2) u_0 has a constant sign and does not vanish. Suppose that $0 < u_m \leq u_0(x) \leq u_M$.

Proposition 3.1 *Suppose that* $\alpha^n_{j+\frac{1}{2}} = (\alpha^n_{j+\frac{1}{2}})_1 = \frac{S^n_{j+\frac{1}{2}}}{s^n_{j+\frac{1}{2}}}$, $\forall j$, $\forall n$, *with*
$S^n_{j+\frac{1}{2}} = max\left(|f'(u^n_j)|, |f'(u^n_{j+1})|\right)$ *and*
$s^n_{j+\frac{1}{2}} = min\left(|f'(u^n_j)|, |f'(u^n_{j+1})\right)|)$.
Denoting by $\alpha^n = \sup\limits_{(j\in\mathbb{Z})} \dfrac{S^n_{j+\frac{1}{2}}}{s^n_{j+\frac{1}{2}}}$, $\alpha_{\max} = \dfrac{fp_M}{fp_m}$,
$fp_m = min(|f'(u_m)|, |f'(u_M)|)$, $fp_M = max(|f'(u_m)|, |f'(u_M)|)$ *and*
$M = sup \mid f'(w) \mid$, $w \in \{w\; in\, \mathbb{R}/\; |w| \leq \alpha_{\max} \parallel u_0 \parallel_\infty\}$.
Then, under condition $\alpha^n M r^n \leq 1$, *SRNHR scheme is* TVD, *satisfy maximum principle and then is* L^∞ *stable.*

Proposition 3.2 *Suppose now that* $\alpha^n_{j+\frac{1}{2}} = (\alpha^n_{j+\frac{1}{2}})_2 = r^n S^n_{j+\frac{1}{2}}$. *Then scheme SRNHR writes*

$$\begin{cases} u^n_{j+\frac{1}{2}} = \frac{1}{2}(u^n_j + u^n_{j+1}) - \frac{r^n}{2}\left[f(u^n_{j+1}) - f(u^n_j)\right] \\ \\ u^{n+1}_j = u^n_j - r^n\left[f(u^n_{j+\frac{1}{2}}) - f(u^n_{j-\frac{1}{2}})\right]. \end{cases}$$

which is secund order Richtmeyer scheme.

Then, the use of limiter theory :

$$\alpha^n_{j+\frac{1}{2}} = \Phi_j(\alpha^n_{j+\frac{1}{2}})_1 + (1-\Phi_j)\,(\alpha^n_{j+\frac{1}{2}})_2 \tag{2}$$

where Φ_j is a limiter function (Superbee, Van-Leer,...).

4 Stationnary states preserving :

For Saint-Venant equations :

$$\begin{cases} \dfrac{\partial h}{\partial t}(x,t) + \dfrac{\partial hu}{\partial x}(x,t) = 0 \\ \dfrac{\partial hu}{\partial t}(x,t) + \dfrac{\partial (hu^2 + \frac{gh^2}{2})}{\partial x}(x,t) = -gh(x,t)\dfrac{dz}{dx}(x) \\ h_0(x), u_0(x), z(x) \text{ given} \end{cases}$$

Proposition 4.1 *Under condition that source term is discretized as*

$$Q_j^n = -\frac{1}{8\Delta x} g(u_{j-1}^n + 2u_j^n + u_{j+1}^n)(z_{j+1} - z_{j-1})$$

SRNHR scheme satisfy exact C-property.

Riemann invariants for Saint-Venant equations are $W_k = u + (-1)^k 2c$.

At step n, for each cell, local Rusanov velocity : $S^n_{j+\frac{1}{2}} = \max_p \left(max\left(|\lambda^n_{p,j}|, |\lambda^n_{p,j+1}|\right)\right)$

If ($W_{k_{j+1}} = W_{k_j}$) **then**

$\theta_{W_k} = 0$

else

Velocity at interface :

$\tilde{c}_{j+\frac{1}{2}} = \sqrt{g\frac{h_j+h_{j+1}}{2}}$,

$\tilde{V}_{j+\frac{1}{2}} = \frac{(u_j+u_{j+1})}{2}$,

$\tilde{\lambda}_{k_{j+\frac{1}{2}}} = \tilde{V}_{j+\frac{1}{2}} + (-1)^k\tilde{c}_{j+\frac{1}{2}}$

If ($\tilde{\lambda}_{k_{j+\frac{1}{2}}} > 0$) **then**

$\theta_{W_k} = \frac{W_{k_j} - W_{k_{j-1}}}{W_{k_{j+1}} - W_{k_j}}$

else

$\theta_{W_k} = \frac{W_{k_{j+2}} - W_{k_{j+1}}}{W_{k_{j+1}} - W_{k_j}}$

end If

end If

$s_{j+\frac{1}{2}} = min(|\tilde{\lambda}_{1_{j+\frac{1}{2}}}|, |\tilde{\lambda}_{2_{j+\frac{1}{2}}}|)$

If ($s_{j+\frac{1}{2}} < \varepsilon$) **then**

$s_{j+\frac{1}{2}} = \varepsilon$

end If

If ($\theta_{W_1} < 0$ or $\theta_{W_2} < 0$) **then**

$\phi = 0$

else

$\phi =$ limiter function

end If

$\alpha^n_{j+\frac{1}{2}} = \frac{S^n_{j+\frac{1}{2}}}{s_{j+\frac{1}{2}}}(1-\phi) + \phi r^n S^n_{j+\frac{1}{2}}$.

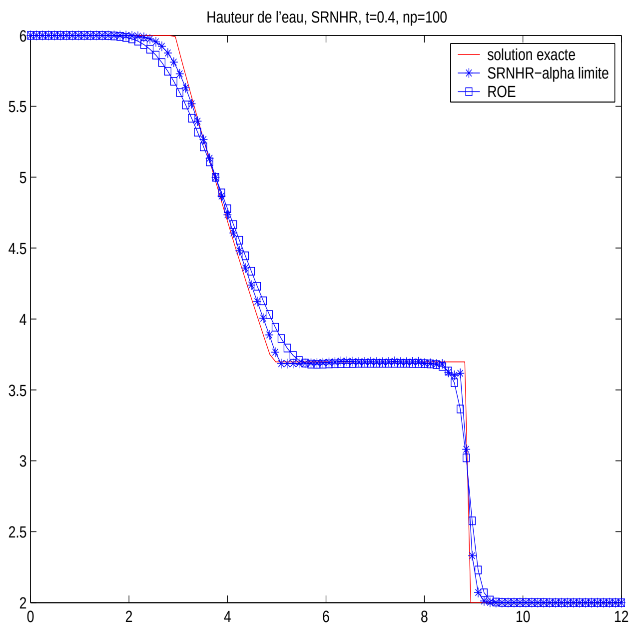
5 $1D$ homogeneous Saint-Venant equations :

Consider a dam break with solution containing a shock wave and a rarefaction wave.

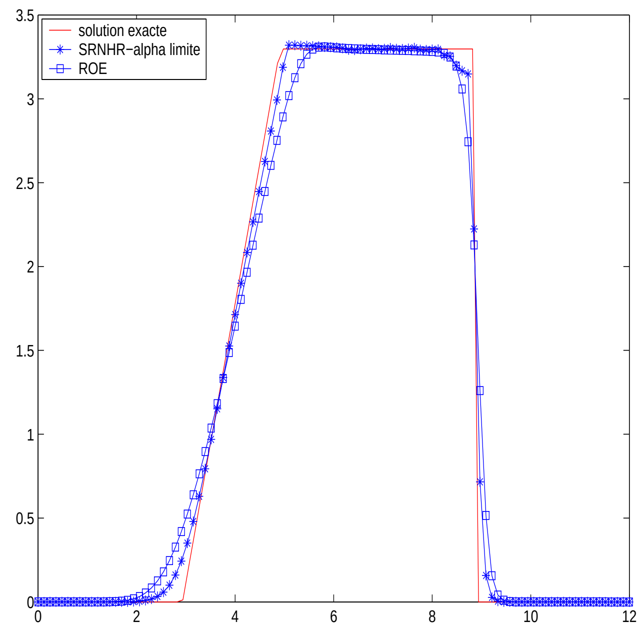
$$\begin{cases} \dfrac{\partial h}{\partial t}(x,t) + \dfrac{\partial (hu)}{\partial x}(x,t) = 0 \\ \dfrac{\partial (hu)}{\partial t}(x,t) + \dfrac{\partial \left(hu^2 + \frac{gh^2}{2} \right)}{\partial x}(x,t) = 0 \end{cases} \tag{3}$$

with initial conditions : $h_0(x) = \begin{cases} 6 & \text{if} \quad x \le 6 \\ 2 & \text{if} \quad x > 6. \end{cases}$, and $u_0(x) = 0, \quad \forall x.$

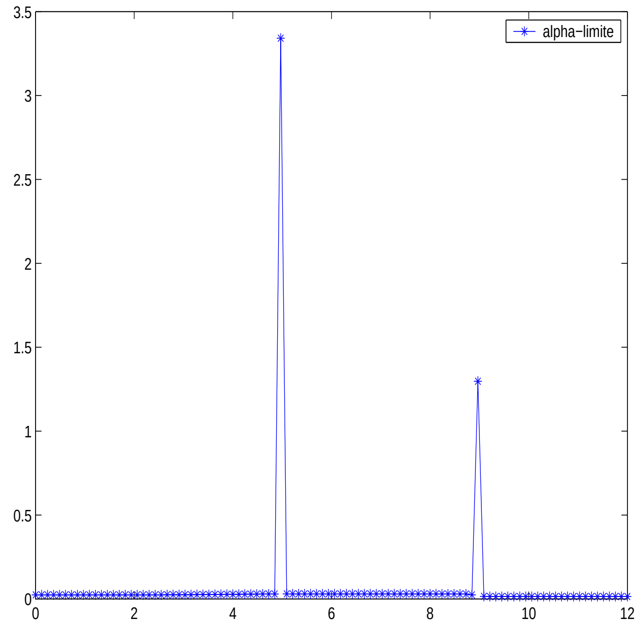
Mesh contains 100 points and results are given at $t = 0.4$.



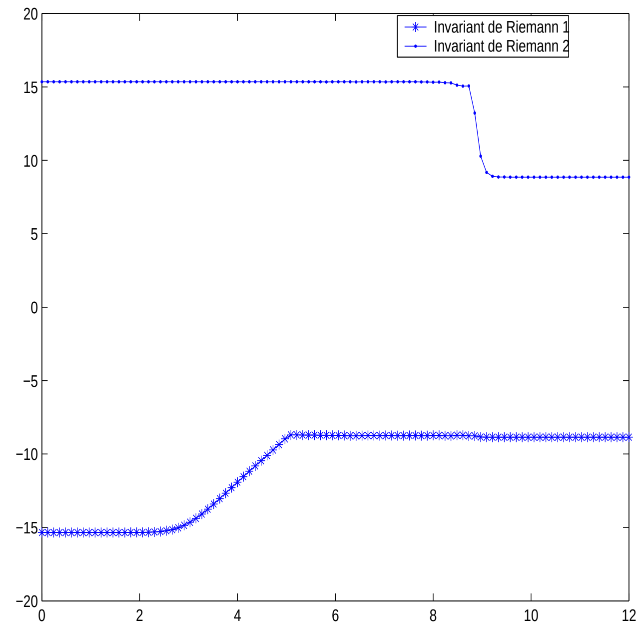
Homogeneous Shallow Water, water level, SRNHR and Roe schemes.



Homogeneous Shallow Water, water velocity, SRNHR and Roe schemes.



Homogeneous Shallow Water, variations of $\alpha^n_{j+\frac{1}{2}}$.



Homogeneous Shallow Water, Riemann invariants.

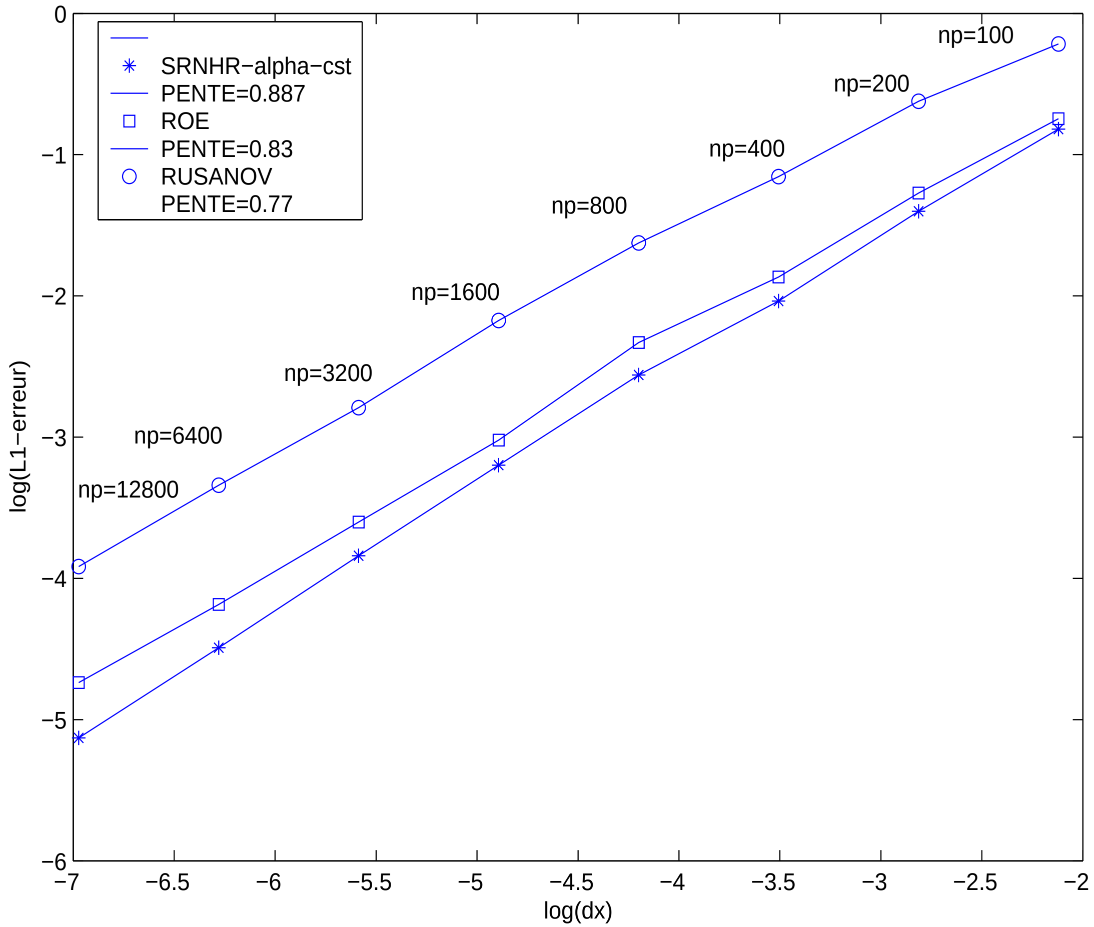


FIG. 1 – Homogeneous Shallow Water, water level, error curve.

6 $1D$ homogeneous Euler equations :

Shock tube with initial conditions

$$\rho_0(x,y) = \begin{cases} 1\,kg/m^3 & \text{si} \quad x \leq 0 \\ 0.01\,kg/m^3 & \text{si} \quad x > 0, \end{cases}$$

$$u_0(x,y) = 0\,m/s, \quad \forall x \in [-10;10],$$

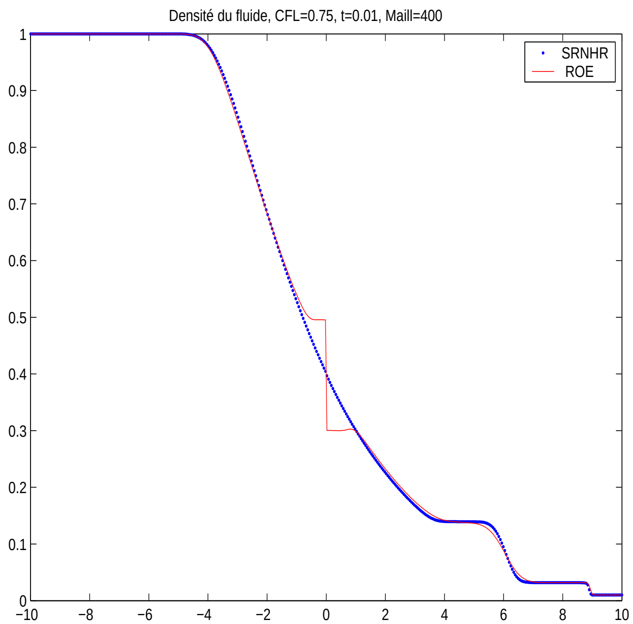
$$P_0(x,y) = \begin{cases} 10^5\,Pa & \text{si} \quad x \leq 0 \\ 10^3\,Pa & \text{si} \quad x > 0, \end{cases}$$

Mesh : 800 points ; $cfl = 0.95$; $t = 0.01$.

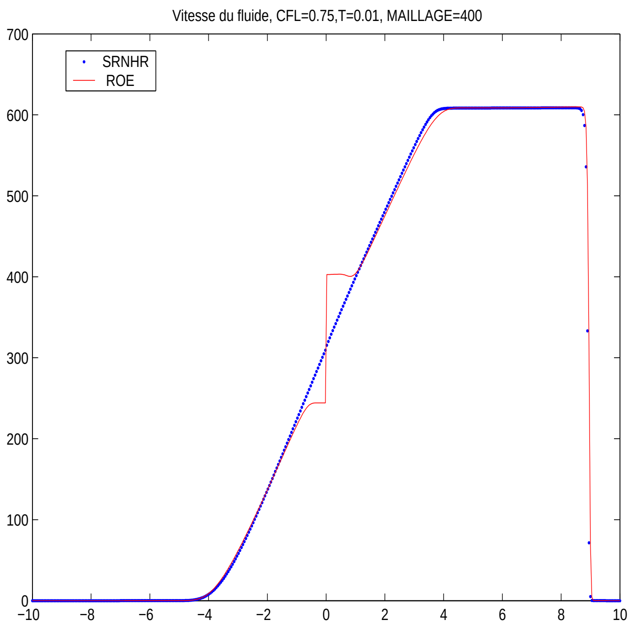
Comparaison between SRNHR scheme and VFRoe scheme.

First characteristic field is a rarefaction wave which contains a sonic point.

Roe scheme needs to add entropic correction.



Fluid density ; SRNHR and Roe schemes.



Fluid velocity ; SRNHR and Roe schemes.

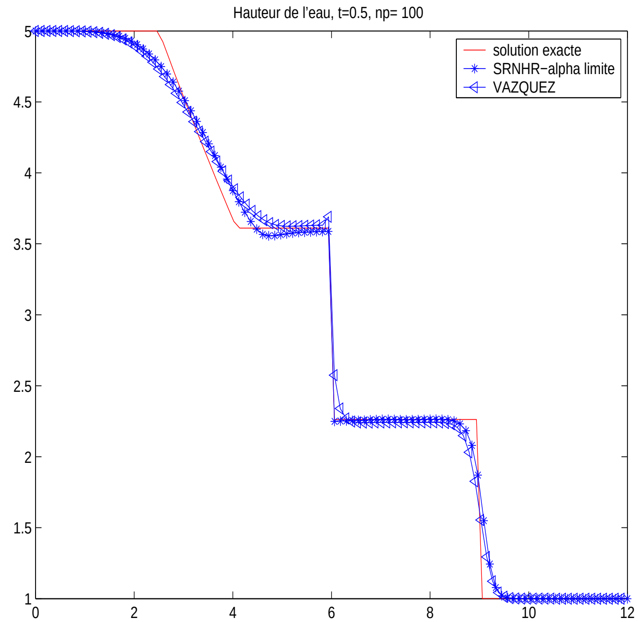
7 $1D$ non homogeneous Saint-Venant equations :

Consider a dam break over a step. Source term describes bottom geometry.

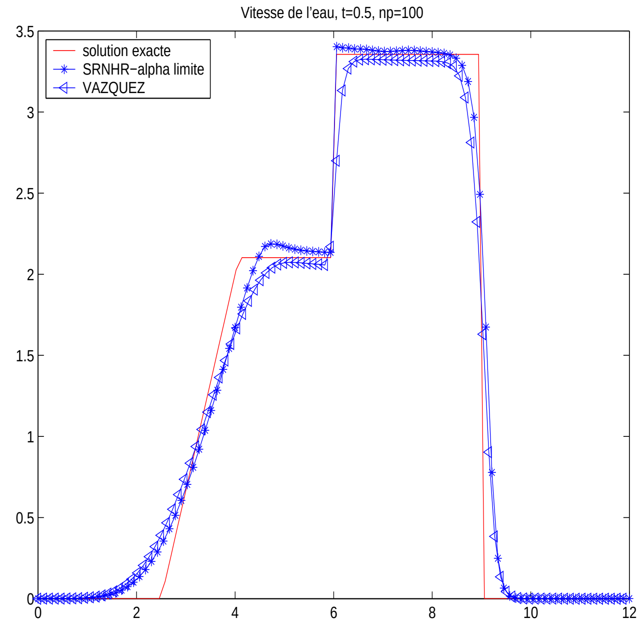
$$\left\{\begin{array}{l}
\dfrac{\partial h}{\partial t}(x,t) + \dfrac{\partial (hu)}{\partial x}(x,t) = 0 \\
\dfrac{\partial (hu)}{\partial t}(x,t) + \dfrac{\partial (hu^2 + \frac{gh^2}{2})}{\partial x}(x,t) = -gh(x,t)\dfrac{dz}{dx}(x) \\
z(x) = \begin{cases} 0 \quad if \quad x \leq 6 \\ 1 \quad if \quad x > 6 \end{cases} \\
h_0(x) = \begin{cases} 5 & \text{if} \quad x \leq 6 \\ 1 & \text{if} \quad x > 6. \end{cases} \\
u_0(x) = 0.
\end{array}\right. \tag{4}$$

Comparaison between SRNHR and Vazquez (equilibrium) schemes.

Mesh : 100 points ; results are given at $t = 0.5$.



Non homogeneous Shallow Water, water level, SRNHR and Vazquez schemes.



Non homogeneous Shallow Water, water velocity, SRNHR and Vazquez schemes.

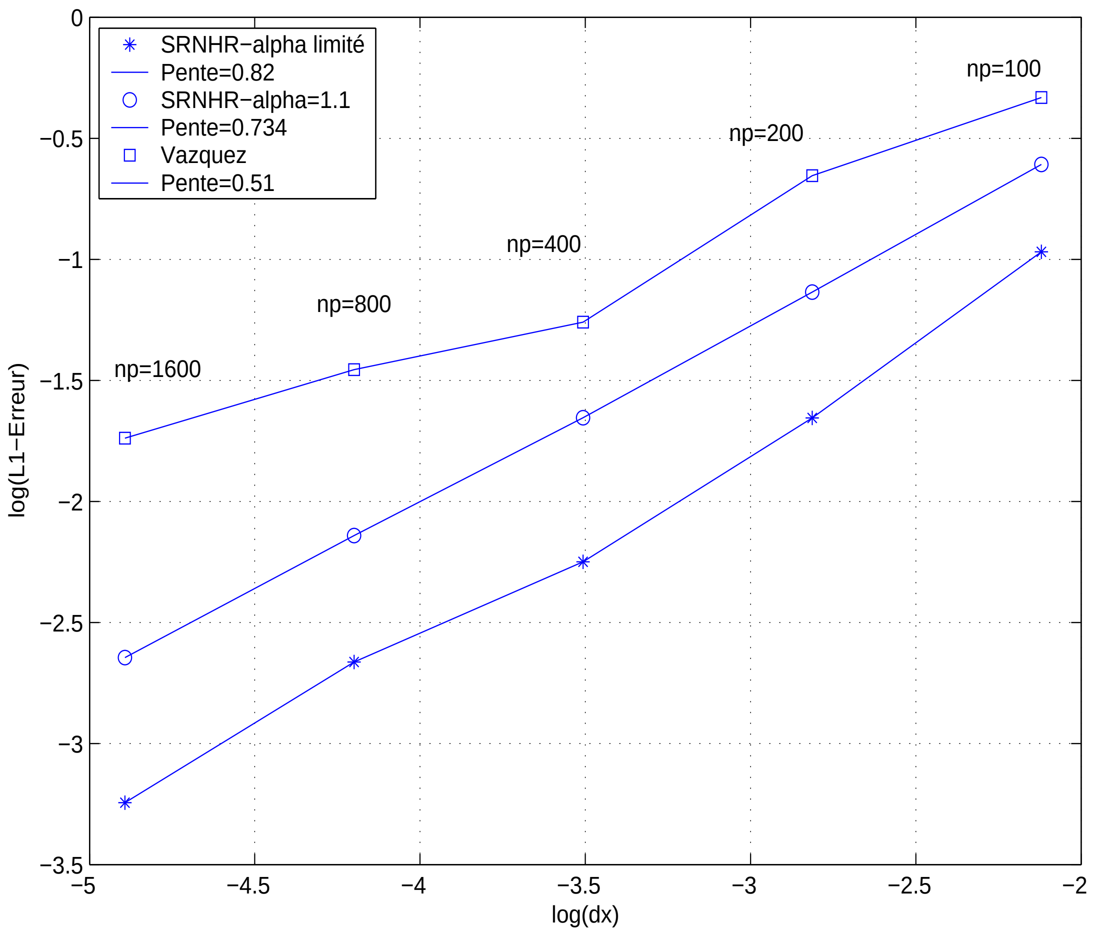


FIG. 2 – Non homogeneous Shallow Water, error curve, (water level), SRNHR and Vazquez schemes.

8 2D SRNHR scheme

Non homogeneous Saint-Venant equations :

$\forall (x, y) \in \Omega \subset \mathbb{R}^2, \quad t \in \mathbb{R}^+,$

$$\begin{cases} \dfrac{\partial h}{\partial t}(x,y,t) + \dfrac{\partial (hu)}{\partial x}(x,y,t) + \dfrac{\partial hv}{\partial y}(x,y,t) = 0 \\ \dfrac{\partial (hu)}{\partial t}(x,y,t) + \dfrac{\partial (hu^2 + \frac{1}{2}gh^2)}{\partial x}(x,y,t) + \dfrac{\partial hv}{\partial y}(x,y,t) = -gh(x,y,t)\dfrac{dz}{dx}(x,y) \\ \dfrac{\partial (hv)}{\partial t}(x,y,t) + \dfrac{\partial hv}{\partial x} + \dfrac{\partial (hv^2 + \frac{1}{2}gh^2)}{\partial y}(x,y,t) = -gh(x,y,t)\dfrac{dz}{dy}(x,y) \end{cases} \tag{5}$$

with initial conditions $\begin{cases} h(x,y,0) = h_0(x,y) \\ (hu)(x,y,0) = (hu)_0(x,y) \\ (hv)(x,y,0) = (hv)_0(x,y) \\ z(x,y) \text{ given.} \end{cases}$

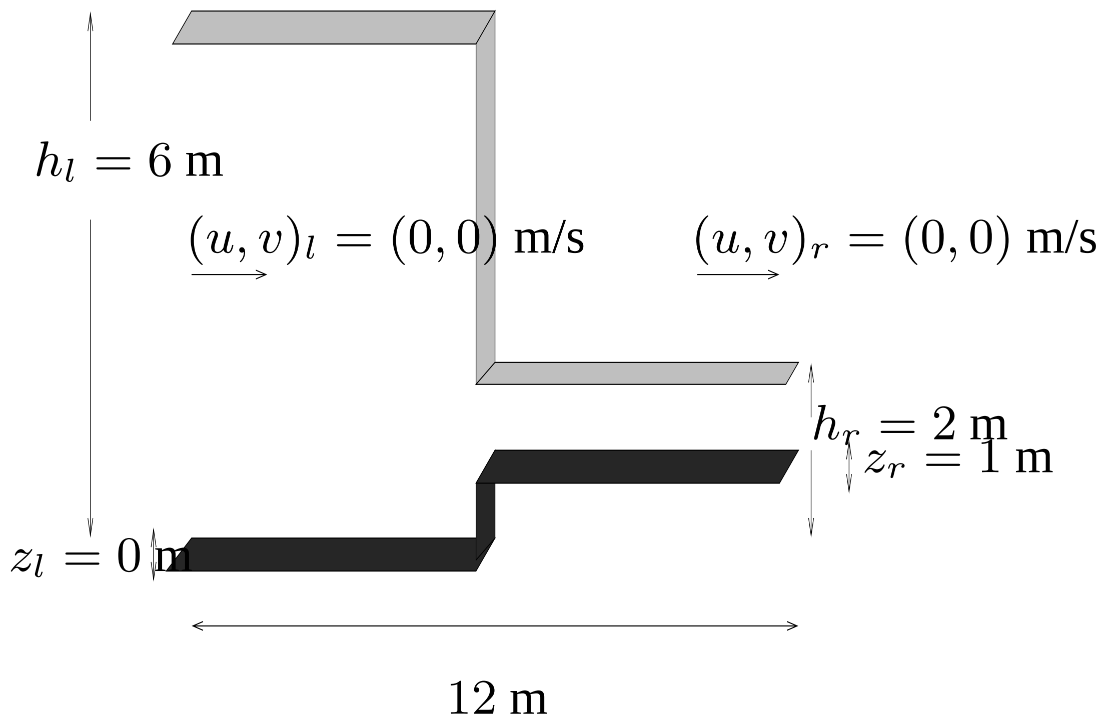
Using projection on normal direction at cell interface (R. Abgrall & al., 2003) :
Denoting by $\vec{V} = (u \quad v)^t$, $\vec{\eta} = (n_x \quad n_y)^t$,
$U = \vec{V} \cdot \vec{\eta} = un_x + vn_y$ and $V = \vec{V} \cdot \vec{\eta}^{\perp} = -un_y + vn_x$, where U is the projection of $\vec{V}$ on $\vec{\eta}$ and V, the projection of $\vec{V}$ on $\vec{\eta}^{\perp}$.

$$\Rightarrow \begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial hU}{\partial \eta} = 0 \\[2em] \dfrac{\partial hU}{\partial t} + \dfrac{\partial (hU^2 + \frac{1}{2}gh^2)}{\partial \eta} + gh\dfrac{dz}{d\eta} = 0 \\[2em] \dfrac{\partial hV}{\partial t} + \dfrac{\partial hUV}{\partial \eta} = 0. \end{cases}$$

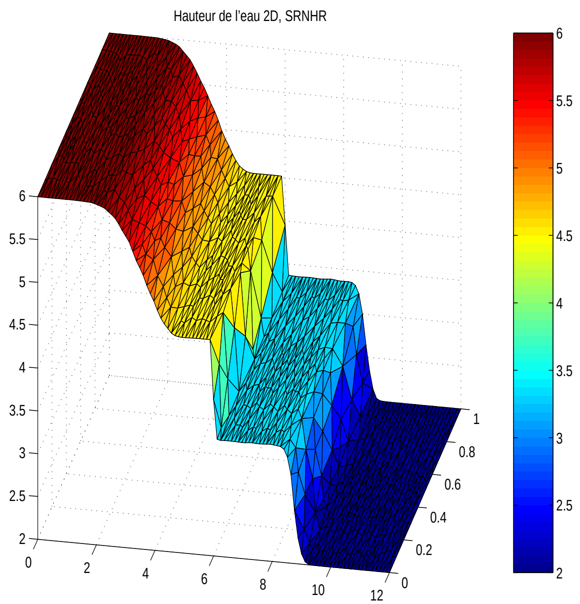
Numerical results :

Initial conditions : $h_0(x,y) = \begin{cases} 6 & \text{if} \quad x \leq 6, \quad \forall y \in [0;1] \\ 2 & \text{if} \quad x > 6, \quad \forall y \in [0;1] \end{cases}$

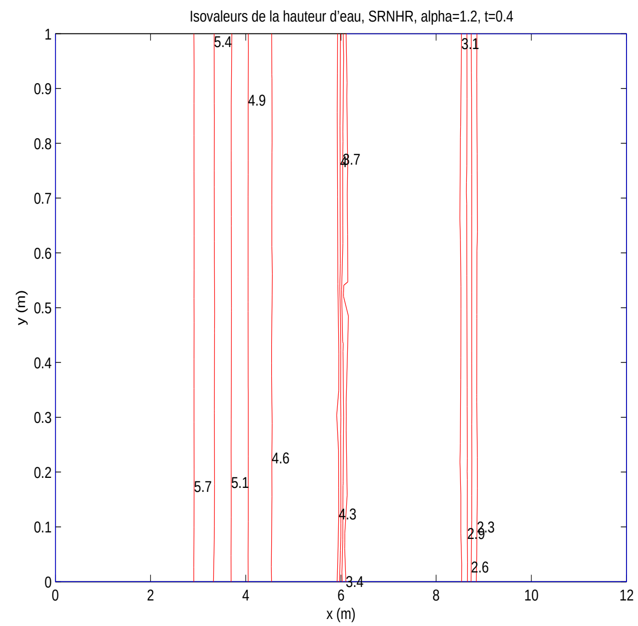
$$u_0(x,y) = v_0(x,y) = 0, \quad \forall x \in [0;12], \quad \forall y \in [0;1].$$



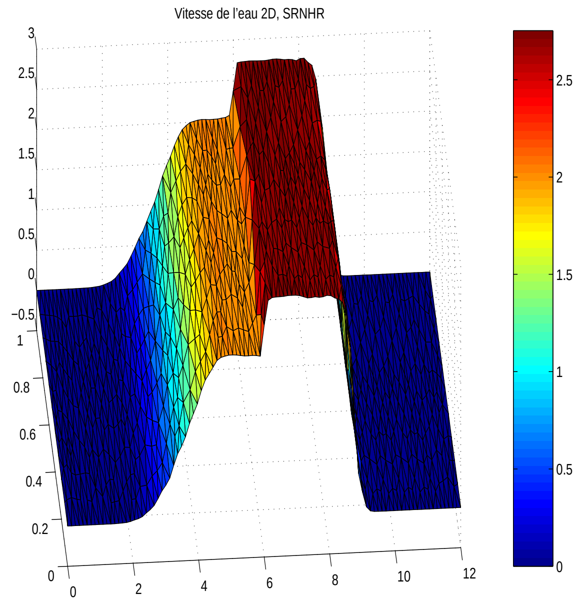
Results with unstructured mesh 100×10.

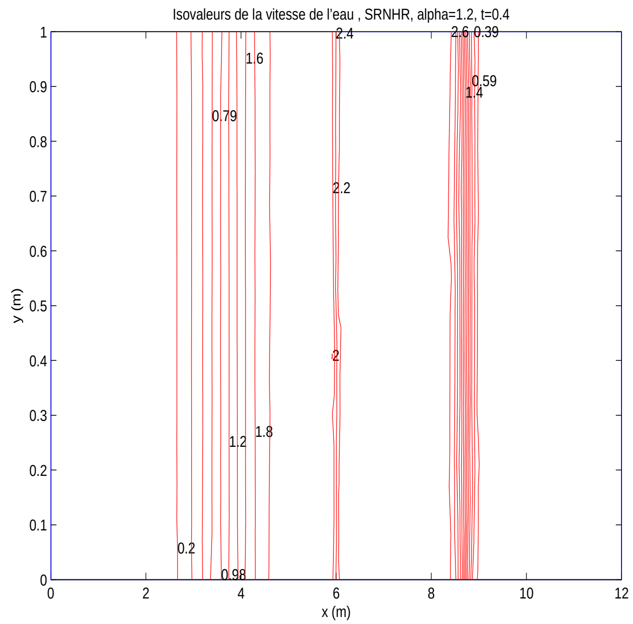


Non homogeneous 2D St-Venant, water level ; SRNHR scheme.



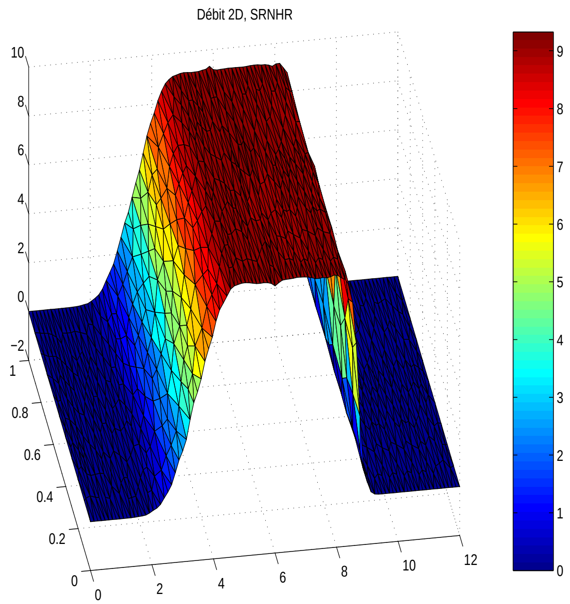
Isolines ; water level ; SRNHR scheme.

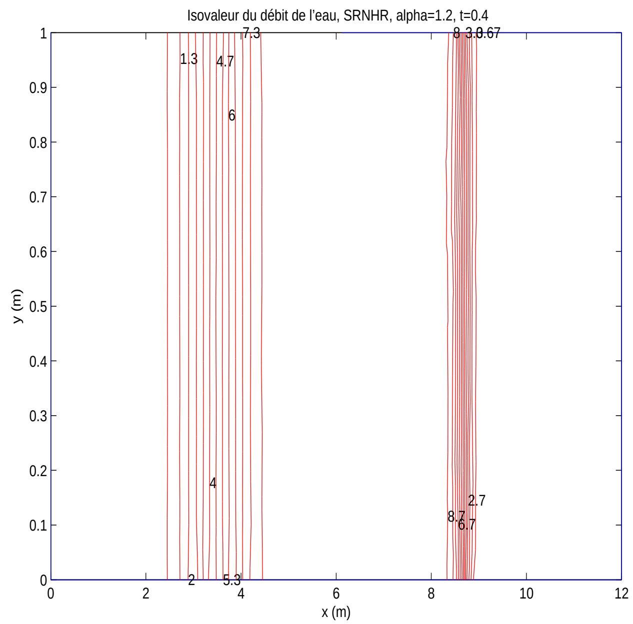




Non homogeneous 2D St-Venant, water velocity ; SRNHR scheme.

Isolines ; water velocity ; SRNHR scheme.





Non homogeneous 2D St-Venant, water flow ; SRNHR scheme.

Isolines ; water flow ; SRNHR scheme.

9 Two phase flow

1D case :

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial f(U)}{\partial x} = Q_1(x,U) + Q_2(x,U) \\ \\ U(x,0) = U_0(x) \end{cases} \tag{6}$$

with

$$U = (\mu_v \rho_v \quad \mu_v \rho_v u_v \quad \mu_l \rho_l \quad \mu_l \rho_l u_l)^t,$$

$$f(U) = (\mu_v \rho_v u_v \quad \mu_v \rho_v u_v^2 \quad \mu_l \rho_l u_l \quad \mu_l \rho_l u_l^2)^t,$$

$$Q_1(x,U) = (0 \quad -\mu_v \frac{\partial P}{\partial x} \quad 0 \quad -\mu_l \frac{\partial P}{\partial x})^t,$$

$$Q_2(x,U) = (0 \quad \mu_v \rho_v g \quad 0 \quad \mu_l \rho_l g)^t.$$

Non hyperbolic and non conservative problem.

1D Ransom Problem

Initial condition :

$\forall x \in [x_0, x_l]$, $\mu_v(t=0) = 0.2$,

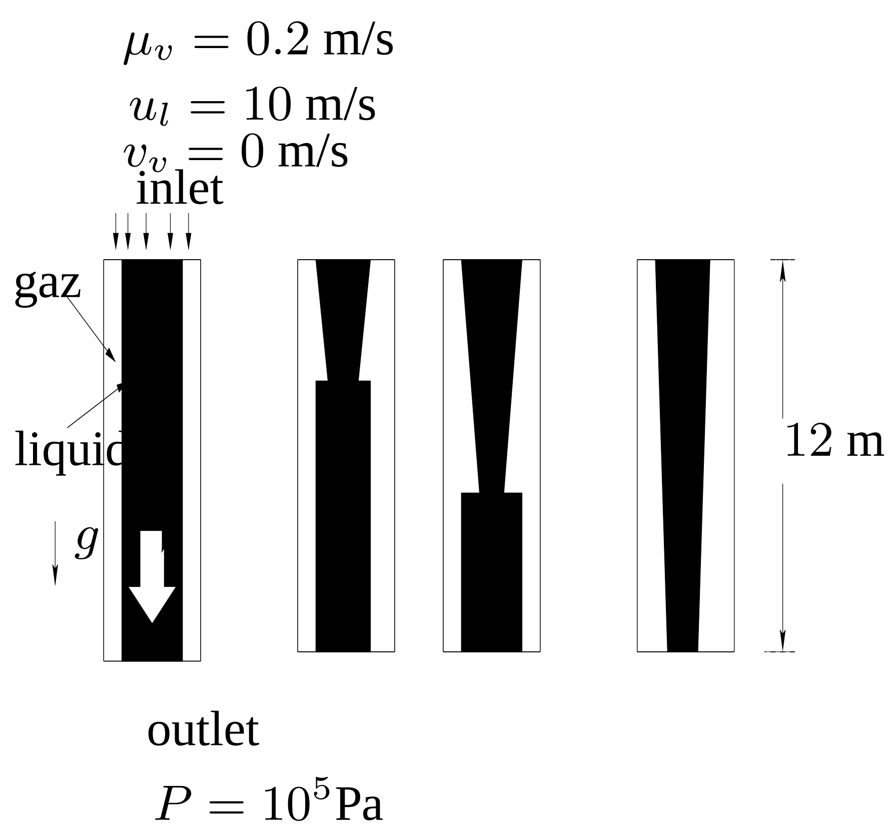
$u_l(t=0) = 10$,

$u_v(t=0) = 0$,

$p(t=0) = 10^5$,

$\rho_v(t=0) = 1$,

$\rho_l(t=0) = 988,0638.$



Boundary conditions :

- inlet ($x_0 = 0$) :

  $\mu_v(0,t) = 0.2$,

  $u_l(0,t) = 10$, $u_v(0,t) = 0$.

- outlet ($x_l = 12$) :

  $p\,(12,t) = 10^5$.

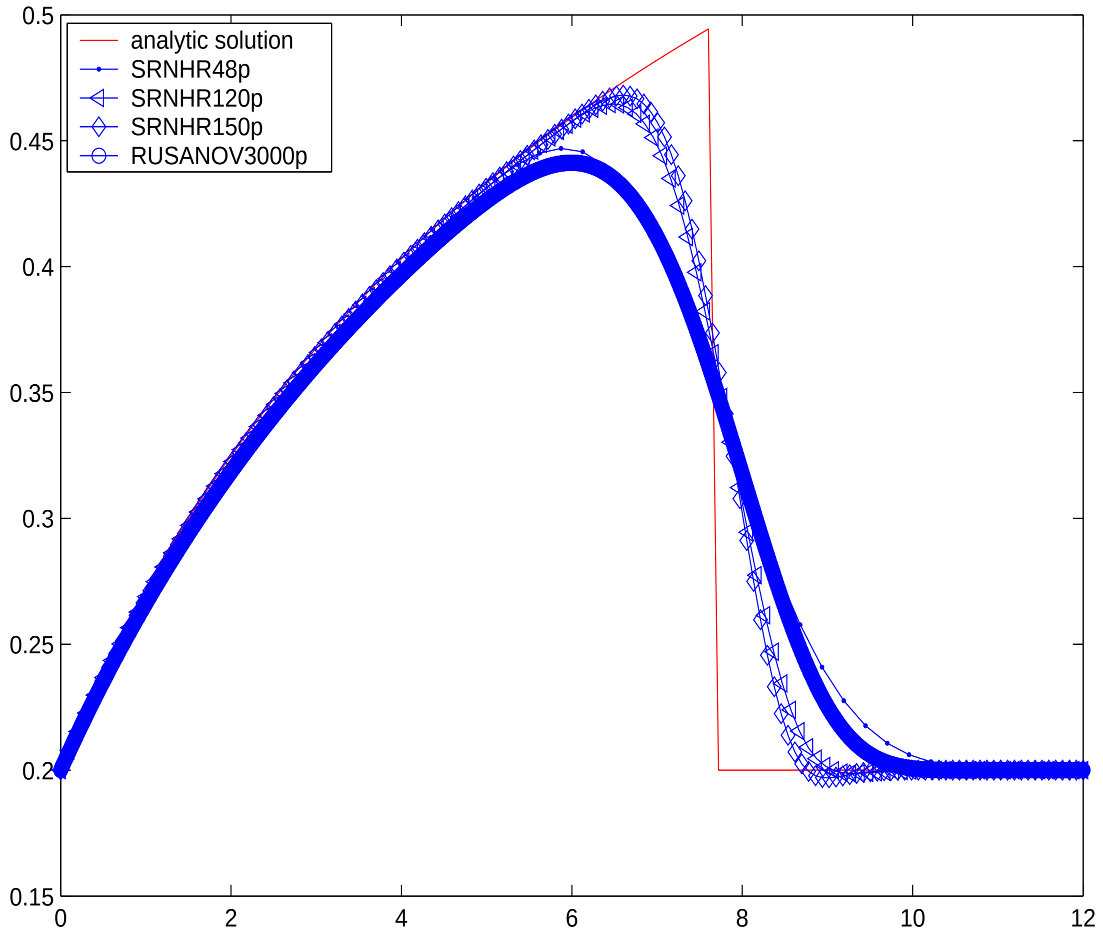


FIG. 3 – Void fraction.

10 Discretization of source term

SRNHR scheme :

$$\begin{cases} U^n_{j+\frac{1}{2}} = (\nu U^n_j + (1-\nu) U^n_{j+1}) - \dfrac{\alpha^n_{j+\frac{1}{2}}}{2 s^n_{j+\frac{1}{2}}} \left[f(U^n_{j+1}) - f(U^n_j) \right] - \dfrac{\alpha^n_{j+\frac{1}{2}}}{2} \dfrac{\Delta x}{s^n_{j+\frac{1}{2}}} \hat{Q}^n_{j+\frac{1}{2}} \\ U^{n+1}_j = U^n_j - r \left[f\left(U^n_{j+\frac{1}{2}} \right) - f\left(U^n_{j-\frac{1}{2}}) \right) \right] + \Delta t^n Q^n_j, \end{cases}$$

$$Q^n_j = \frac{1}{2\Delta x} \left(\mu^n_j \right) \left(P_{j+1} - P_{j-1} \right) \text{ (second step).}$$

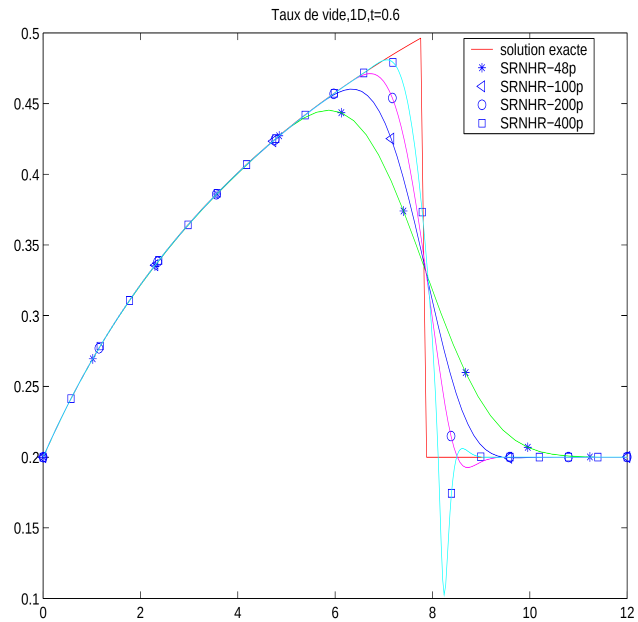
$$\hat{Q}^n_{j+\frac{1}{2}} = \frac{\tilde{\mu}^n_{j+\frac{1}{2}}}{2} \frac{(P_{j+1} - P_j)}{h} \text{ (first step).}$$

$\tilde{\mu}^n_{j+\frac{1}{2}}$: intermediate value between μ^n_j and μ^n_{j+1}.

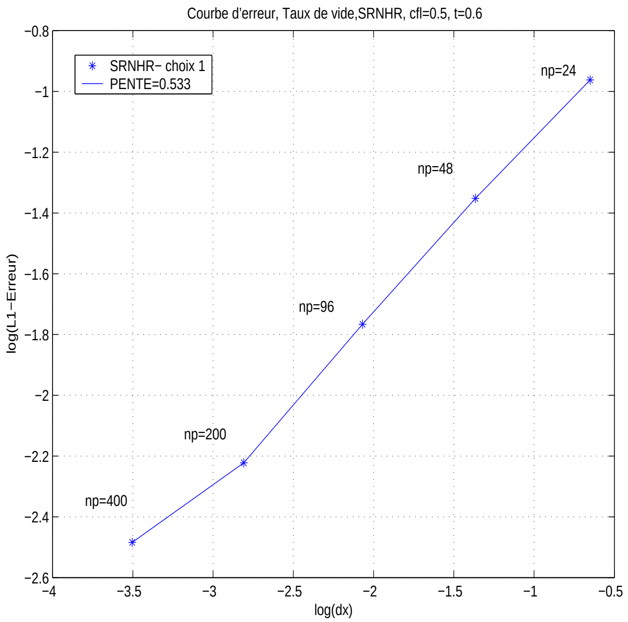
To preserve stationnary states, 2 choices :

$\tilde{\mu}^n_{j+\frac{1}{2}} = \frac{1}{2}(\mu^n_j + \mu^n_{j+1})$: results given before.

$\tilde{\mu}^n_{j+\frac{1}{2}} = \mu^n_{j+\frac{1}{2}}$ obtained with $U^n_{j+\frac{1}{2}}$ computed at same step $\longrightarrow$ better in this case.



Void fraction ; $\tilde{\mu}^n_{j+\frac{1}{2}} = \mu^n_{j+\frac{1}{2}}$.



Error curve ; Void fraction ; $\tilde{\mu}^n_{j+\frac{1}{2}} = \mu^n_{j+\frac{1}{2}}$.

11 Model with interfacial pressure

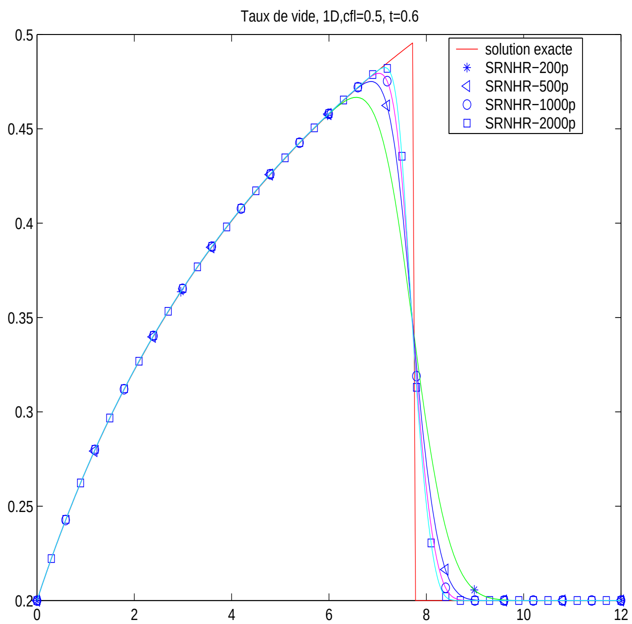
Source term writes now :

$$Q_1(x,W) = (0 \quad -\mu_v \frac{\partial P}{\partial x} - (P-P_i) \partial\mu_v \partial x \quad 0 \quad -\mu_l \frac{\partial P}{\partial x} - (P-P_i)\frac{\partial \mu_l}{\partial x})^t,$$

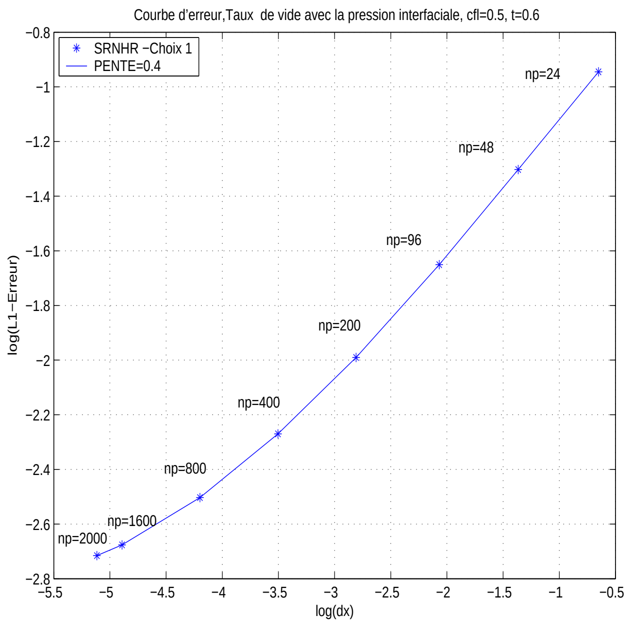
$$Q_2(x,W) = (0 \quad \mu_v \rho_v g \quad 0 \quad \mu_l \rho_l g)^t,$$

with $P - P_i = \rho_v (u_v - u_l)^2$: interfacial pressure.

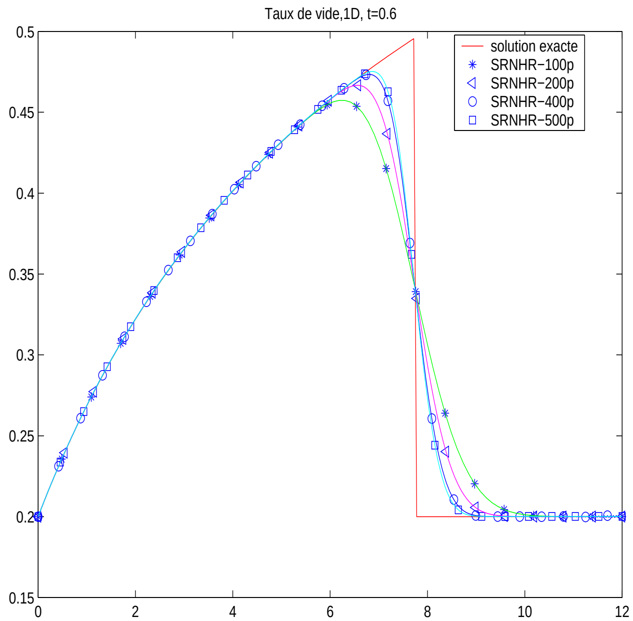
$\longrightarrow$ Hyperbolic problem



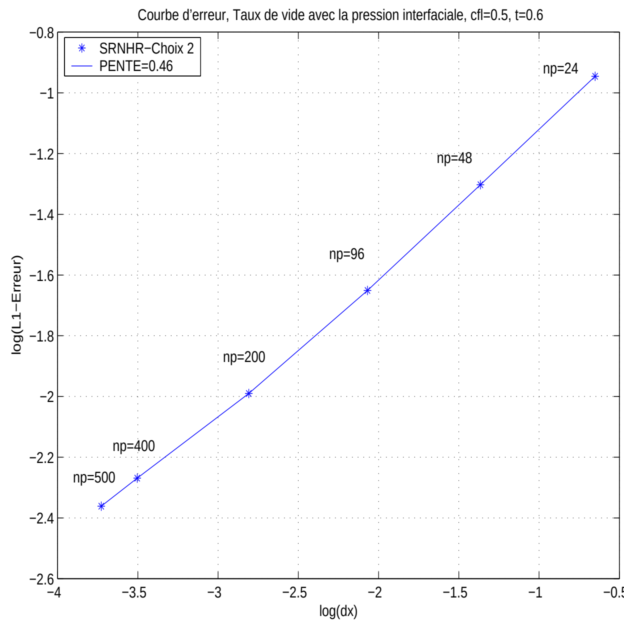
Void Fraction (with interfacial pressure) ; $\tilde{\mu}^n_{j+\frac{1}{2}} = \mu^n_{j+\frac{1}{2}}$.



Error curve ; Void Fraction (with interfacial pressure) ; $\tilde{\mu}^n_{j+\frac{1}{2}} = \mu^n_{j+\frac{1}{2}}$.



Void Fraction (with interfacial pressure) ; $\tilde{\mu}^n_{j+\frac{1}{2}} = \frac{1}{2}(\mu^n_j + \mu^n_{j+1})$.



Error curve ; Void Fraction (with interfacial pressure) ; $\tilde{\mu}^n_{j+\frac{1}{2}} = \frac{1}{2}(\mu^n_j + \mu^n_{j+1})$.

	$\tilde{\mu}^n_{j+\frac{1}{2}} = \frac{1}{2}(\mu^n_j + \mu^n_{j+1})$	$\tilde{\mu}^n_{j+\frac{1}{2}} = \mu^n_{j+\frac{1}{2}}$
Classical model	Refinement until 150 points	Refinement until 400 points
Interfacial pressure	Refinement until 500 points	Refinement until 2000 points

2D case :

$$\left\{\begin{array}{l}\dfrac{\partial W}{\partial t}+\dfrac{\partial F(W)}{\partial x}+\dfrac{\partial G(W)}{\partial y}=Q_1(x,,y,W)+Q_2(x,y,W)\\ W(x,y,0)=W_0(x,y),\end{array}\right. \quad (7)$$

with $W(x,y,t)=(\mu_l\rho_l \;\; \mu_l\rho_l u_l \;\; \mu_l\rho_l v_l \;\; \mu_v\rho_v \;\; \mu_v\rho_v u_v \;\; \mu_v\rho_v v_v)^t$,

$$F(W(x,y,t))=(\mu_l\rho_l u_l \;\; \mu_l\rho_l u_l^2 \;\; \mu_l\rho_l u_l v_l \;\; \mu_v\rho_v u_v \;\; \mu_v\rho_v u_v^2 \;\; \mu_v\rho_v u_v v_v)^t,$$

$$G(W(x,y,t))=(\mu_l\rho_l v_l \;\; \mu_l\rho_l u_l v_l \;\; \mu_l\rho_l v_l^2 \;\; \mu_v\rho_v v_v \;\; \mu_v\rho_v u_v v_v \;\; \mu_v\rho_v v_v^2)^t,$$

$$Q_1(x,y,W)=(0 \quad -\mu_l\frac{\partial P}{\partial x} \quad -\mu_l\frac{\partial P}{\partial y} \quad 0 \quad -\mu_v\frac{\partial P}{\partial x} \quad -\mu_v\frac{\partial P}{\partial y})^t,$$

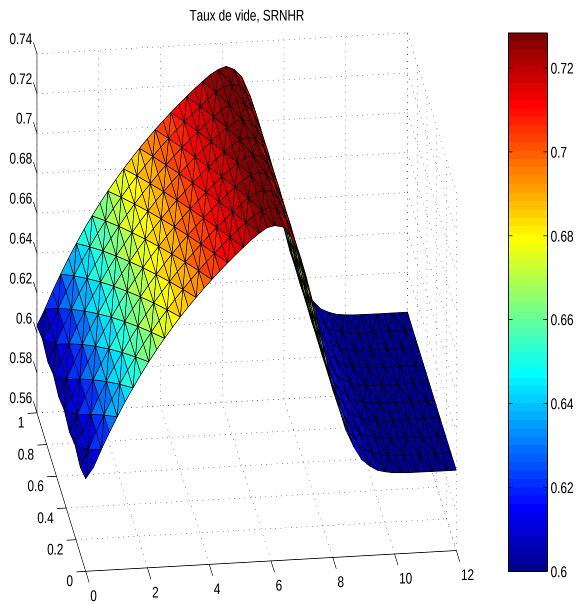
$$Q_2(x,y,W)=(0 \;\; \mu_l\rho_l g \;\; 0 \;\; 0 \;\; \mu_v\rho_v g \;\; 0)^t.$$

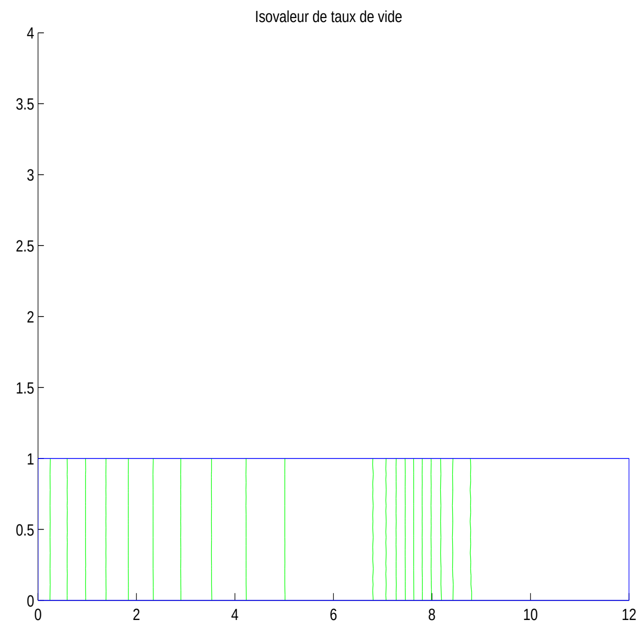
ρ_k, μ_k, u_k, v_k : density, void fraction, velocities.

P : commun pressure.

$P=A_v\rho_v^\gamma$, $\rho_l=K_l P^a$, A_v, γ, K_l, a : constantes.

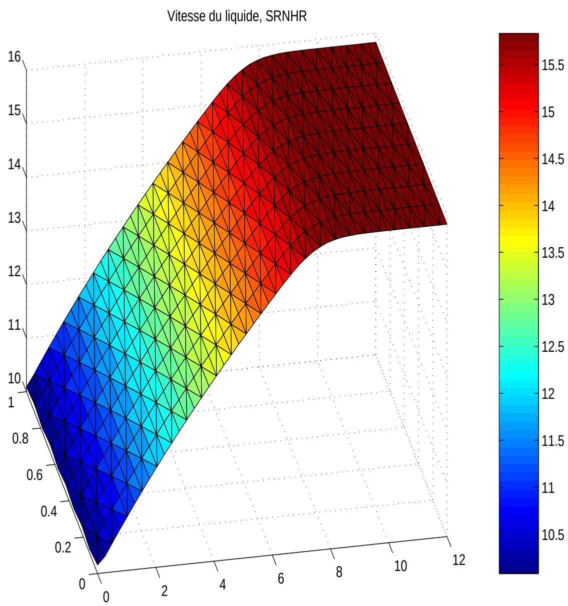
$\mu_v = 0.6$. UK Drag mesh 48×10.



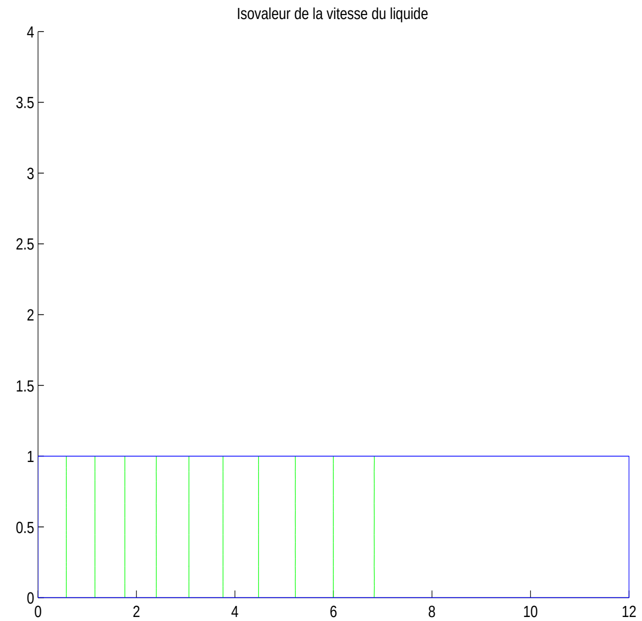


Void fraction ; SRNHR scheme ($\alpha = 2$).

Void fraction ; Isolines ; SRNHR scheme ($\alpha = 2$).



Liquid velocity ; SRNHR scheme ($\alpha = 2$).



Liquid velocity ; Isolines ; SRNHR scheme ($\alpha = 2$).

Conclusion :

- Robust and efficient scheme for non homogeneous systems.
- Do not need calculus of jacobien fields decomposition.
- Accurate results obtained with few mesh points.