Combined Characteristics and Finite Volume Methods for Dam-break Problems

New Approach

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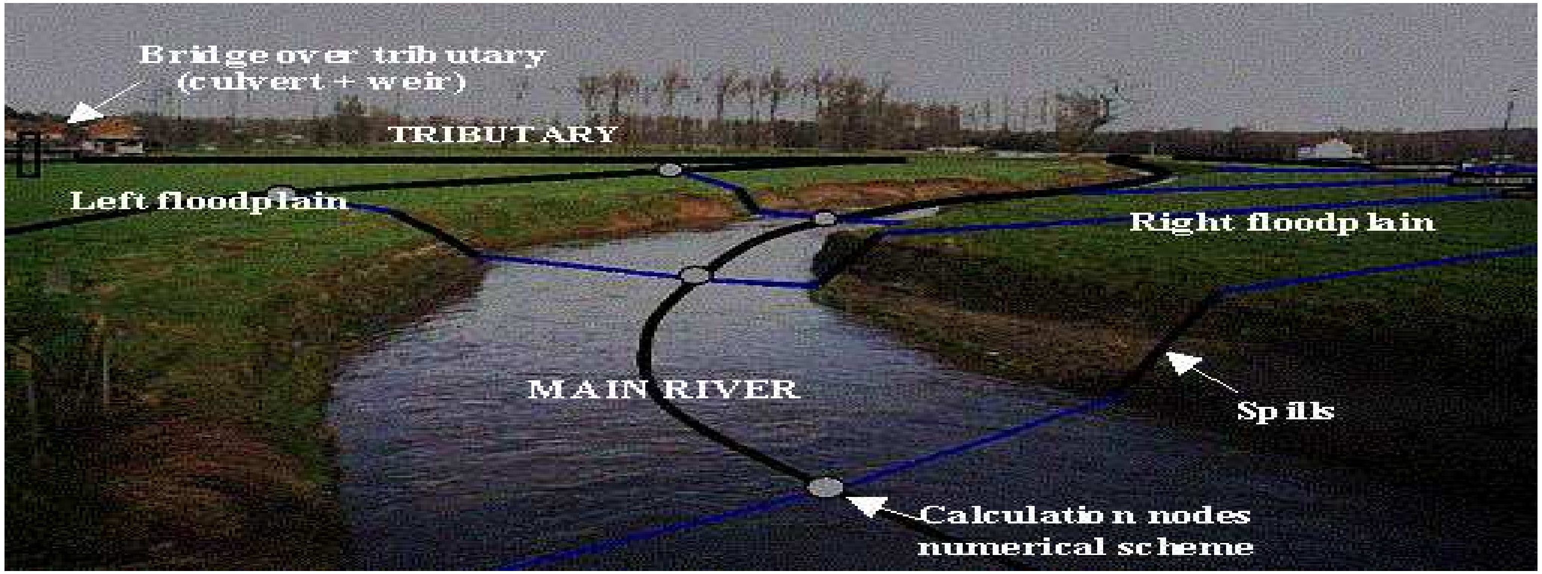
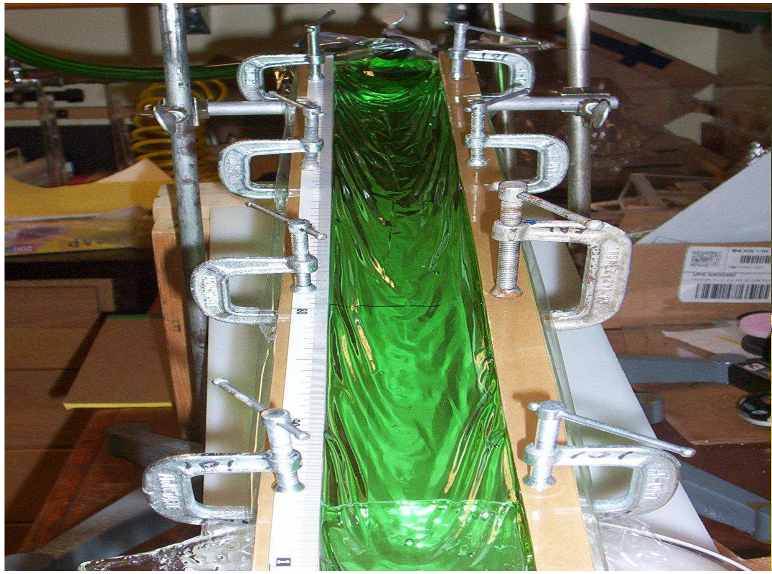
Overview


Bridge over tributary
(culvert + weir)
TRIBUTARY
Left floodplain
Right floodplain
MAIN RIVER
Spills
Calculation nodes
numerical scheme



Outline

- Mathematical Models
  - Shallow Water Equations with Horizontal Variable Density
  - Dam-break Problems
- Combined Characteristics and Finite Volume Methods
  - Formulation of the Finite Volume Method
  - Formulation of the Characteristics Method
  - A Simple Linear Analysis
- Numerical Results

Equations for Dam-break Problems

SWE with Horizontal Variable Density

We consider the one-dimensional shallow water equations written in conservative as

$$\begin{aligned} \partial_t(\rho h) + \partial_x(\rho h u) &= 0, \\ \partial_t(\rho h u) + \partial_x\left(\rho h u^2 + \frac{1}{2}\rho g h^2\right) &= -g\rho h \partial_x Z. \end{aligned}$$

If the density ρ is constant, the equations reduce to

$$\begin{aligned} \partial_t h + \partial_x(hu) &= 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{1}{2}gh^2\right) &= -gh\partial_x Z. \end{aligned}$$

Both systems can reformulated in a compact form as

$$\mathbf{W}_t + \mathbf{F}(\mathbf{W})_x = \mathbf{S}(\mathbf{W}).$$

Advective Formulation of the SWE

The SWE equations

$$\begin{aligned} \partial_t (\rho h) + \partial_x (\rho h u) &= 0, \\ \partial_t (\rho h u) + \partial_x \left(\rho h u^2 + \frac{1}{2}\rho g h^2 \right) &= -g \rho h \partial_x Z, \end{aligned}$$

can also be reformulated in an advective form as

$$\begin{aligned} D_t (\rho h) + \rho h \partial_x u &= 0, \\ D_t u + g \partial_x (h + Z) &= -\frac{g}{2\rho} h \partial_x \rho, \end{aligned}$$

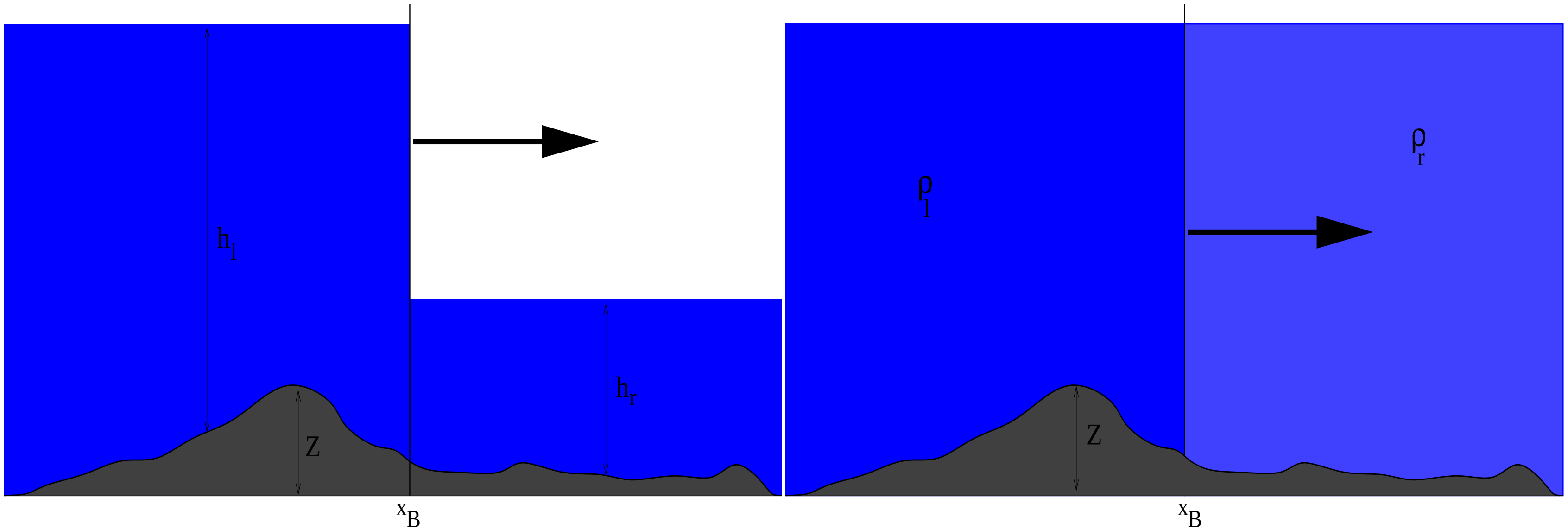
where D_t denotes the total derivative defined by

$$D_t \omega = \partial_t \omega + u \partial_x \omega.$$

Dam-break Problems

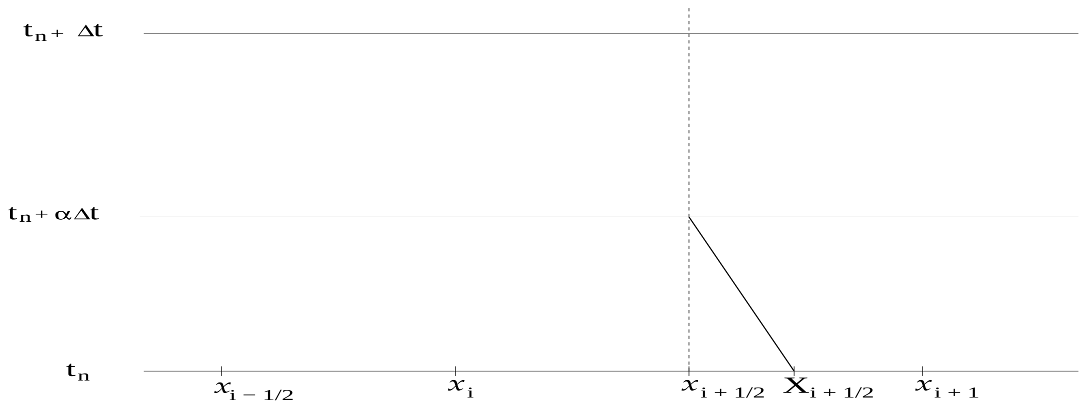
Dam-break problems can be formed either by discontinuous initial conditions for the water height or discontinuous density function within the spatial domain

$$h(x,0) = \begin{cases} h_l, & \text{if} \quad x \leq x_B, \\ h_r, & \text{if} \quad x > x_B. \end{cases} \qquad \rho(x) = \begin{cases} \rho_l, & \text{if} \quad x \leq x_B, \\ \rho_r, & \text{if} \quad x > x_B. \end{cases}$$



Combined Characteristics and Finite Volume Methods

Formulation of the Finite Volume Method



Integrating the system

$$\mathbf{W}_t + \mathbf{F}(\mathbf{W})_x = \mathbf{0},$$

with respect to time and space over the time-space control domain $[t_n, t_{n+1}] \times [x_{i-1/2}, x_{i+1/2}]$, one obtains the following discrete problem

$$\mathbf{W}_i^{n+1} = \mathbf{W}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}(\mathbf{W}_{i+1/2}^n) - \mathbf{F}(\mathbf{W}_{i-1/2}^n)\right),$$

where $\mathbf{W}_i^n$ is the time-space average of the solution $\mathbf{W}$ in the domain $[x_{i-1/2}, x_{i+1/2}]$ at time t_n and $\mathbf{F}(\mathbf{W}_{i\pm1/2}^n)$ are the numerical fluxes at $x = x_{i\pm1/2}$ and time t_n.

$$\mathbf{W}_{i\pm1/2}^n?$$

Formulation of the Characteristics Method

We reconstruct the numerical fluxes $\mathbf{W}^n_{i\pm1/2}$ using the method of characteristics applied to the SWE rewritten in physical variables $\mathbf{U} = (\rho h, u)^T$

$$\begin{aligned} D_t(\rho h) + \rho h \partial_x u &= 0, \\ D_t u + g\partial_x(h+Z) &= -\frac{g}{2\rho} h\partial_x \rho. \end{aligned}$$

Thus, the associated characteristic curves are solutions of the initial-value problem

$$\begin{aligned} \frac{dX_{i+1/2}(\tau)}{d\tau} &= u_{i+1/2}\Big(\tau, X_{i+1/2}(\tau)\Big), \quad \tau \in [t_n, t_n+\Delta t], \\ X_{i+1/2}(t_n+\Delta t) &= x_{i+1/2}. \end{aligned}$$

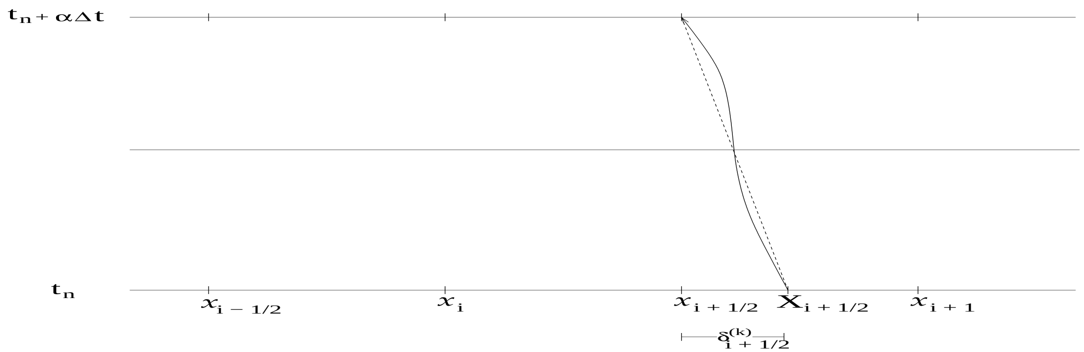
Note that $X_{i+1/2}(\tau)$ is the departure point at time τ of a particle that will arrive at point $x_{i+1/2}$ in time $t_n + \Delta t$. The solutions $X_{i+1/2}(\tau)$ can be expressed as

$$X_{i+1/2}(t_n) = x_{i+1/2} - \int_{t_n}^{t_n+\Delta t} u_{i+1/2}\left(X_{i+1/2}(\tau)\right) d\tau. \tag{1}$$

The integral in (1) can be calculated using the simple iteration

$$\delta^{(m+1)}_{i+1/2} = \alpha \Delta t u\left(t_n, x_{i+1/2} - \delta^{(m)}_{i+1/2}\right), \qquad m = 0, 1, \ldots. \tag{2}$$

Formulation of the Characteristics Method



Once the characteristics curves $X_{i+1/2}(t_n)$ are known, the intermediate solutions are reconstructed as

$$\mathbf{U}^n_{i+1/2} = \mathbf{U}\left(t_n + \alpha\Delta t, x_{i+1/2}\right) = \tilde{\mathbf{U}}\left(t_n, X_{i+1/2}(t_n)\right),$$

where $\tilde{\mathbf{U}}\left(t_n, X_{i+1/2}(t_n)\right)$ is the solution at the characteristic foot computed by interpolation from the gridpoints of the control volume where the departure point resides *i.e.*

$$\tilde{\mathbf{U}}\left(t_n, X_{i+1/2}(t_n)\right) = \mathcal{P}\Big(\mathbf{U}\left(t_n, X_{i+1/2}(t_n)\right)\Big),$$

where $\mathcal{P}$ represents the interpolating polynomial. For instance a Lagrange-based interpolation polynomials can be formulated as

$$\mathcal{P}\Big(\mathbf{U}\left(t_n, X_{i+1/2}(t_n)\right)\Big) = \sum_k l_k(X_{i+1/2})\mathbf{U}^n_k, \qquad \text{with} \qquad l_k(x) = \prod_{\substack{q=0\\q\neq k}} \frac{x - x_q}{x_k - x_q}.$$

A Simple Linear Analysis

Let us consider the linear problem

$$\partial_t u + c\partial_x u = 0,$$

where c is a constant. The proposed characteristic finite volume scheme can be interpreted as the predictor stage

$$U^n_{i+1/2} = u\left(t_n + \alpha\Delta t, x_{i+1/2}\right) = \mathcal{P}\Big(U\left(t_n, X_{i+1/2}(t_n)\right)\Big), \qquad (3)$$

followed by the corrector stage

$$U^{n+1}_i = U^n_i - \frac{\alpha\Delta t}{\Delta x}\left(f(U^n_{i+1/2}) - f(U^n_{i-1/2})\right). \qquad (4)$$

Using a linear interpolating polynomial $\mathcal{P}$ for the linear problem, we have the following results:

LEMMA **1** *Let Δt satisfy the condition*

$$\frac{1}{2\alpha} \leq |c|\frac{\Delta t}{\Delta x} \leq \frac{1}{\sqrt{2\alpha}}.$$

Then the characteristic finite volume scheme (3) *and* (4) *is stable and TVD.*

C-property

Applied to the shallow water equations the characteristic finite volume scheme gives

$$\begin{aligned} h_i^{n+1} &= h_i^n - \frac{\Delta t}{\Delta x}\left(q_{i+1/2}^n - q_{i-1/2}^n\right), \\ q_i^{n+1} &= q_i^n - \frac{g}{2}\frac{\Delta t}{\Delta x}\left(\left(h_{i+1/2}^n\right)^2 - \left(h_{i-1/2}^n\right)^2\right) - \frac{g}{2}\frac{\Delta t}{\Delta x}\hat{h}_i^n\left(Z_{i+1} - Z_{i-1}\right), \end{aligned}$$

where the intermediate solutions are given by

$$\begin{aligned} h_{i+1/2}^{n+1} &= \tilde{h}_{i+1/2}^n - \alpha\Delta t\tilde{h}_{i+1/2}^n\left(u_{i+1} - u_i\right), \\ u_{i+1/2}^{n+1} &= \tilde{u}_{i+1/2}^n - \alpha\Delta t g\left(h_{i+1}^n + Z_{i+1} - h_i^n + Z_i\right). \end{aligned}$$

Using a linear interpolating polynomial $\mathcal{P}$ for the linear problem, we have the following results:

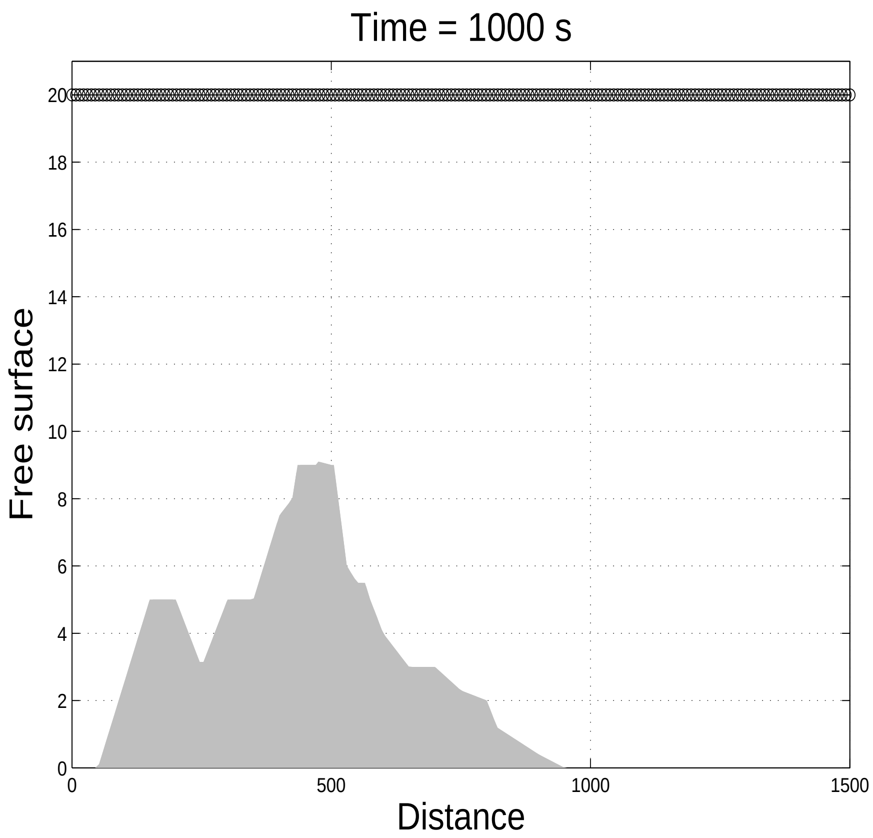
LEMMA **2** *If the source term is approximated by*

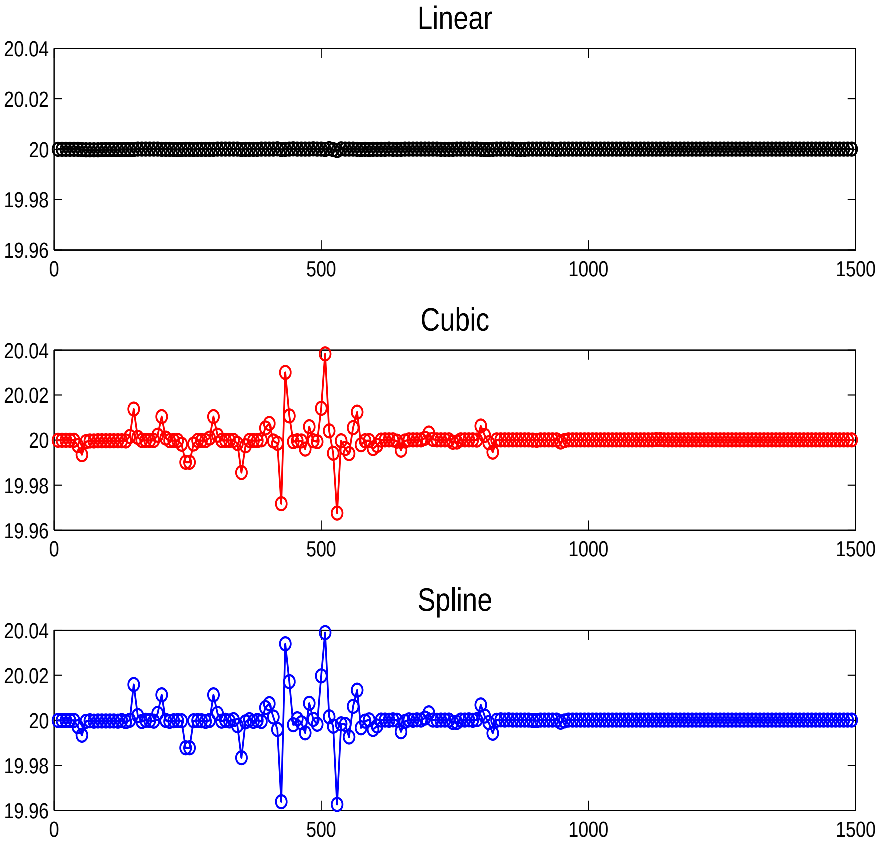
$$\hat{h}_i^n = \frac{1}{4}\left(h_{i+1}^n + 2h_i^n + h_{i-1}^n\right).$$

Then the characteristic finite volume scheme stasfies the C-property.

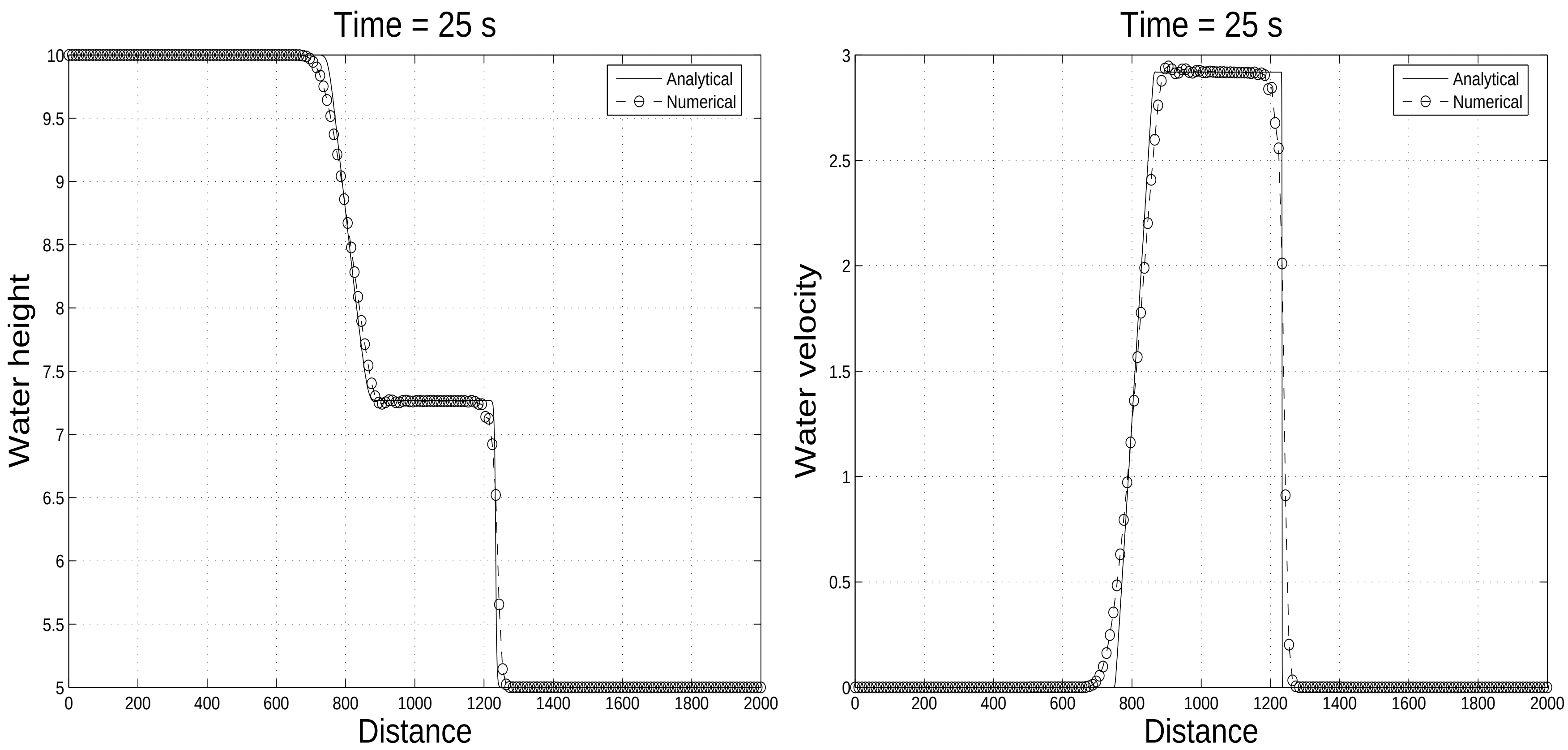
Numerical Results

Verification of the C-property

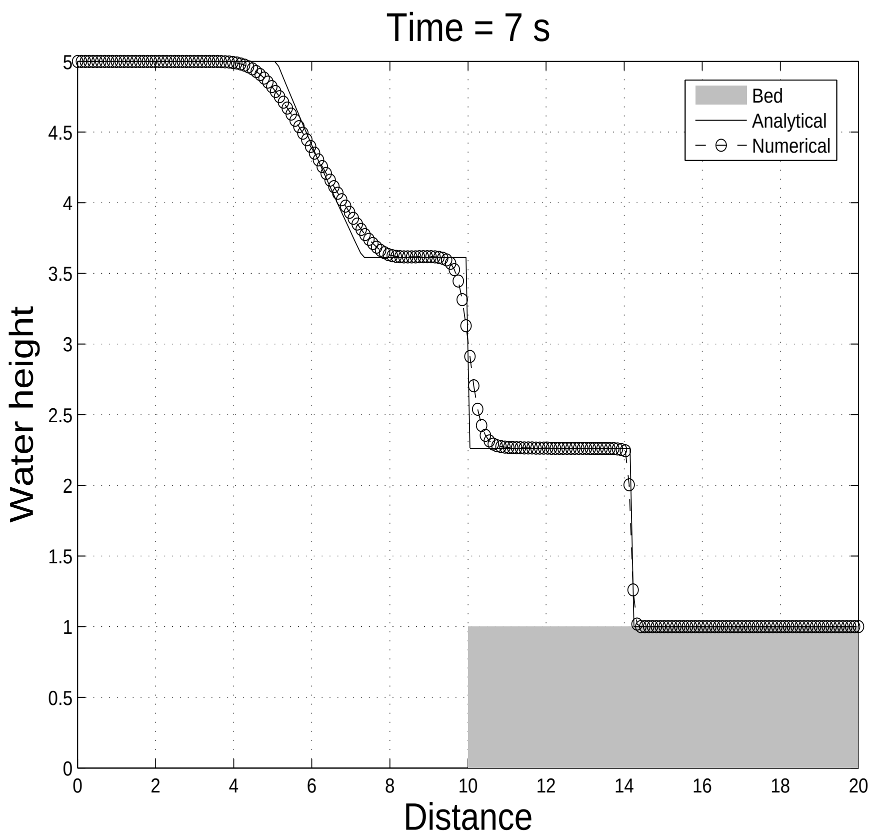


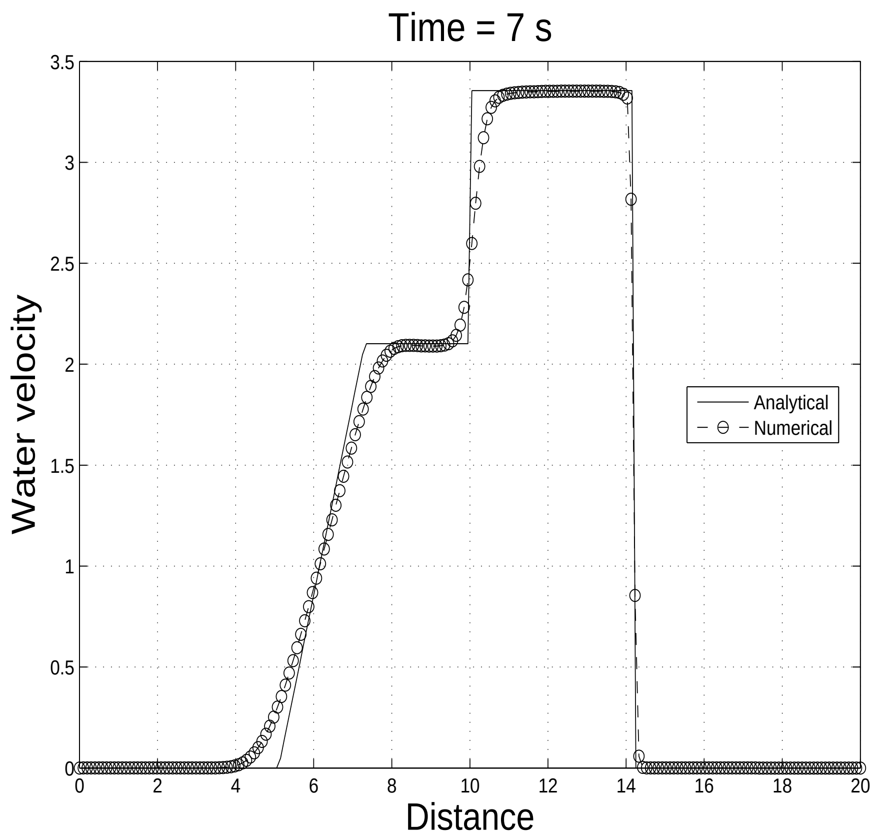


Dam-break on a wet bed

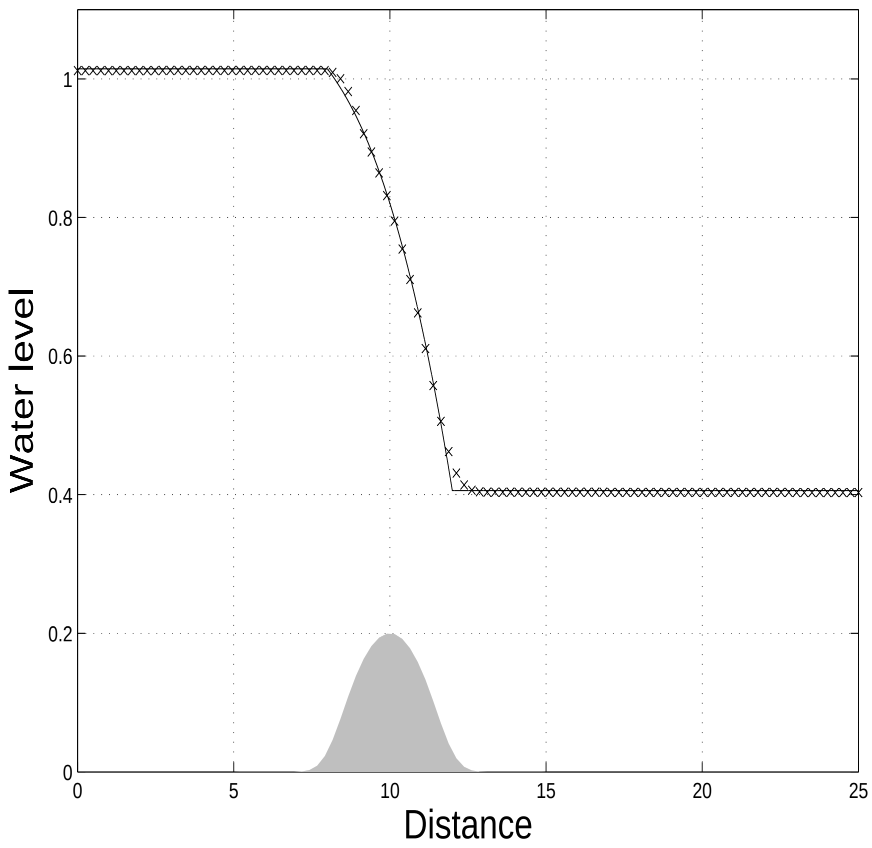


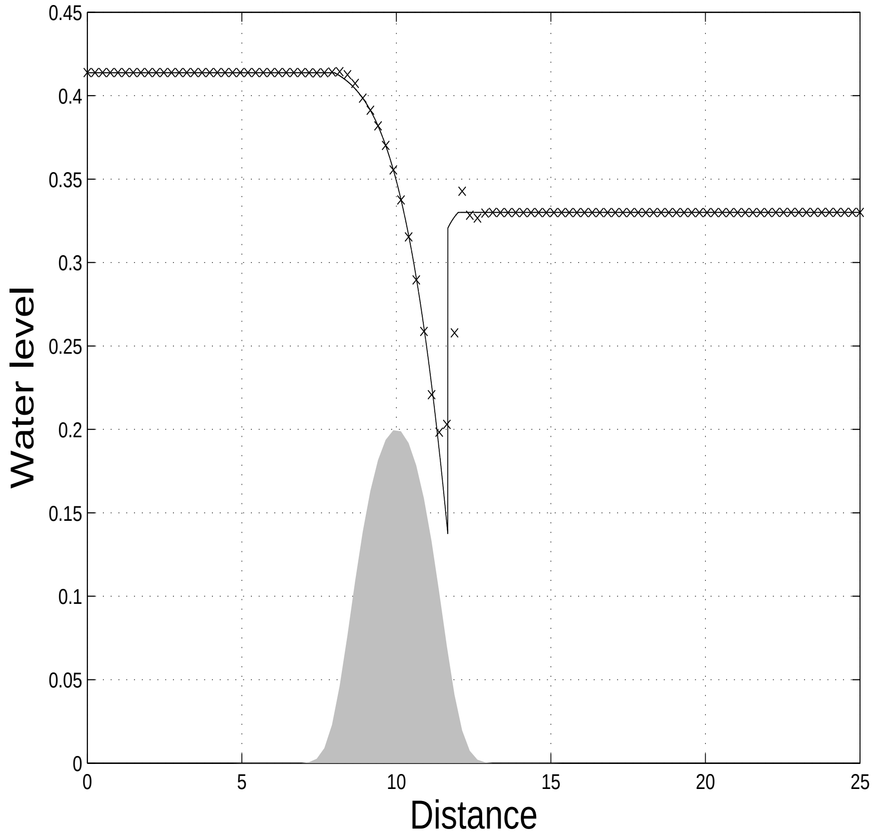
Dam-break over a step-type bed



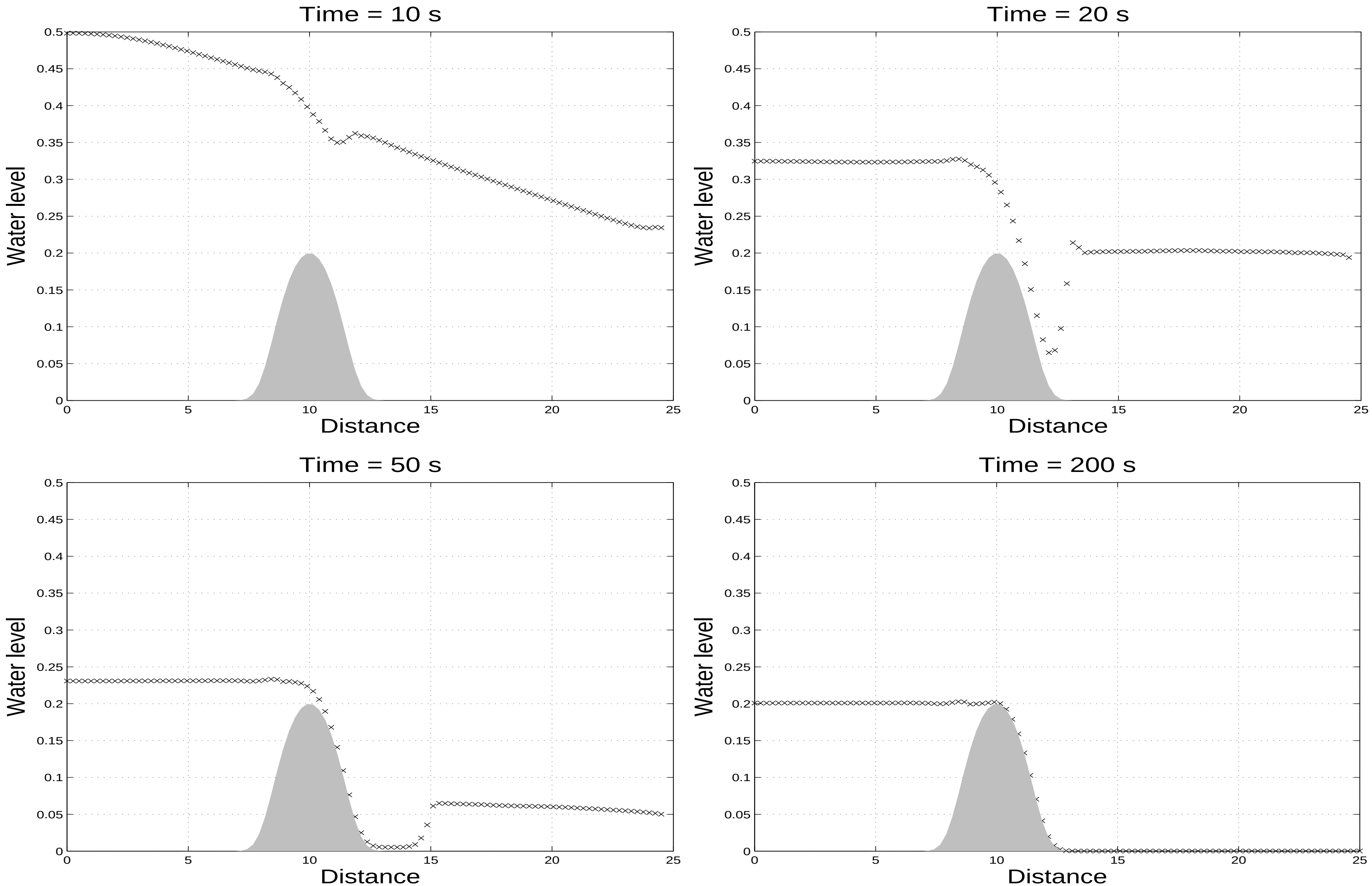


Drain on a non-flat bottom





Drain on a non-flat bottom



Summary

Summary

- Combined characteristics and finite volume method performs well for numerical solution of dam-break problems.
- Combined characteristics and finite volume method does not require Riemann-problem solvers.
- Combined characteristics and finite volume method is conservative and satisfy the C-property.

- Nonlinear analysis.
- Extension to two-dimensional problems.

Thank You.