EFFICIENT SCHEMES AND ADAPTIVE NUMERICAL SIMULATIONS FOR WATER PROBLEMS

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Summary of the talk:

- Introduction
- Presentation of *SRNH* scheme
- Application to pollutant transport in the strait of Gibraltar
- Application to sediment transport problems
- Limitations of the classical Shallow Water model
- A Multilayer Shallow Water model
- A Characteristic Finite Volume scheme
- First results with Multilayer Shallow Water model
- Future work

Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on conservative systems and exact Jacobian eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of two steps schemes designed for non conservative systems.

SRNH SCHEME FOR NON CONSERVATIVE SYSTEMS

SRNH scheme for $1D$ non homogeneous systems of balance laws:

$$\left\{ \frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} = Q(x, W) \text{ in } X = D\times]0, T[\right. \tag{1}$$

$$\begin{cases} W^n_{i+\frac{1}{2}} & = \frac{1}{2}\left(W^n_{i+1} + W^n_i\right) - \frac{1}{2}\,\mathrm{sgn}\left[\mathcal{B}^n_{i+\frac{1}{2}}\right]\left(W^n_{i+1} - W^n_i\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}^n_{i+\frac{1}{2}}\right|^{-1} Q^n_{i+\frac{1}{2}} \\ W^{n+1}_i & = W^n_i - r\left(F(W^n_{i+\frac{1}{2}}) - F(W^n_{i-\frac{1}{2}})\right) + \Delta t Q^n_i \end{cases} \tag{2}$$

with : $\mathcal{B}^n_{i+\frac{1}{2}} = \left(\mathcal{R}\Lambda^\star\mathcal{R}^{-1}\right)\left(V\left(W^n_i, W^n_{i+1}\right)\right)$ is an approximation of the jacobian matrix calculated at the average state $V\left(W^n_i, W^n_{i+1}\right)$.

Application of *SRNH* scheme to non homogeneous 2D Shallow Water flows :

The system considered may by written as follows :

$$\begin{cases} h_{,t} + (hu)_{,x} + (hv)_{,y} = 0 \\ (hu)_{,t} + (hu^2)_{,x} + (huv)_{,y} + g\left(\frac{h^2}{2}\right)_{,x} = -gh(Z_f)_{,x} \\ (hv)_{,t} + (huv)_{,x} + (hv^2)_{,y} + g\left(\frac{h^2}{2}\right)_{,y} = -gh(Z_f)_{,y}, \end{cases} \tag{3}$$

where h is the water level, $\mathbf{u} = {}^t(u, v)$ the water velocity and Z_f the bottom height.

To calculate the predictor phase of *SRNH* scheme, one projects the equations on each interface e_{ij}:

$$(U_\eta)_t + (F_\eta)_{,\eta} = Q(x, y, U_\eta) \tag{4}$$

with

$$U_\eta = (h, hu_\eta, hu_\tau)^T, \qquad F_\eta = \left(hu_\eta, hu_\eta^2 + g\frac{h^2}{2}, hu_\eta u_\tau\right)^T,$$

$$\text{et} \qquad Q(x, y, U_\eta) = (0, -gh(Z_f)_{,\eta}, 0)^T,$$

$u_\eta = \mathbf{u} \cdot \eta$, $u_\tau = \mathbf{u} \cdot \tau$, η and τ normal and tangential vector to the interface.

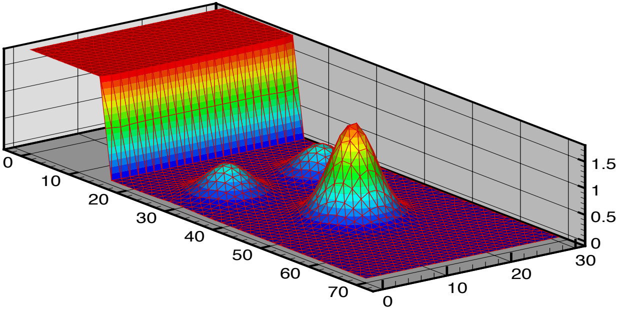
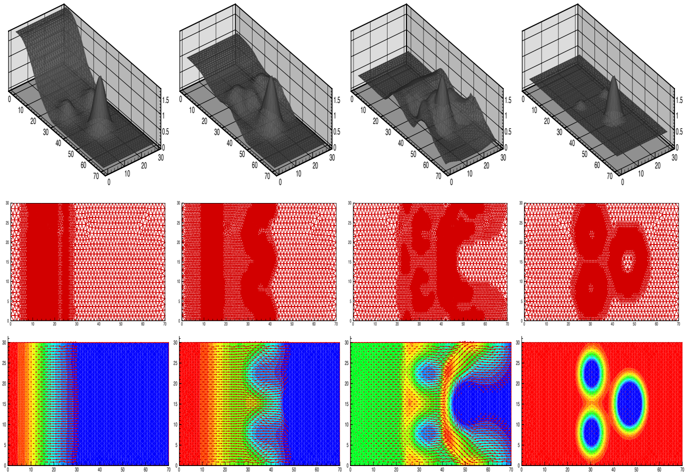


Figure: Initial conditions for the dam-break over three mounds.



Water free-surface (first row), Adapted meshes (second row) and velocity vectors (third row) at different simulation times. From left to right $t = 2$, 6, 12 and 300 s.

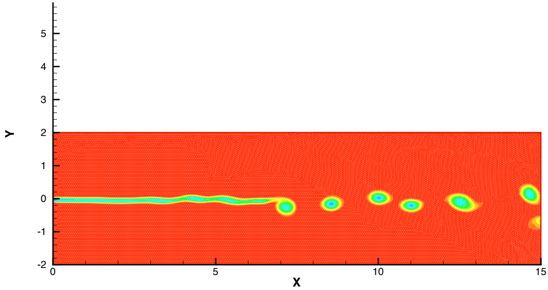


Figure: Vorticity in SW mixing layer using Smagorinsky model and LES

POLLUTANT TRANSPORT IN THE STRAIT OF GIBRALTAR

For simplicity in presentation we write the equations in a conservative form as:

$$\partial_t \mathbf{W} + \partial_x \Big(\mathbf{F}(\mathbf{W}) - \tilde{\mathbf{F}}(\mathbf{W})\Big) + \partial_y \Big(\mathbf{G}(\mathbf{W}) - \tilde{\mathbf{G}}(\mathbf{W})\Big) = \mathbf{Q}(\mathbf{W}), \quad (5)$$

where:

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hv \\ hC \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{W}) = \begin{pmatrix} 0 \\ -gh(S_{0x}+S_{fx}) \\ -gh(S_{0y}+S_{fy}) \\ hQ \end{pmatrix},$$

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \\ huC \end{pmatrix}, \qquad \mathbf{G}(\mathbf{W}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \\ hvC \end{pmatrix},$$

$$\tilde{\mathbf{F}}(\mathbf{W}) = \left(0, 0, 0, D_{xx}\partial_x (hC) + D_{xy}\partial_y (hC)\right)^T$$

$$\tilde{\mathbf{G}}(\mathbf{W}) = \left(0, 0, 0, D_{yx}\partial_x (hC) + D_{yy}\partial_y (hC)\right)^T$$

$S_{fx} = M^2 \dfrac{u\sqrt{u^2+v^2}}{h^{4/3}}$, $S_{fy} = M^2 \dfrac{v\sqrt{u^2+v^2}}{h^{4/3}}$, where M is the Manning roughness coefficient.

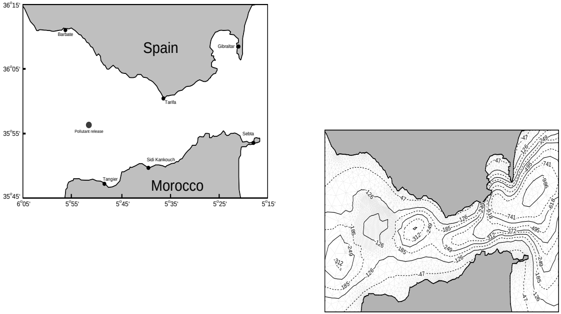


Figure: Definition of the strait of Gibraltar (left) and its bathymetry (right).

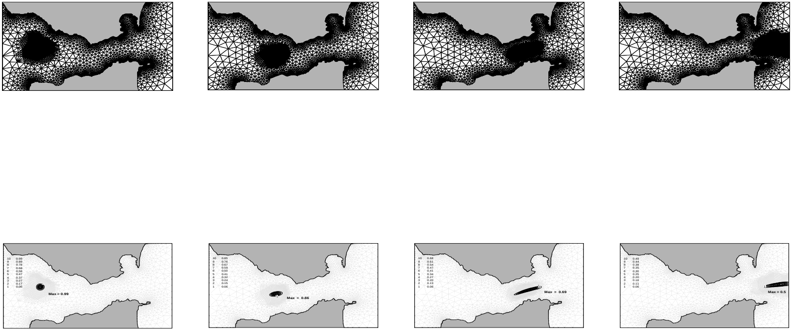


Figure: Adapted meshes (first row) and pollutant concentration (second row) at different simulation times.

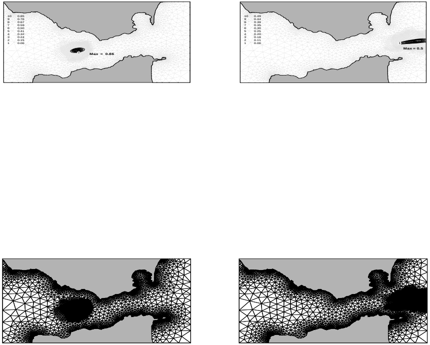


Figure: Adapted meshes (first row) and pollutant concentration (second row) at different simulation times.

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NUMERICAL MODELING OF BED LOAD AND SEDIMENT TRANSPORT

Application of *SRNH* scheme to $2D$ bedload transport model:

$$
\begin{aligned}
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0, \\
\partial_t(hu) + \partial_x\left(hu^2 + \frac{1}{2}gh^2\right) + \partial_y\left(huv\right) &= -gh\partial_x Z, \\
\partial_t(hv) + \partial_x\left(huv\right) + \partial_y\left(hv^2 + \frac{1}{2}gh^2\right) &= -gh\partial_y Z,
\end{aligned} \qquad (6)
$$

with the bed-load equation

$$\partial_t Z + \xi\partial_x q_1 + \xi\partial_x q_2 = 0, \qquad (7)$$

with $\xi = \frac{1}{1-\varepsilon}$ and the sediment transport fluxes are:

$$q_1 = Au\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad q_2 = Av\left(\sqrt{u^2+v^2}\right)^{m-1}, \qquad (8)$$

Numerical results

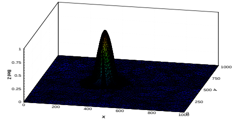
We use a simplified test example of the evolution of an initially hump-shaped bed in a squared channel. The channel is of length 1000 m and the initial bed is defined as

$$Z(0,x,y) = \begin{cases} \sin^2\left(\dfrac{(x-500)\,\pi}{200}\right)\sin^2\left(\dfrac{(x-400)\,\pi}{200}\right), & \text{if} \quad (x,y)\in\Omega, \\ 0, & \text{elsewhere}, \end{cases}$$

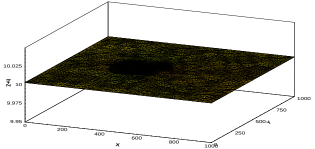
where $\Omega = [500,700]\times[400,600]$ and the initial water level and initial velocity field are

$$h(0,x,x) = 10 - Z(0,x,y), \quad u(0,x,y) = \frac{Q}{h(0,x,y)}, \quad v(0,x,y) = 0, \tag{9}$$

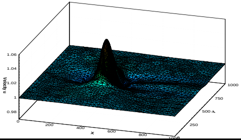
where $m = 3$, the porosity $\epsilon = 0.4$ and the discharge $Q = 10\ m^2/s$.



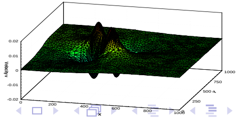
Bed Z
x
y



10.025
10
9.975
9.95
x
y



Velocity u
x
y



Velocity v
x
y

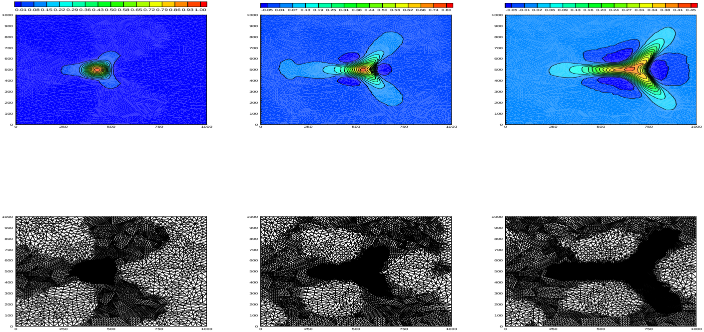


Figure: Bed and mesh evolution for the coupled approach using $A = 1$ at three different times. From left to right $t = 50$, 300 and 600 s.

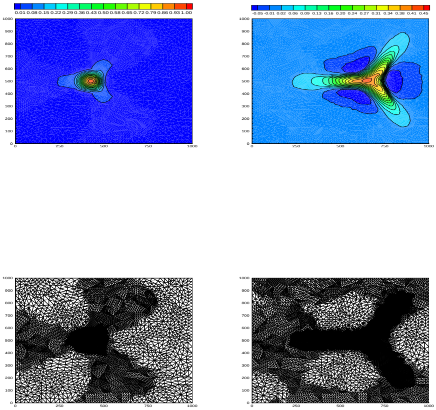


Figure: Bed and mesh evolution for the coupled approach using $A = 1$ at two different times. From left to right $t = 50$ and $t = 600\ s$.

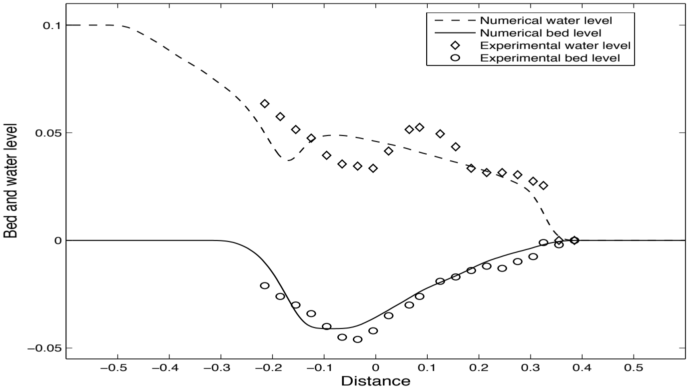


Figure: Comparison between numerical results and measurements for the bed and water level.

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NUMERICAL MODELING OF TRANSIENT FLOWS INVOLVING EROSION AND DEPOSITION OF SEDIMENT [4]

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} &= \frac{E-D}{1-p}, \\
\frac{\partial (hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{1}{2}gh^2\right) &= -gh\left(\frac{\partial Z}{\partial x} - S_f\right) - \frac{(\rho_s - \rho_w)g}{2\rho}h^2\frac{\partial c}{\partial x} - \\
&\quad \frac{(\rho_0 - \rho)(E-D)u}{\rho(1-p)}, \\
\frac{\partial (hc)}{\partial t} + \frac{\partial (huc)}{\partial x} &= E - D, \\
\frac{\partial Z}{\partial t} + A\xi\frac{\partial u^3}{\partial x} &= -\frac{E-D}{1-p},
\end{aligned}
$$

Variables and parameters

h, u, Z and c: water depth, velocity, bed elevation, and sediment concentration.

g: gravitational acceleration

p: bed sediment porosity

E and D: sediment entrainment and deposition fluxes

$\xi = \frac{1}{1-p}$

$\rho = \rho_w(1-c) + \rho_s c$: density of the water-sediment mixture

$\rho_0 = \rho_w p + \rho_s(1-p)$: density of the saturated bed

A: Grass constant for the sediment transport flux

$S_f = \frac{n_b^2 u^2}{h^{4/3}}$: friction slope

n_b: Manning roughness coefficient.

Deposition of sediments

For deposition of non-cohesive sediment, we use the relation :

$$D = w(1 - C_a)^m C_a \tag{11}$$

w: settling velocity of a single particle in water at rest
d: average diameter of sediment particles
m: exponent related to high sediment concentrations
C_a: near-bed volumetric sediment concentration
$C_a = \alpha_c c$ where α_c is a coefficient larger than unity.

For the entrainment of cohesive material the following relation is used:

$$E = \begin{cases} \varphi(\theta - \theta_c) u h^{-1} d^{-0.2}, & \text{if } \theta \geq \theta_c \\ 0, & \text{else} \end{cases} \tag{12}$$

φ controles the erosion forces, θ_c is the critical value of Shield parameter for the initiation of sediment motion.

The Shields parameter $\theta = \dfrac{u_*^2}{sgd}$, with $u_*^2 = \sqrt{\dfrac{f}{8}}\,|u|$, is the friction velocity.

Vector form presentation of the system (10)

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \mathbf{S}(\mathbf{W}). \tag{13}$$

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hc \\ Z \end{pmatrix}, \qquad \mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huc \\ A\xi u^3 \end{pmatrix},$$

$$
\mathbf{S} = \begin{pmatrix} \dfrac{E-D}{1-p} \\ -gh(\dfrac{\partial Z}{\partial x} + S_f) - \dfrac{(\rho_s - \rho_w)g}{2\rho} h^2 \dfrac{\partial c}{\partial x} - \dfrac{(\rho_0 - \rho)(E-D)u}{\rho(1-p)} \\ E-D \\ -\dfrac{E-D}{1-p} \end{pmatrix}.
$$

Solution procedure

The system (10) can be reformulated in an advective form as

$$\frac{\partial \mathcal{W}}{\partial t} + \mathcal{B}(\mathcal{W})\frac{\partial \mathcal{W}}{\partial x} = \mathcal{G}(\mathcal{W}), \qquad (14)$$

$$\mathcal{B}(\mathcal{W}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ gh - u^2 - \dfrac{(\rho_s - \rho_w)}{2\rho} ghc & 2u & \dfrac{(\rho_s - \rho_w)}{2\rho} gh & gh \\ -uc & c & u & 0 \\ -3A\xi \dfrac{u^3}{h} & 3A\xi \dfrac{u^2}{h} & 0 & 0 \end{pmatrix},$$

$$
\mathcal{W} = \begin{pmatrix} h \\ u \\ c \\ Z \end{pmatrix}, \qquad \mathcal{G}(\mathcal{W}) = \begin{pmatrix} \dfrac{E-D}{1-p} \\ -ghS_f - \dfrac{\rho_0(E-D)}{\rho(1-p)h}u \\ E-D \\ -\dfrac{E-D}{1-p} \end{pmatrix}.
$$

Bed erosion after dam break

We consider the test example of dam-break flow over movable beds studied in [4].
Here, the experiment is carried out in a rectangular channel 1000 m long.

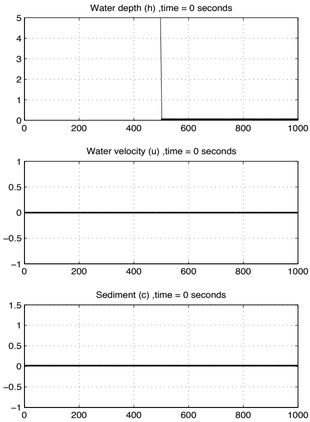


Figure: Initial conditions used in the computations.

Upstream initial water depth $h_l = 5\ m$
Sediment porosity $p = 0.28$
Exner constant $A = 0$
$\Delta x = 10\ m$
Results displayed at time $t = 20\ s$.

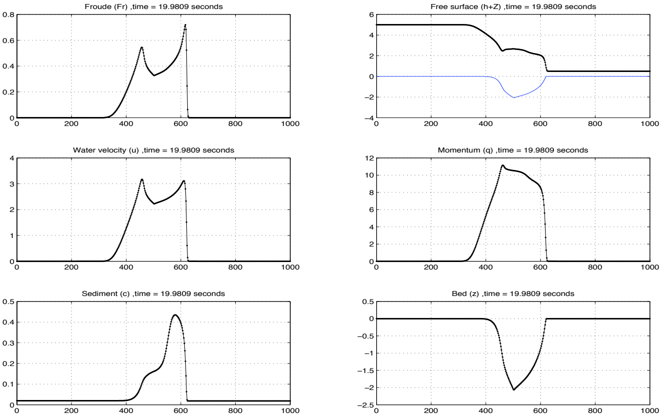


Figure: Computed results at t=20 s.

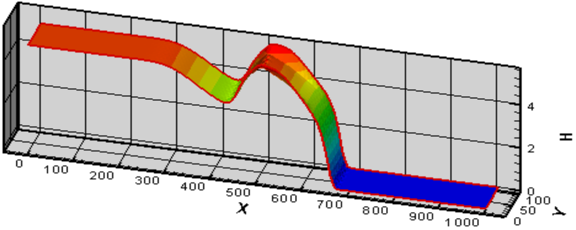


Figure: Water depht in 2D

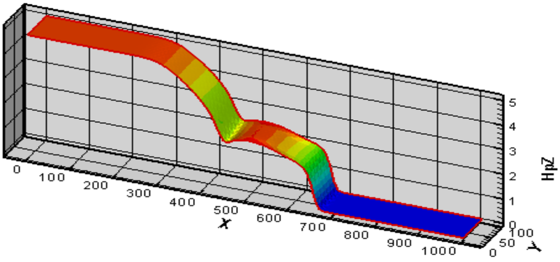


Figure: Free surface in 2D

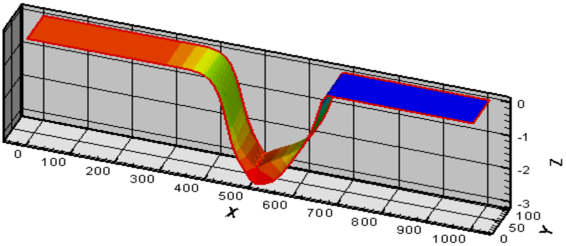
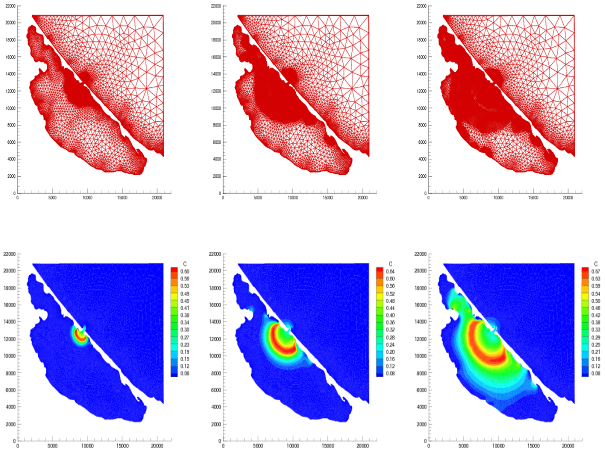


Figure: Bed erosion in 2D



Transport of sediments with erosion and deposition in North Morocco Nador Lagoun

Qualitative results:

The erosion magnitude and wavefront location are well predicted by the numerical model.

As expected, a hydraulic jump is formed near the initial dam place and propagates upstream.

The effect of erosion due to the dam break is well reproduced.

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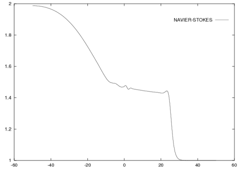
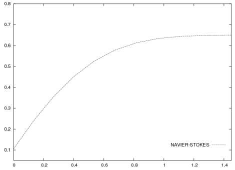
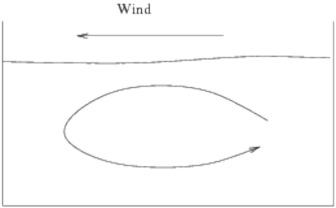
LIMITATIONS OF THE CLASSICAL SHALLOW WATER MODEL

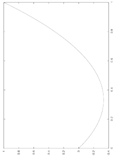
- Strong friction on the bottom $\rightsquigarrow$ Vertical dependency for u





- Wind stress in a lake $\rightsquigarrow$ Wind-induced circulation process

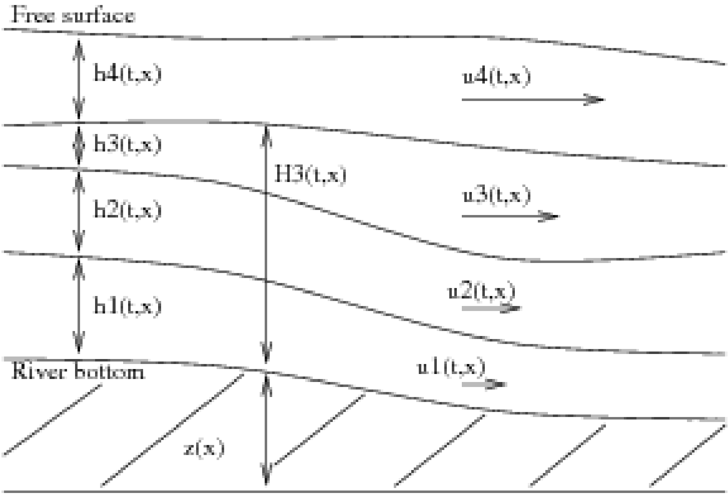




TOWARDS A MULTILAYER MODEL

- A model
  - that is of SW-type (2d and hyperbolic)
  - that takes into account vertical effects in the flow
- Derivation
  - Formal asymptotic analysis of NS equ. under SW assumption
  - Vertical discretization of P_0-type of the horizontal velocity
  - Vertical integration of approximated NS equations by layer
- Consequences
  - Each layer has its own velocity
  - Coupling between the layers through
    - Pressure term (global coupling)
    - Mass exchange
      ("local" coupling : interface condition for mass equation)
    - Viscous effect
      (local coupling : interface condition for momentum equation)

Multilayer approach



Vertical integration on a layer

$$\frac{\partial}{\partial t}h_\alpha + \frac{\partial}{\partial x}h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2},$$

$$\frac{\partial}{\partial t}h_\alpha \bar{u}_\alpha + \frac{\partial}{\partial x}h_\alpha \bar{u_\alpha}^2 + gh_\alpha \frac{\partial}{\partial x}H$$

$$= -gh_\alpha \frac{\partial}{\partial x}z_b + u_{\alpha+1/2}G_{\alpha+1/2} - u_{\alpha-1/2}G_{\alpha-1/2}$$

with mass exchange term

$$G_{\alpha+1/2} = \frac{\partial}{\partial t}z_{\alpha+1/2} + u_{\alpha+1/2}\frac{\partial}{\partial x}z_{\alpha+1/2} - w_{\alpha+1/2}$$

$$G_{1/2} = 0 \qquad \text{bottom}$$

$$G_{N+1/2} = 0 \qquad \text{free surface}$$

and $h_\alpha(t,x) = l_\alpha H(t,x),\qquad \sum_{\alpha=1}^{N} l_\alpha = 1$

Vertical integration on a layer

$$
\begin{aligned}
&\frac{\partial}{\partial t} h_\alpha + \frac{\partial}{\partial x} h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2}, \\
&\frac{\partial}{\partial t} h_\alpha \bar{u}_\alpha + \frac{\partial}{\partial x} h_\alpha \bar{u_\alpha}^2 + g h_\alpha \frac{\partial}{\partial x} H \\
&\qquad\qquad = -g h_\alpha \frac{\partial}{\partial x} z_b + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}
\end{aligned}
$$

with mass exchange term

$$
\begin{aligned}
G_{\alpha+1/2} &= \sum_1^\alpha \frac{\partial}{\partial t} h_\beta + \frac{\partial}{\partial x} h_\beta \bar{u}_\beta \\
G_{1/2} &= 0 \qquad \text{bottom} \\
G_{N+1/2} &= 0 \qquad \text{free surface}
\end{aligned}
$$

and $h_\alpha(t,x) = l_\alpha H(t,x)$, $\qquad \sum_{\alpha=1}^N l_\alpha = 1$

Multilayer SW model

$$\frac{\partial}{\partial t}H + \sum_{\alpha=1}^{N} l_\alpha \frac{\partial}{\partial x} H\bar{u}_\alpha = 0,$$

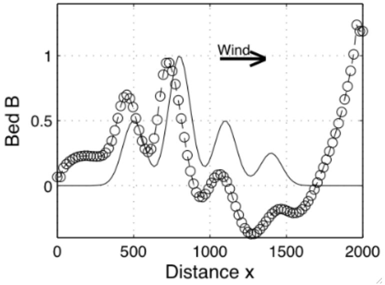
$$\frac{\partial}{\partial t}H\bar{u}_\alpha + \frac{\partial}{\partial x}H\bar{u_\alpha}^2 + \frac{\partial}{\partial x}\left(\frac{gH^2}{2}\right)$$
$$= -gH\frac{\partial}{\partial x}z_b + u_{\alpha+1/2}G_{\alpha+1/2} - u_{\alpha-1/2}G_{\alpha-1/2}$$

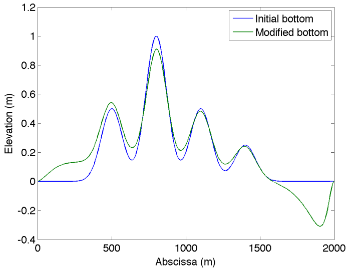
with explicit mass exchange term

$$G_{\alpha+1/2} = \sum_{1}^{\alpha} \frac{l_\beta}{l_\alpha}\left(\frac{\partial}{\partial t}H + \frac{\partial}{\partial x}H\bar{u}_\beta\right)$$

and interface velocity $u_{\alpha+1/2}$ computed by upwinding (following the sign of $G_{\alpha+1/2}$)

Wind-induced circulation over moveable bottom

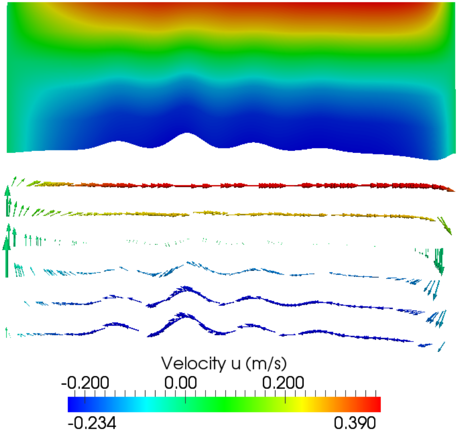




Classical SW model

Multilayer SW model

Wind-induced circulation over moveable bottom



A CHARACTERISTIC BASED FINITE VOLUME SCHEME

To formulate the Finite Volume Characteristics (FVC) scheme we consider a scalar homogeneous equation of a nonlinear conservation law given by

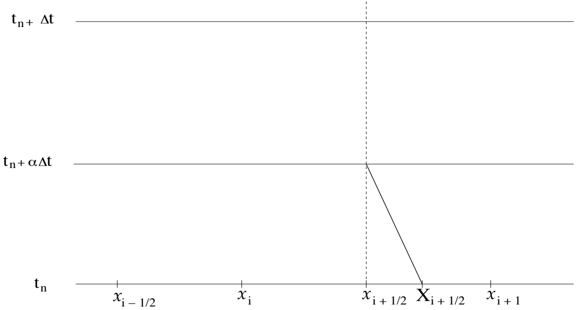
$$\partial_t U + \partial_x F(U) = 0. \tag{15}$$

This equation can be rewritten in an advective form as

$$\partial_t U + V \partial_x U = 0, \qquad V = F'(U).$$

Integrating the equation (15) with respect to time and space over the time-space control domain $[t_n, t_{n+1}] \times [x_{i-1/2}, x_{i+1/2}]$, we obtain the following discrete equation

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F(U_{i+1/2}^n) - F(U_{i-1/2}^n)\right).$$



We reconstruct the intermediate states $U^n_{i\pm 1/2}$ using the method of characteristics. Here, the characteristic curves associated with the equation are solutions of the initial-value problem

$$\frac{dX_{i+1/2}(\tau)}{d\tau} = V_{i+1/2}(\tau, X_{i+1/2}(\tau)), \quad \tau \in [t_n, t_n + \alpha\Delta t],$$

$$X_{i+1/2}(t_n + \alpha\Delta t) = x_{i+1/2},$$

where $V_{i+1/2} = F'(U_{i+1/2})$ and $\alpha \in]0, 1]$ is a parameter to be selected.

Interpolation Procedure

Once the characteristic curves $X_{i+1/2}(t_n)$ are known, the numerical fluxes are reconstructed using

$$U^n_{i+1/2} = u\left(t_n + \alpha\Delta t, x_{i+1/2}\right) = \tilde{U}\left(t_n, X_{i+1/2}(t_n)\right),$$

where $\tilde{U}\left(t_n, X_{i+1/2}(t_n)\right)$ is the solution at the characteristic foot computed by interpolation from the gridpoints of the control volume where the departure point resides *i.e.*

$$\tilde{U}\left(t_n, X_{i+1/2}(t_n)\right) = \mathcal{P}\Big(U\left(t_n, X_{i+1/2}(t_n)\right)\Big),$$

where $\mathcal{P}$ represents the interpolating polynomial. For instance, a Lagrange-based interpolation polynomials can be formulated as

$$\mathcal{P}\Big(U\left(t_n, X_{i+1/2}(t_n)\right)\Big) = \sum_k l_k(X_{i+1/2})U^n_k,$$

with l_k are the Lagrange basis polynomials given by

$$l_k(x) = \prod_{\substack{q=0\\ q\neq k}} \frac{x - x_q}{x_k - x_q}.$$

APPLICATION TO SHALLOW WATER EQUATIONS

Let us consider the shallow water equations

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -gh\partial_x Z \end{pmatrix}.$$

This system can be written in an advective form as

$$\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + u\partial_x \begin{pmatrix} h \\ u \end{pmatrix} = \begin{pmatrix} -h\partial_x u \\ -g\partial_x (h+Z) \end{pmatrix}.$$

Then, we calculate the characteristic curves $X_{i+1/2}(\tau)$ as

$$\begin{aligned} \frac{dX_{i+1/2}(\tau)}{d\tau} &= u\Big(\tau, X_{i+1/2}(\tau)\Big), \quad \tau \in [t_n, t_n + \alpha\Delta t], \\ X_{i+1/2}(t_n + \alpha\Delta t) &= x_{i+1/2}, \end{aligned}$$

where u is the velocity of the water flow.

The predictor stage in FVC method:

$$
\begin{aligned}
h^n_{i+1/2} &= \tilde{h}^n_{i+1/2} - \alpha\nu\tilde{h}^n_{i+1/2}\left(u^n_{i+1} - u^n_i\right),\\
q^n_{i+1/2} &= \tilde{q}^n_{i+1/2} - \alpha\nu\tilde{q}^n_{i+1/2}\left(u^n_{i+1} - u^n_i\right) - \frac{1}{2}\alpha\nu g\left(\left(h^2\right)^n_{i+1} - \left(h^2\right)^n_i\right)\\
&\quad -\alpha\nu g\tilde{h}^n_{i+1/2}\left(Z_{i+1} - Z_i\right),
\end{aligned}
$$

where $q = hu$ is the water discharge, $\nu = \dfrac{\Delta t}{\Delta x}$ and

$$
\tilde{h}^n_{i+1/2} = h\left(t_n, X_{i+1/2}(t_n)\right), \qquad \tilde{q}^n_{i+1/2} = q\left(t_n, X_{i+1/2}(t_n)\right),
$$

The corrector stage in the method:

$$
\begin{aligned}
h^{n+1}_i &= h^n_i - \nu\left(h^n_{i+1/2}u^n_{i+1/2} - h^n_{i-1/2}u^n_{i-1/2}\right),\\
q^{n+1}_i &= q^n_i - \nu\left(\left(hu^2 + \frac{1}{2}gh^2\right)^n_{i+1/2} - \left(hu^2 + \frac{1}{2}gh^2\right)^n_{i-1/2}\right)\\
&\quad -\frac{1}{2}\nu g\hat{h}^n_i\left(Z_{i+1} - Z_{i-1}\right).
\end{aligned}
$$

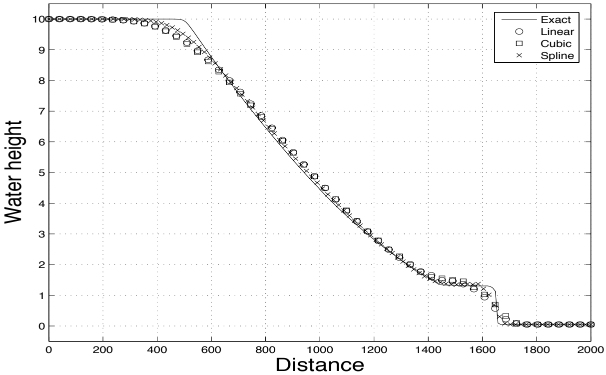


Figure: Comparative results for water height in dam-break on wet bed at $t = 50\ s$ using $\Delta x = 20\ m$, $h_r/h_l = 0.005$.

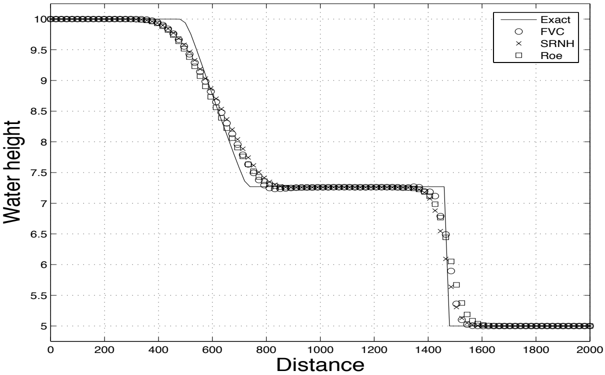


Figure: Comparative results for water height in dam-break on wet bed at $t = 50\ s$ using $\Delta x = 20\ m$, $h_r/h_l = 0.5$.

Table: Computational times in seconds for dam-break on wet bed at $t = 50\ s$ using $h_r/h_l = 0.005$ and different gridpoints.

Gridpoints	Roe method	SRNH method	FVC method
500	8.746	13.193	1.008
1000	34.780	52.655	2.707
2000	134.152	210.620	15.756
4000	534.124	834.055	61.096
8000	2178.701	3378.303	249.209

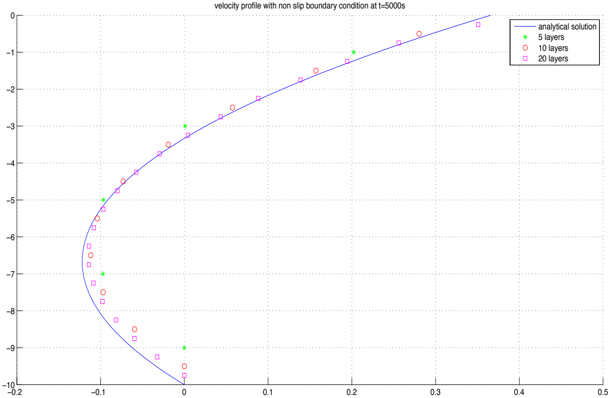


Figure: Comparison between exact solution and FVC scheme for a simplified multilayer model

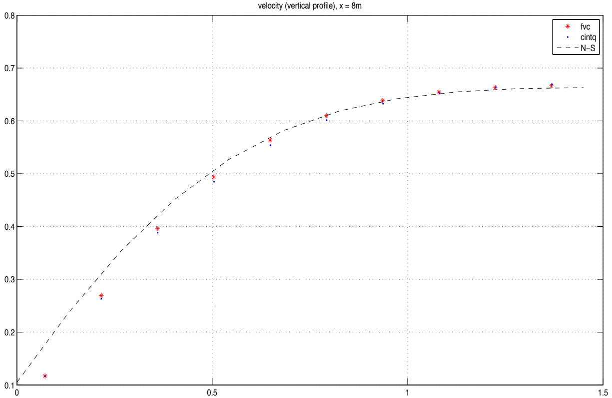


Figure: Comparison between Navier-Stokes solution, FVC and kinetic scheme for SW multilayer model

FUTURE WORK

- 2D Multilayer SW Models
- Extension of FVC to 2D problems
- SW models with variable density

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