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Efficiency of finite volume solvers for inhomogeneous systems.

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Summary of the talk

- Introduction
- Some non homogeneous systems
- Presentation of $SRNH$ scheme
- Application of $SRNH$ scheme to academic tests
- Analysis of the convergence stagnation problem
- Application of SRNH scheme to a realistic situation: the transport of pollutant in the strait of Gibraltar
- Conclusion and future work

Introduction: Complex fluid flow phenomena such as combustion, multiphase flows or flows submitted to external forces, are represented by stiff or ill posed inhomogeneous systems (e.g. multiphase systems can have non hyperbolic regions). It is therefore not easy to extend the usual Riemann solvers based on system eigenvalues and eigenvectors computations. To propose an alternative, we consider in this work a particular class of non conservative systems. We assume that the solution of the associated Riemann problem is self-similar. Assuming this hypothesis, a new Non Homogeneous Riemann Solver ($SRNH$), using approximate states instead of approximate fluxes, was developed. The new scheme depends on a local parameter allowing to control numerical diffusion. The stability analysis of the scheme, first in the scalar case then in the case of systems of conservation laws, leads to a new formulation of the scheme which is based on the sign of genuine or approximate jacobian of the system considered.

1 Some non homogeneous systems

1.1 $1D$ Two phase flows (non hyperbolic model):

$$\begin{cases} \dfrac{\partial W(x,t)}{\partial t} + \dfrac{\partial F(W(x,t))}{\partial x} + S_1(x,W) = S_2(x,W) \\ W(x,0) = W_0(x), \end{cases} \tag{1}$$

$$W(x,t) = (\alpha_v \rho_v, \alpha_v \rho_v u_v, \alpha_l \rho_l, \alpha_l \rho_l u_l)^T$$

$$F(W(x,t)) = \left(\alpha_v \rho_v u_v, \alpha_v \rho_v u_v^2, \alpha_l \rho_l u_l, \alpha_l \rho_l u_l^2\right)^T$$

$$S_1(x,W) = \left(0, \alpha_v \frac{\partial p}{\partial x}, 0, \alpha_l \frac{\partial p}{\partial x}\right)^T$$

$$S_2(x,W) = (0, \alpha_v \rho_v g, 0, \alpha_l \rho_l g)^T, \qquad p = C_v \rho_v^\gamma = C_l \rho_l^\beta$$

1.2 $2D$ Shallow Water equations with irregular topography:

$$\begin{cases} h_{,t} + (hu)_{,x} + (hv)_{,y} = 0 \\ (hu)_{,t} + (hu^2)_{,x} + (huv)_{,y} + g\left(\frac{h^2}{2}\right)_{,x} = -gh(Z_f)_{,x} \\ (hv)_{,t} + (huv)_{,x} + (hv^2)_{,y} + g\left(\frac{h^2}{2}\right)_{,y} = -gh(Z_f)_{,y}, \end{cases} \tag{2}$$

where h is the water elevation, $\mathbf{u} = {}^t(u, v)$ the velocity, and Z_f the bottom function.

1.3 Non-isentropic Euler equations in a duct with variable section

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho A \\ \rho A u \\ \rho A E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho A u \\ \rho A(u^2 + p/\rho) \\ \rho A u H \end{bmatrix} = \begin{bmatrix} 0 \\ p\dfrac{dA}{dx} \\ 0 \end{bmatrix} \tag{3}$$

$\gamma = 1.4$

$$E = \frac{p}{(\gamma - 1)\rho} + \frac{u^2}{2} \qquad H = \frac{\gamma p}{(\gamma - 1)\rho} + \frac{u^2}{2} \tag{4}$$

$A(x)$ is the duct section.

2 Presentation of the SRNH scheme

Integrating a first time the system in the square:
$R =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[\times [t_n, t_{n+1}[$, gives :

$$\begin{aligned} W_i^{n+1} &= W_i^n - r_n \left[F\left(Rs(0, W_i^n, W_{i+1}^n) \right) - F\left(Rs(0, W_{i-1}^n, W_i^n) \right) \right] \\ &+ \Delta t_n Q_i^n, \end{aligned} \tag{5}$$

where $r_n = \dfrac{\Delta t_n}{\Delta x}$,
Q_i^n is an approximation of $\dfrac{1}{\Delta t_n \Delta x} \int\limits_R Q(x, W)\, dxdt$.

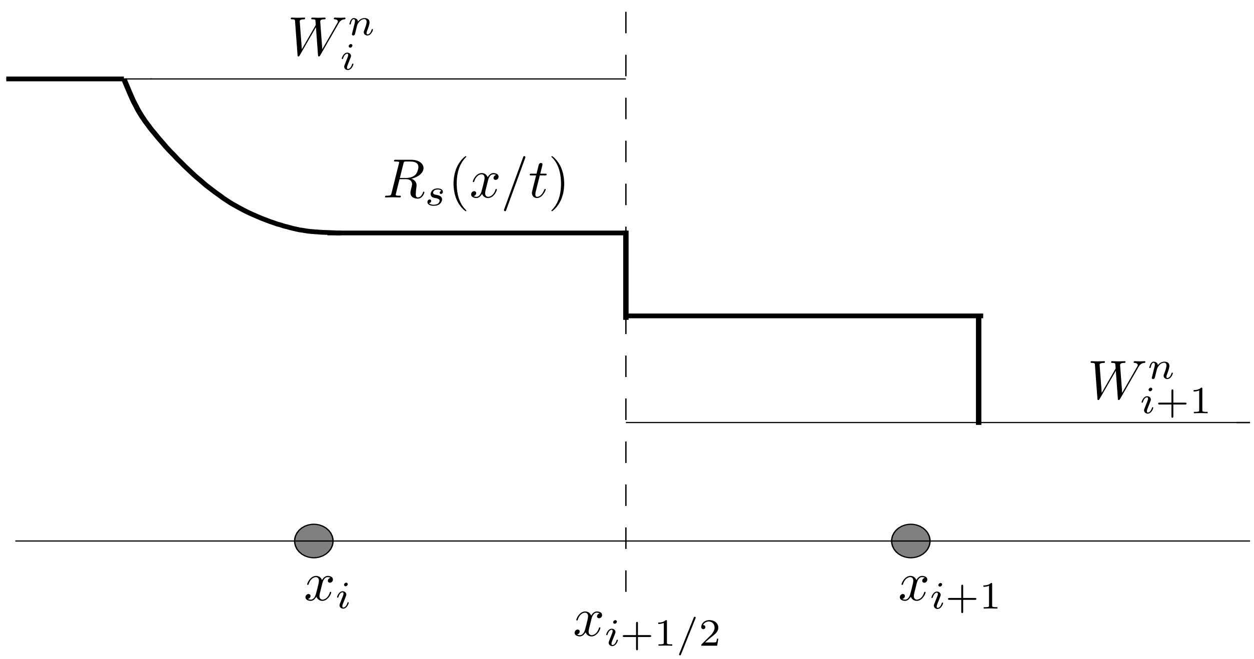


Figure 1: The Riemann problem solution R_s at a cell interface.

Let $W^n_{i+\frac{1}{2}}$ be an approximation of $Rs\left(0, W^n_i, W^n_{i+1}\right)$.

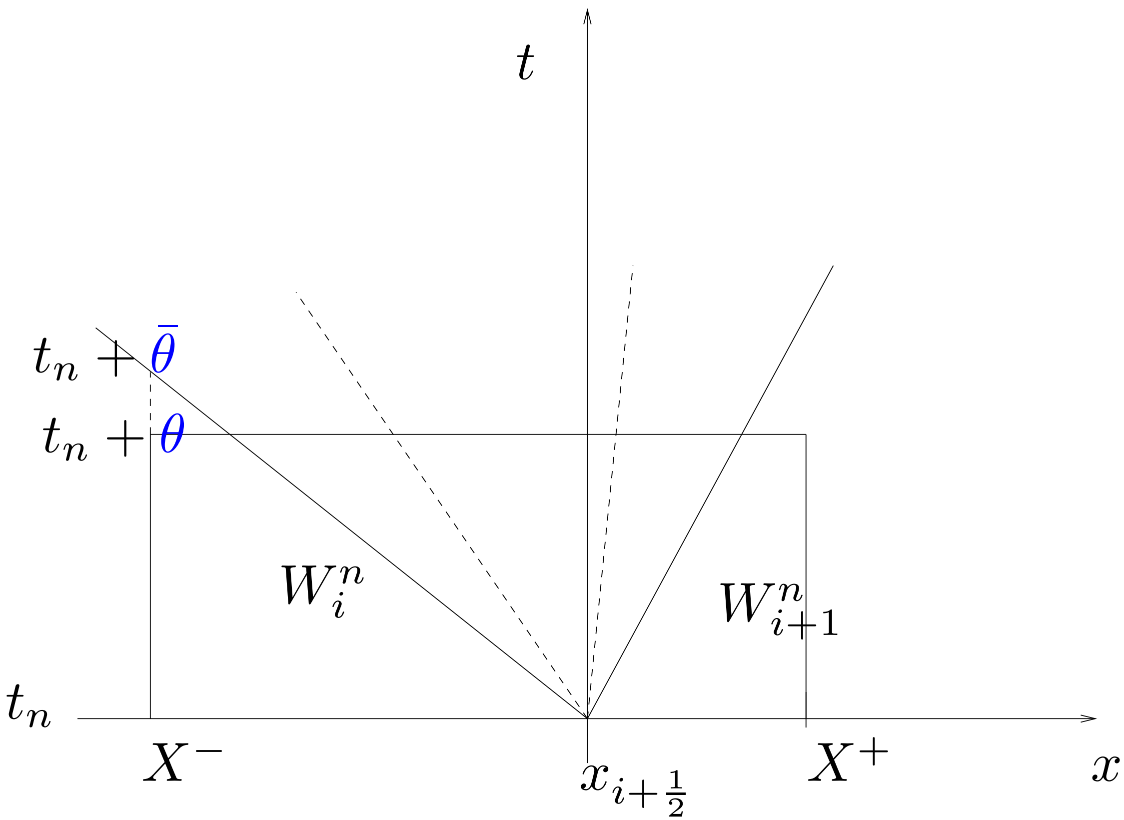


Figure 2: Staggered box around the interface $x_{i+\frac{1}{2}}$

With the choice : $X^- = x_i$ and $X^+ = x_{i+1}$, one gets a generalized expression :

$$\begin{aligned} W^n_{i+\frac{1}{2}} &= \frac{1}{2}(W^n_i + W^n_{i+1}) - \frac{\theta}{\Delta x}\left[F(W^n_{i+1}) - F(W^n_i)\right] \\ &+ \theta \; Q^n_{i+\frac{1}{2}}, \end{aligned}$$

where

$$Q^n_{i+\frac{1}{2}} = G\left(W^n_i, W^n_{i+1}\right)\left[\frac{E(x_{i+1}, W^n_{i+1}) - E(x_i, W^n_i)}{\Delta x}\right].$$

A possible choice is: $\theta = \dfrac{\alpha^n_{i+\frac{1}{2}}}{2}\Delta t$ (see **Benkhaldoun 02**).

Here, to make the extension of $SRNH$ scheme to $2D$ easier, one writes: $\theta = \alpha^n_{i+\frac{1}{2}}\bar{\theta}$ where $\bar{\theta}$ is defined by the local Rusanov velocity (see figure 2):

$$\bar{\theta} = \frac{\Delta x}{2S^n_{i+\frac{1}{2}}},$$

where $S^n_{i+\frac{1}{2}} = \max_{p=1\ldots m}\left(\max\left(\left|\lambda^n_{i,p}\right|, \left|\lambda^n_{i+1,p}\right|\right)\right)$.

On gets the following expression of the intermediate state :

$$\begin{aligned} W^n_{i+\frac{1}{2}} &= \frac{1}{2}(W^n_i + W^n_{i+1}) - \frac{\alpha^n_{i+\frac{1}{2}}}{2S^n_{i+\frac{1}{2}}}\left[F(W^n_{i+1}) - F(W^n_i)\right] \\ &+ \frac{\alpha^n_{i+\frac{1}{2}}}{2S^n_{i+\frac{1}{2}}}G\left(W^n_i, W^n_{i+1}\right)\left[E(x_{i+1}, W^n_{i+1}) - E(x_i, W^n_i)\right]. \end{aligned}$$

3 SRNH final form for non linear systems

Considering a local Roe linearisation, one obtains that necessarily $\alpha^n_{i+\frac{1}{2}} = S^n_{i+\frac{1}{2}} \left|\Lambda^\star \left(V\left(W^n_i, W^n_{i+1}\right)\right)\right|^{-1}$, and the $SRNHS$ scheme writes:

$$\begin{cases} W^n_{i+\frac{1}{2}} & = \frac{1}{2}\left(W^n_{i+1} + W^n_i\right) - \frac{1}{2}\,\mathrm{sgn}\left[\mathcal{B}^n_{i+\frac{1}{2}}\right]\left(W^n_{i+1} - W^n_i\right) \\ & \quad + \frac{\Delta x}{2}\left|\mathcal{B}^n_{i+\frac{1}{2}}\right|^{-1} Q^n_{i+\frac{1}{2}} \\ W^{n+1}_i & = W^n_i - r\left(F(W^n_{i+\frac{1}{2}}) - F(W^n_{i-\frac{1}{2}})\right) + \Delta t Q^n_i \end{cases} \tag{6}$$

with : $\mathcal{B}^n_{i+\frac{1}{2}} = \left(\mathcal{R}\Lambda^\star\mathcal{R}^{-1}\right)\left(V\left(W^n_i, W^n_{i+1}\right)\right)$ is a pseudo jacobian matrix calculated at the average state $V\left(W^n_i, W^n_{i+1}\right)$.

4 Application to the $1D$ Shallow Water equations with irregular topography

Let us consider the Shallow water equations :

$$\begin{cases} \dfrac{\partial W}{\partial t} + \dfrac{\partial F(W)}{\partial x} = Q\left(x, W\right), \ (x,t) \in \mathcal{D} \times \mathbb{R}_+^\times, \ \mathcal{D} \subset \mathbb{R} \\ W(x,0) = W_0(x), \ x \in \mathcal{D} \end{cases} \tag{7}$$

$$W(x,t) = \left(h(x,t), hu(x,t)\right)^T$$

$$F(W(x,t)) = \left(hu(x,t), hu^2(x,t) + \frac{1}{2} g h^2(x,t)\right)^T$$

$$Q(x, W(x,t)) = \left(0, -gh(x,t)\frac{dz(x)}{dx}\right)^T.$$

The $SRNHS$ scheme for problem (7) may be written:

$$\begin{cases} W^n_{i+\frac{1}{2}} & = \frac{1}{2}\left(W^n_{i+1} + W^n_i\right) - \frac{1}{2}\,\mathrm{sgn}\left[\mathcal{B}^n_{i+\frac{1}{2}}\right]\left(W^n_{i+1} - W^n_i\right) \\ & + \frac{\Delta x}{2}\left|\mathcal{B}^n_{i+\frac{1}{2}}\right|^{-1} Q^n_{i+\frac{1}{2}} \\ W^{n+1}_i & = W^n_i - r\left(F(W^n_{i+\frac{1}{2}}) - F(W^n_{i-\frac{1}{2}})\right) + \Delta t Q^n_i \end{cases} \tag{8}$$

with

$$Q^n_{i+\frac{1}{2}} = -\frac{g}{2\Delta x}\left(h^n_i + h^n_{i+1}\right)\begin{bmatrix} 0 \\ z_{i+1} - z_i \end{bmatrix},$$

and

$$Q^n_i = \frac{1}{\Delta t \Delta x}\int_{t^n}^{t^{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} Q(x, W(x,t))dxdt.$$

Définition 1. *$W(x,t)$ is a static stationary solution of the system if* $\dfrac{\partial W}{\partial t} = 0$ *and* $u(x,t) = 0$. *In this case, one has*
$h(x,t) + z(x) =$ *canstant.*

Définition 2. *A finite volume scheme is said to verify the exact $\mathcal{C}$-property* ***(Bermudez & Vazquez 1999)****, if it preserves the equilibrium state:*

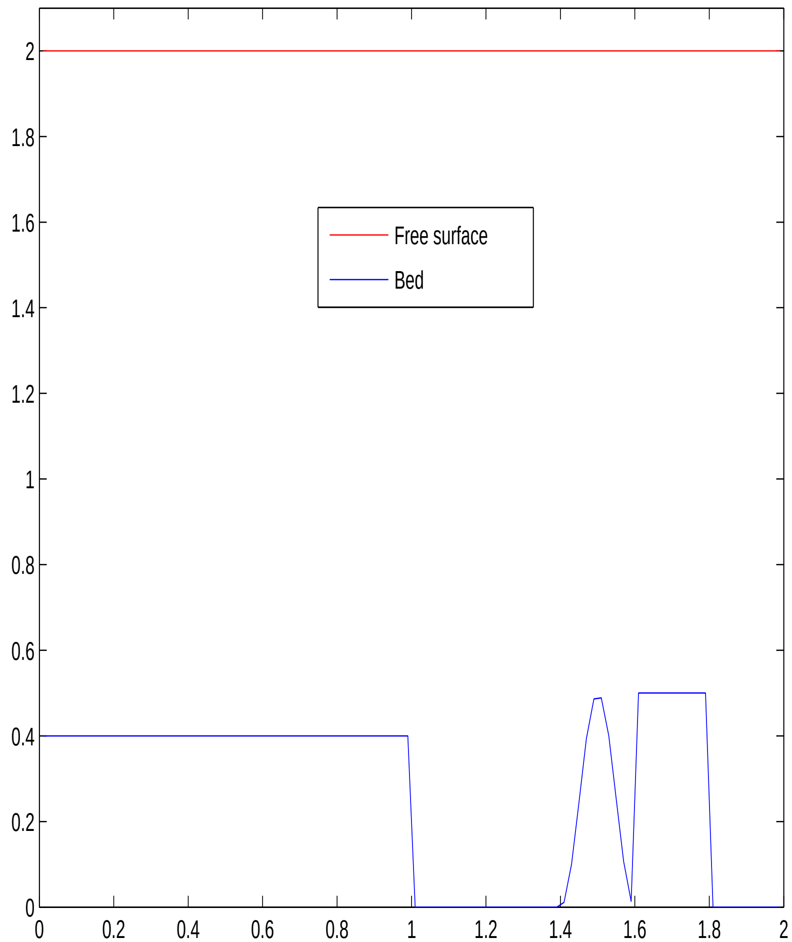
$$h_i^n + z_i = c \qquad \text{and} \qquad u_i^n = 0 \quad \forall (i,n) \in \mathbb{Z} \times \mathbb{N}.$$

Proposition 1. *If the source term, in the second step of the scheme, is discretized as follows :* $(Q_i^n)_1 = 0$, *and*

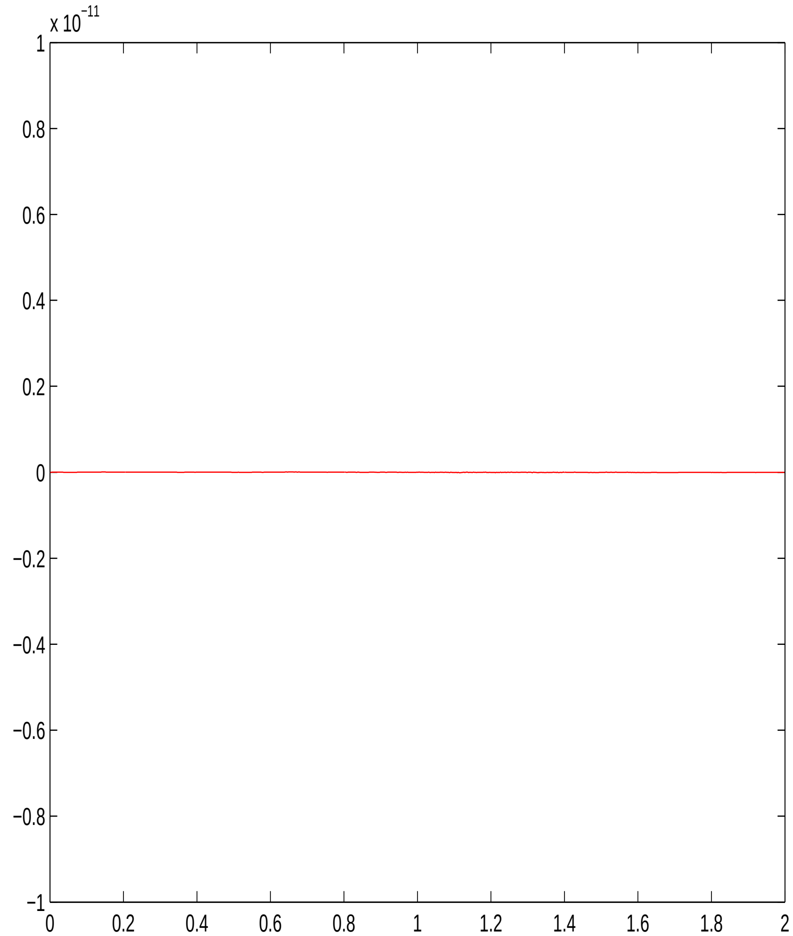
i) $(Q_i^n)_2 = -\dfrac{g}{4\Delta x}\left(h^n_{i+\frac{1}{2}} + h^n_{i-\frac{1}{2}}\right)(z_{i+1} - z_{i-1})$, *or*

ii) $(Q_i^n)_2 = -\dfrac{g}{8\Delta x}\left(h^n_{i+1} + 2h^n_i + h^n_{i-1}\right)(z_{i+1} - z_{i-1})$

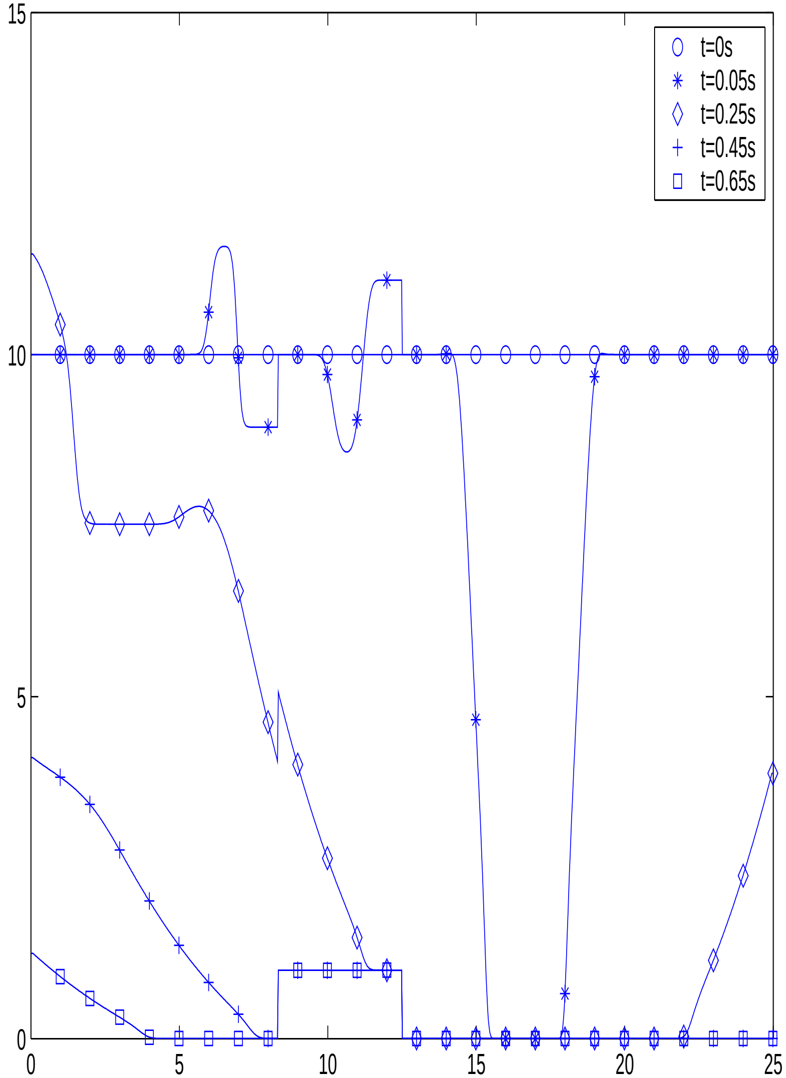
then the scheme (8) respects the exact $\mathcal{C}$-property.



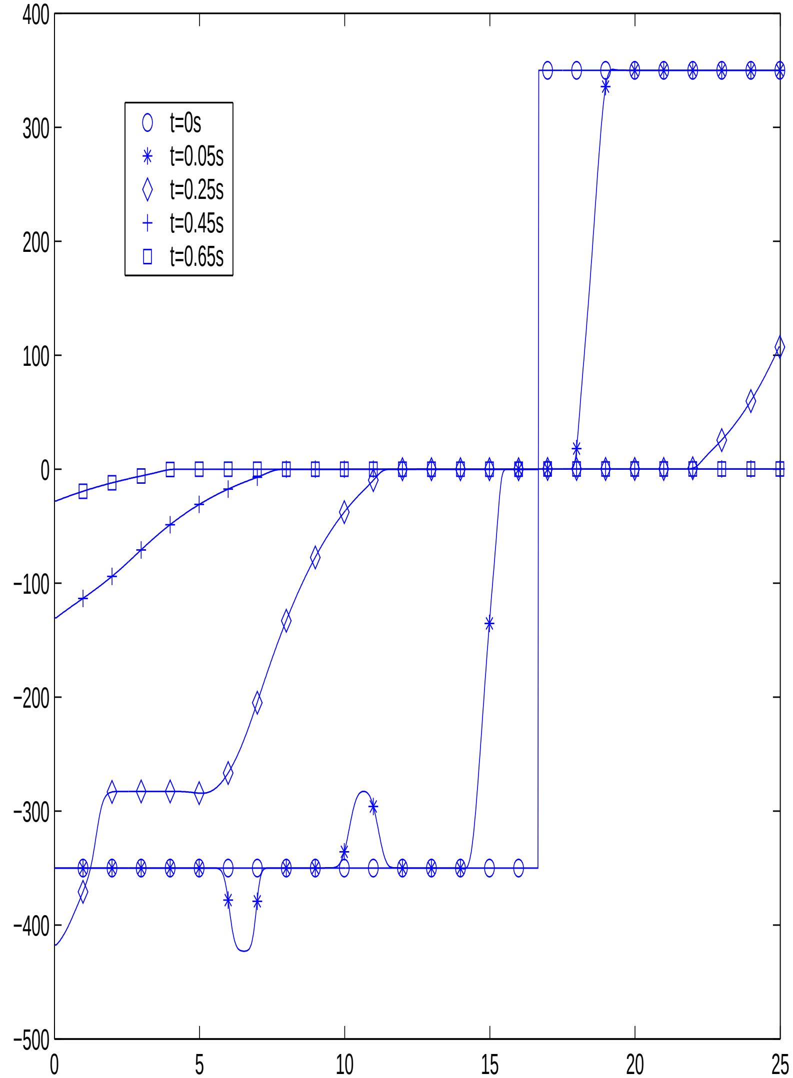
Bed and free surface, $t = 10s$



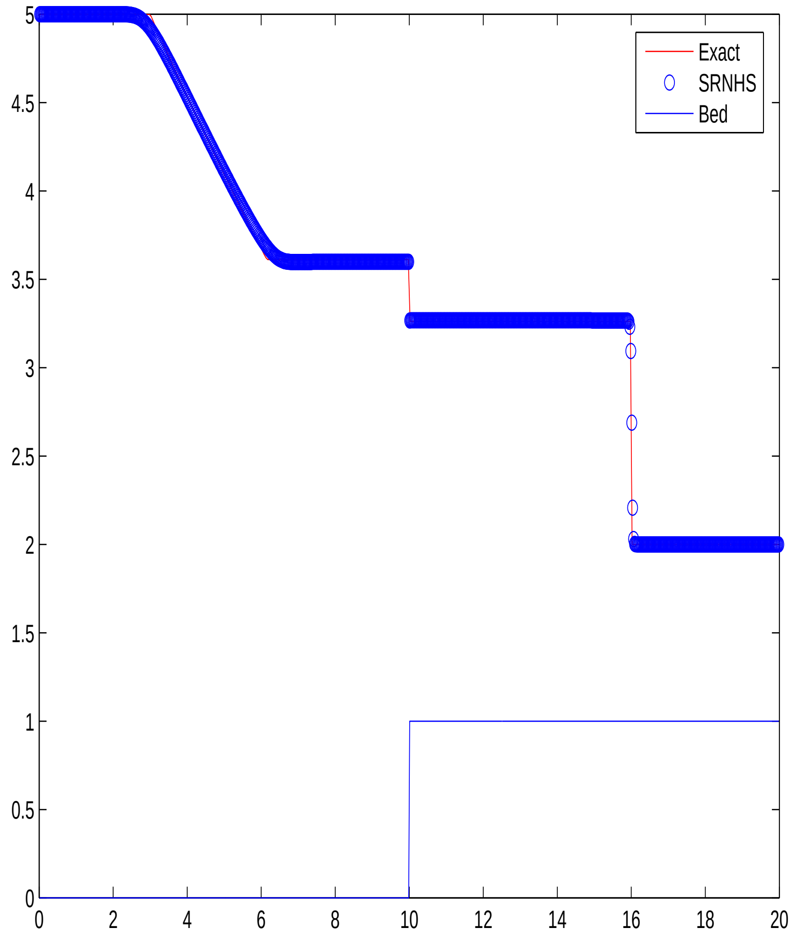
Water momentum $t = 10s$



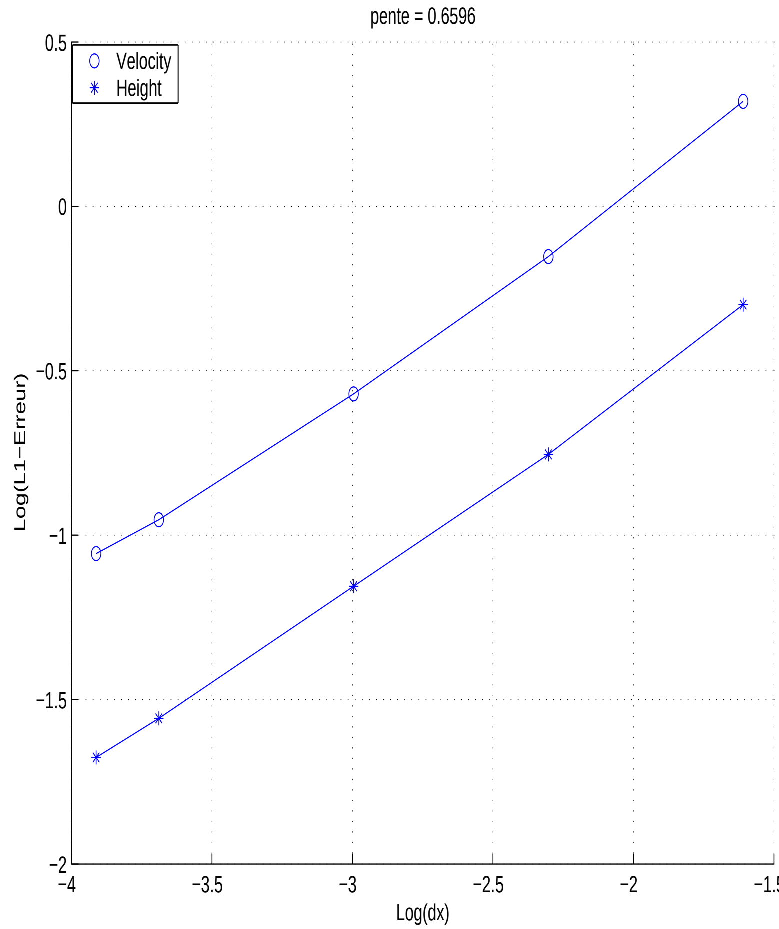
Water free surface cfl=0.5, 1000 points (vacuum occurance)



Water momentum cfl=0.5, 1000 points



Free surface on a step, t=1s ,CFL=0.75



Error plot, cfl=0.75, slope $\simeq 0.65$

5 Application to 1D non-isentropic Euler equations in a duct of variable cross section

The governing (Euler) equations can be written:

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho A \\ \rho A u \\ \rho A E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho A u \\ \rho A(u^2 + p/\rho) \\ \rho A u H \end{bmatrix} = \begin{bmatrix} 0 \\ p\dfrac{dA}{dx} \\ 0 \end{bmatrix} \tag{9}$$

where ρ, u and p are the gas density, velocity and pressure respectively. $A(x)$ is the cross section of the duct, and E and H represent the total energy and total enthalpy.

$$E = \frac{p}{(\gamma - 1)\rho} + \frac{u^2}{2} \qquad\qquad H = \frac{\gamma p}{(\gamma - 1)\rho} + \frac{u^2}{2} \tag{10}$$

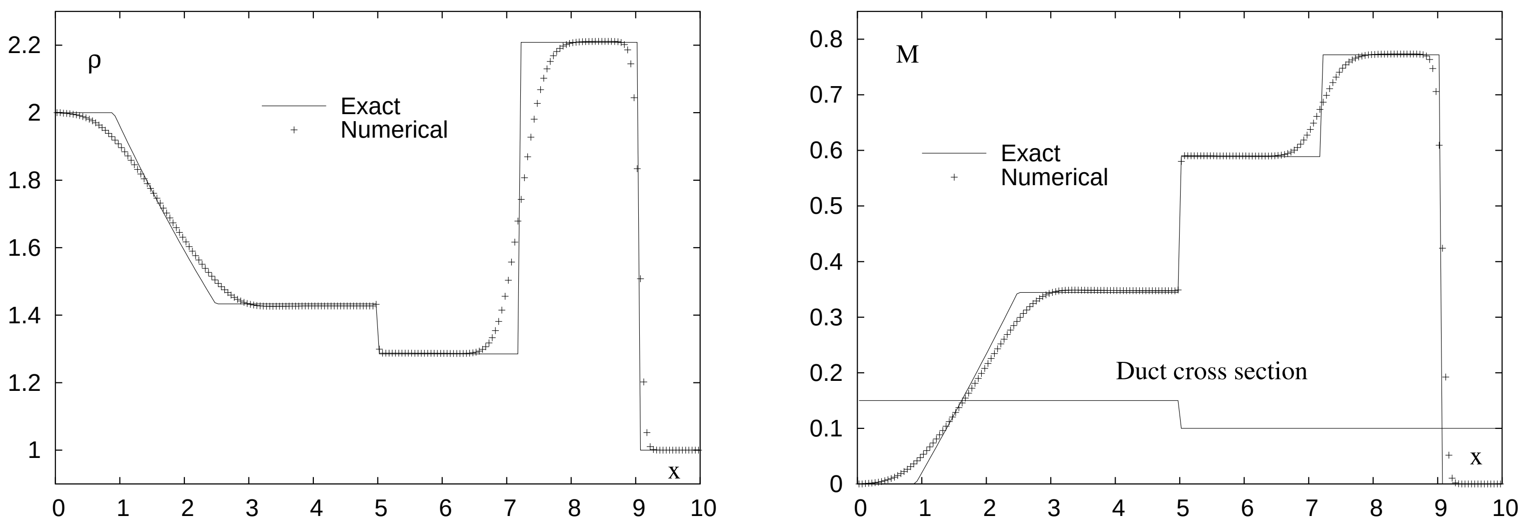


Figure 3: Shock tube problem with discontinuous cross section computed with 200 cells. Density (left) and Mach number (right) at t=2s (the duct cross section area is plotted also as a dotted line in the Mach plot).

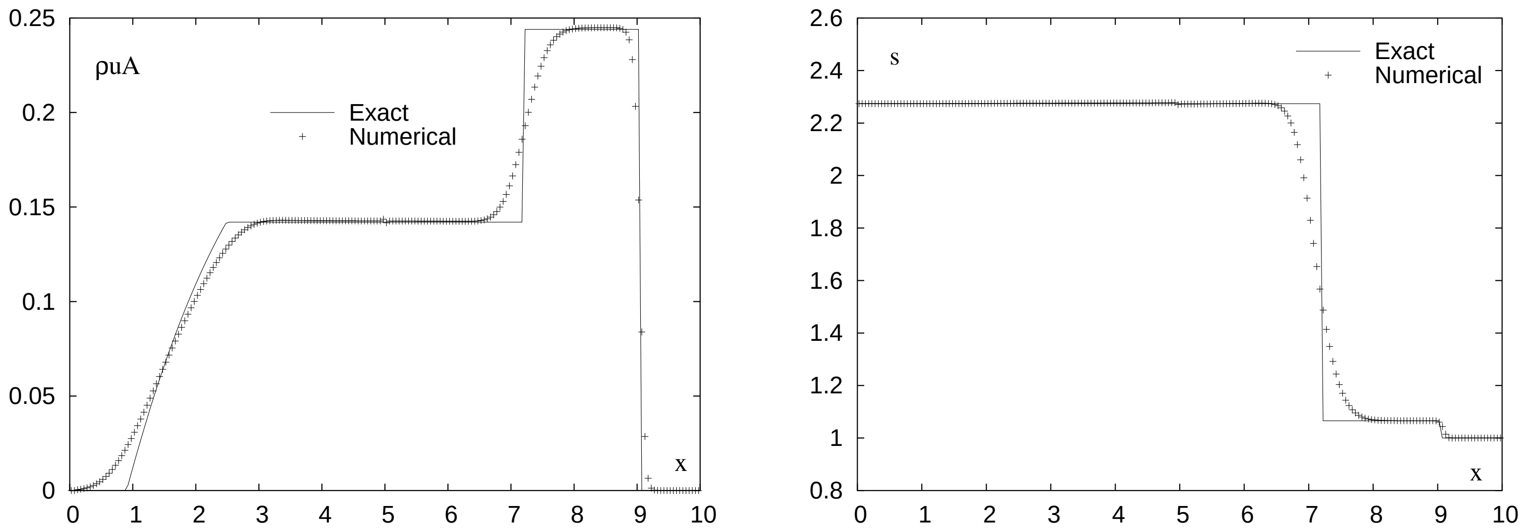


Figure 4: Shock tube problem with discontinuous cross section computed with 200 cells. Mass flow (left) and entropy (right) at t=2s. Both quantities must be constant across the cross section discontinuity.

	Constant State 1		Constant State 2		Constant State 3	
	Exact	Num.	Exact	Num.	Exact	Num.
ρ	1.433	1.427	1.285	1.287	2.208	2.211
u	0.661	0.666	1.105	1.107	1.105	1.107
p	3.764	3.747	3.231	3.237	3.231	3.238
M	0.345	0.347	0.589	0.590	0.772	0.773

Table 1: Comparison between exact and numerically computed constant states on a 200 cell mesh

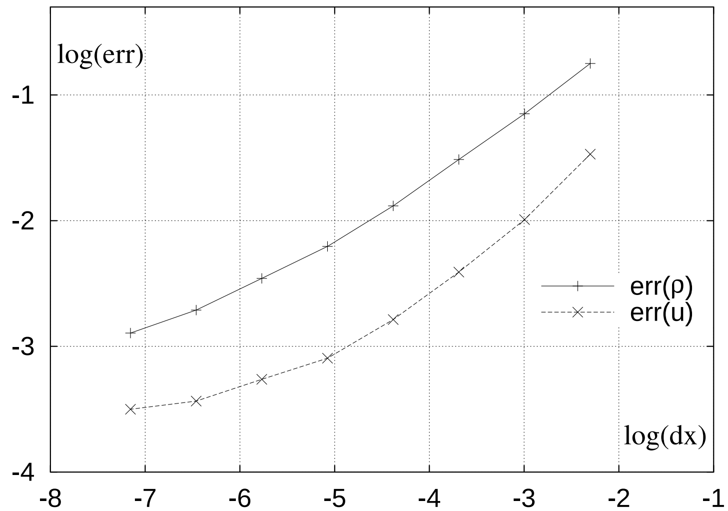


Figure 5: L_1 Convergence plot of the density and velocity for the shock tube problem with cross section discontinuity. In this case the convergence decay is more clearly visible than for the Shallow Water Equations.), with both curves rapidly approaching a stagnation condition.

6 Application of $SRNH$ scheme to 1D two-fluid model

Let us consider the two-fluid model :

$$\begin{cases} \dfrac{\partial W(x,t)}{\partial t} + \dfrac{\partial F(W(x,t))}{\partial x} + S_1(x,W) = S_2(x,W) \\ W(x,0) = W_0(x), \end{cases} \tag{11}$$

$$W(x,t) = (\alpha_v\rho_v, \alpha_v\rho_v u_v, \alpha_l\rho_l, \alpha_l\rho_l u_l)^T$$

$$F(W(x,t)) = \left(\alpha_v\rho_v u_v, \alpha_v\rho_v u_v^2, \alpha_l\rho_l u_l, \alpha_l\rho_l u_l^2\right)^T$$

$$S_1(x,W) = \left(0, \alpha_v\frac{\partial p}{\partial x}, 0, \alpha_l\frac{\partial p}{\partial x} + \delta(p - p_l^i)\frac{\partial \alpha_l}{\partial x}\right)^T$$

$$S_2(x,W) = (0, \alpha_v\rho_v g, 0, \alpha_l\rho_l g)^T, \qquad p - p_l^i = \alpha_v\rho_l(u_l - u_v)^2$$

the subscript k is either v for vapour or l for liquid, and δ is a non negative real parameter.

Numerical algorithm:

We use the splitting strategy presented in [**Benkhaldoun 02**]. The gravity source term is treated in a first step, to get $\hat{W}$ from W^n

$$\begin{cases} \dfrac{\partial \hat{W}}{\partial t} = S_2(\hat{W}) \\ \hat{W}(x, t^n) = W^n(x), \end{cases}$$

and using $SRNH$ scheme, we solve

$$\begin{cases} \dfrac{\partial W\,(x,t)}{\partial t} + \mathcal{A}(W)\dfrac{\partial W\,(x,t)}{\partial x} = 0 \\ W(x,0) = \hat{W}_0(x), \end{cases} \tag{12}$$

where $\mathcal{A}(W) = \nabla F\,(W) + C(W)$ and $C(W)\dfrac{\partial W\,(x,t)}{\partial x} = S_1(x,W)$.

Case $\delta = 0$

$$\mathcal{A}(W) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -u_v^2 + \frac{\gamma p}{\rho_v} & 2u_v & \frac{\gamma p}{\rho_l} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\alpha_l}{\alpha_v}\frac{\gamma p}{\rho_v} & 0 & -u_l^2 + \frac{\alpha_l}{\alpha_v}\frac{\gamma p}{\rho_l} & 2u_l \end{pmatrix}.$$

The $SRNHS$ scheme writes

$$\begin{cases} W^n_{i+\frac{1}{2}} = \frac{1}{2}\left(W_i^n + W_{i+1}^n\right) - \frac{1}{2} sgn\left(\mathcal{A}(\bar{W})\right)\left(W_{i+1}^n - W_i^n\right) \\ W_i^{n+1} = W_i^n - r\left(F\left(W^n_{i+\frac{1}{2}}\right) - F\left(W^n_{i-\frac{1}{2}}\right)\right) + \Delta t (S_1)_i^n. \end{cases} \tag{13}$$

$\bar{W}$ is a Roe state.

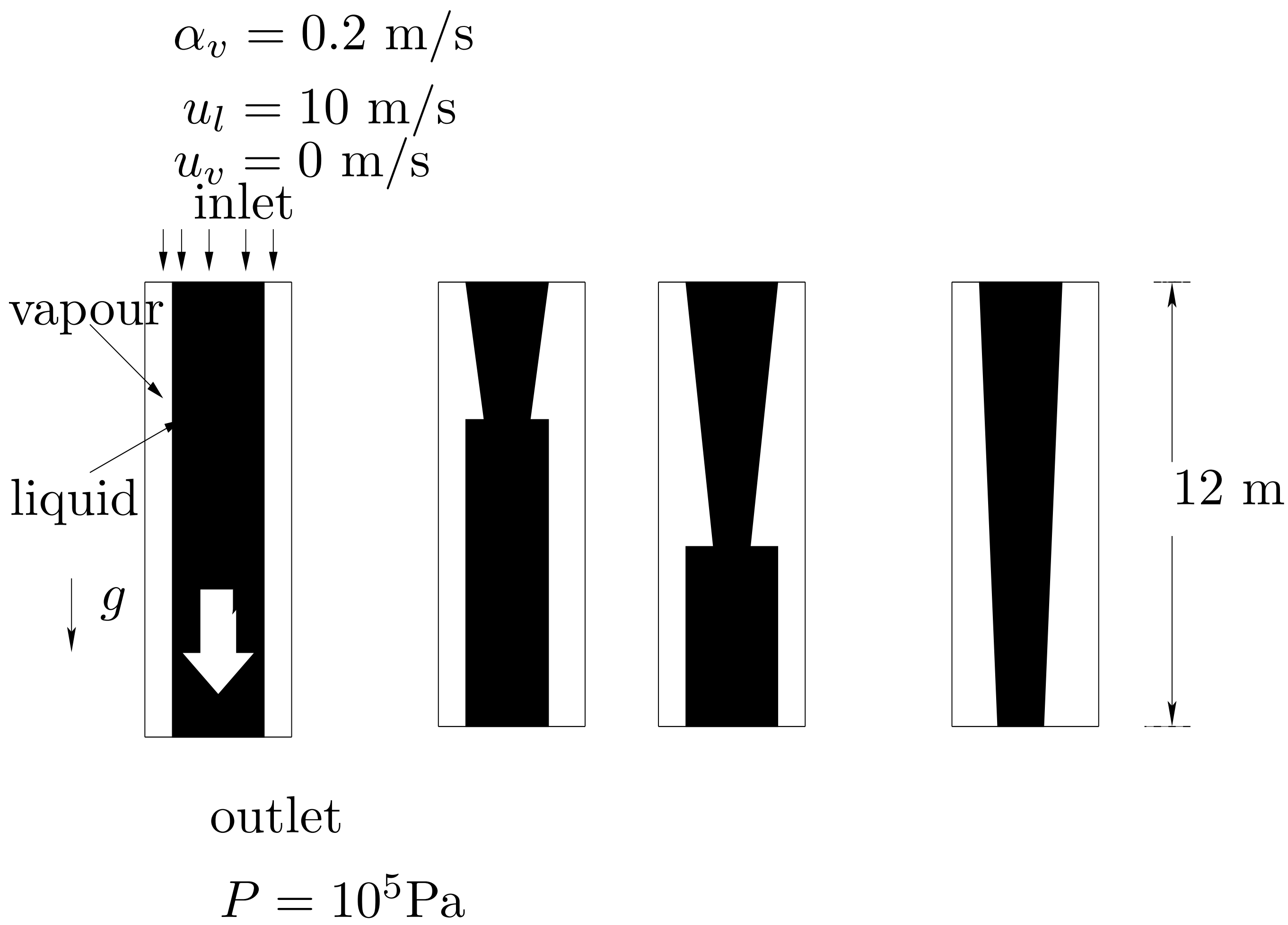
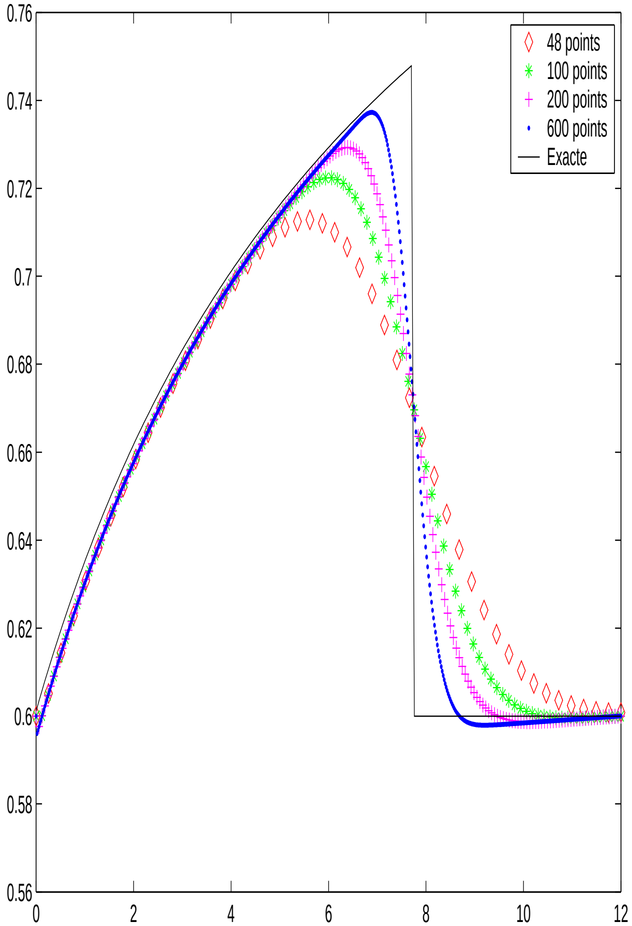
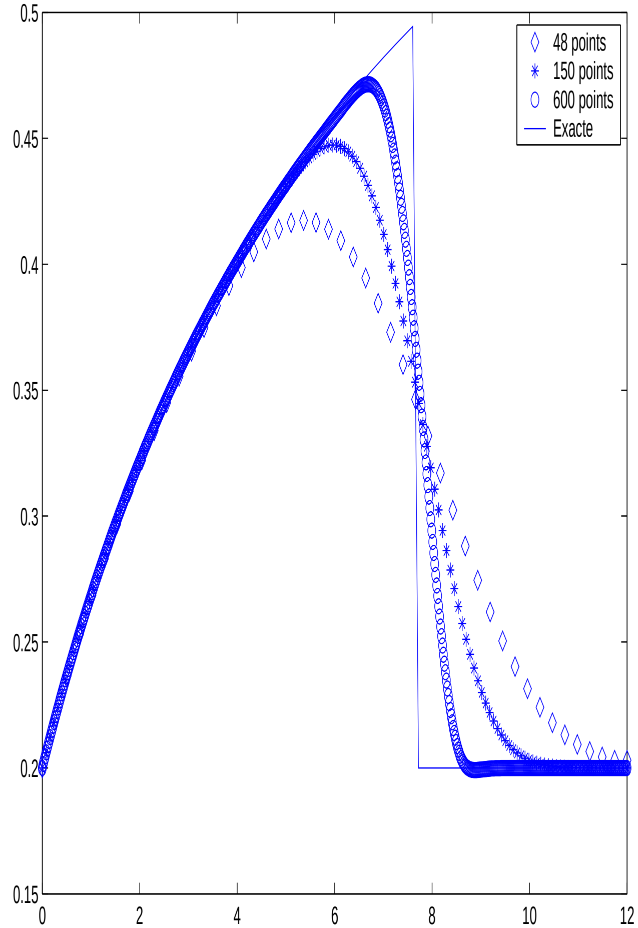


Figure 6: Ransom problem: variation of liquid quantity according to time



Void fraction ($\alpha_{v,0} = 0.6$), $\delta = 0$, Alouges's method



Void fraction ($\alpha_{v,0} = 0.2$), $\delta = 5 \times 10^{-4}$, Toumi's method

7 Application of $SRNH$ scheme to non homogeneous 2D Shallow Water flows

The system considered may by written as follows :

$$\begin{cases} h_{,t} + (hu)_{,x} + (hv)_{,y} = 0 \\ (hu)_{,t} + (hu^2)_{,x} + (huv)_{,y} + g\left(\frac{h^2}{2}\right)_{,x} = -gh(Z_f)_{,x} \\ (hv)_{,t} + (huv)_{,x} + (hv^2)_{,y} + g\left(\frac{h^2}{2}\right)_{,y} = -gh(Z_f)_{,y}, \end{cases} \tag{14}$$

where h is the water level, $\mathbf{u} = {}^t(u, v)$ the water velocity and Z_f the bottom height.

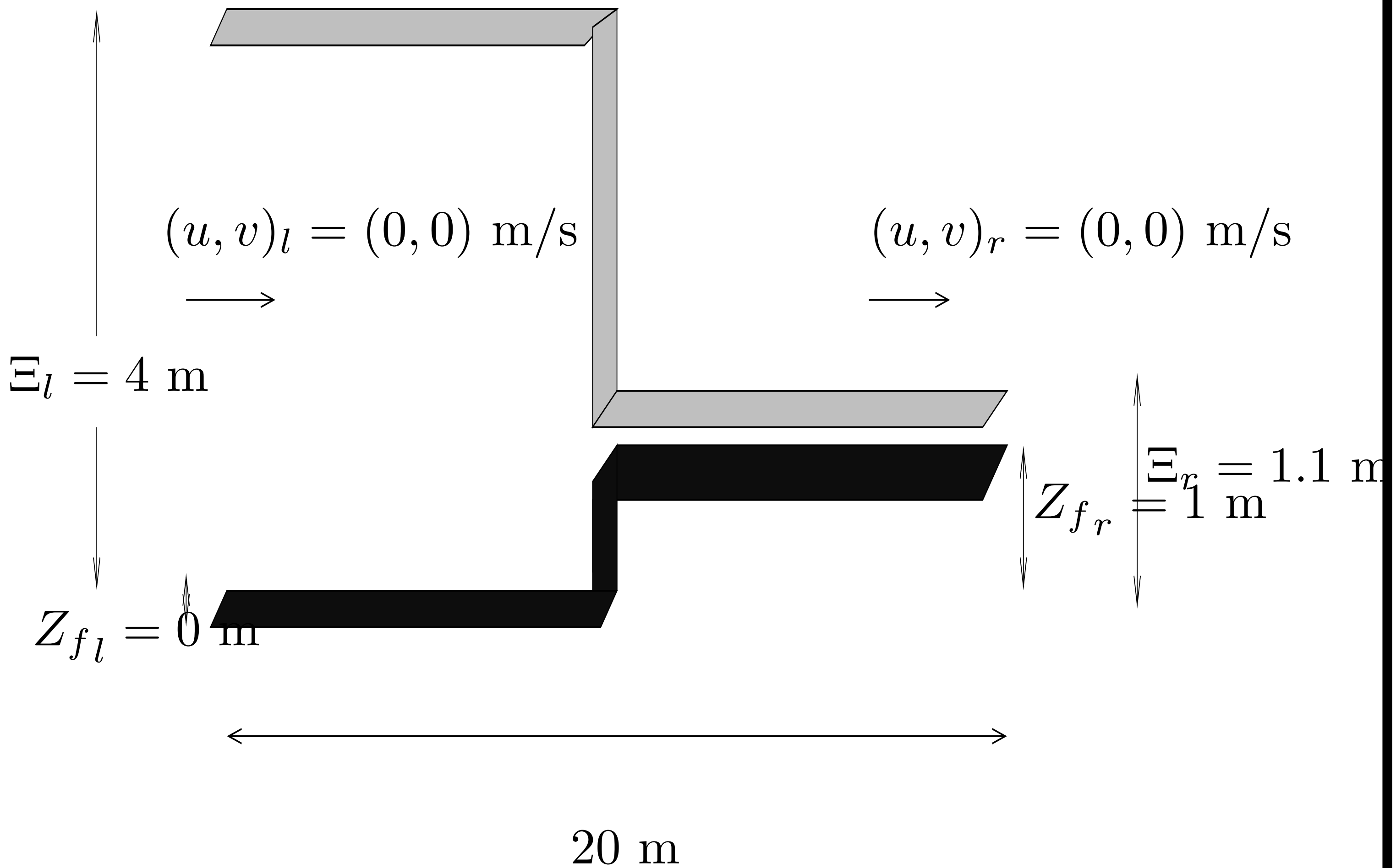


Figure 7: River bed , $\Xi = h + Z_f$

To calculate the predictor phase of $SRNHS$ scheme, one projects the equations on each interface e_{ij}, and gets the following system (**Abgrall 03)**

$$(U_\eta)_t + (F_\eta)_{,\eta} = Q(x, y, U_\eta) \tag{15}$$

with

$$U_\eta = (h, hu_\eta, hu_\tau)^T, \qquad F_\eta = \left(hu_\eta, hu_\eta^2 + g\frac{h^2}{2}, hu_\eta u_\tau\right)^T,$$

$$\text{et} \qquad Q(x, y, U_\eta) = (0, -gh(Z_f)_{,\eta}, 0)^T,$$

$u_\eta = \mathbf{u} \cdot \eta$, $u_\tau = \mathbf{u} \cdot \tau$, η and τ the normal and the tangential vector to the interface, and $(.)_{,\eta}$ the derivate along the normal vector η.

In this case, the predictor phase of scheme $SRNHS$ may be written as follows :

$$U_{ij}^n = \frac{1}{2}\left(U_i^n + U_j^n\right) - \frac{1}{2}\,\mathrm{sgn}\left(\nabla F_\eta\left(\bar{U}\right)\right)\left(U_j^n - U_i^n\right) + \frac{1}{2}\left|\nabla F_\eta\left(\bar{U}\right)\right|^{-1} Q_{ij}^n, \qquad (16)$$

where

$$Q_{ij}^n = -\frac{g}{2}\left(h_i + h_j\right)\left(Z_{f\,j} - Z_{f\,i}\right)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad (17)$$

and $\bar{U}$ the Roe state.

The corrector phase may be written as follows :

$$W_i^{n+1} = W_i^n - \frac{\Delta t^n}{A_i} \sum_{j \in N_i} G\left(W_i^n, W_j^n, Q_{ij}^n, \eta_{ij}^n\right) + \Delta t Q_i^n, \qquad (18)$$

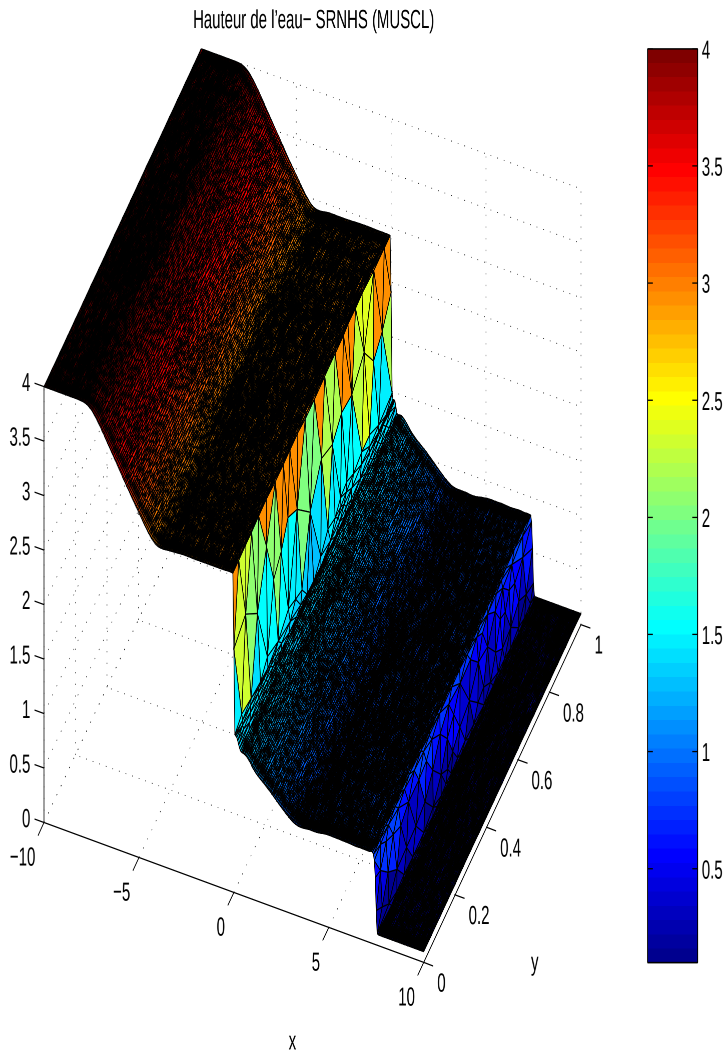
with

$$G\left(W_i^n, W_j^n, Q_{ij}^n, \eta_{ij}^n\right) = F\left(W_{ij}^n\right) \cdot \eta_{ij}$$

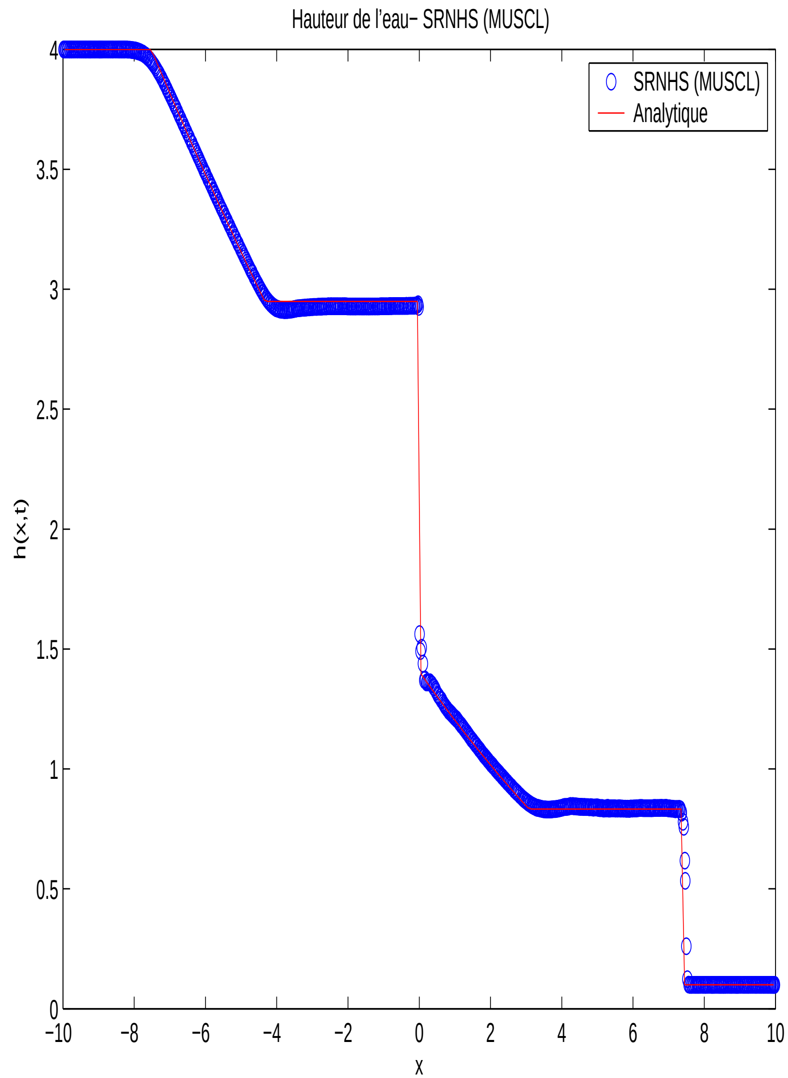
$$W_{ij}^n = \left(h_{ij}^n, (hu_\eta)_{ij}^n\, \eta_x - (hu_\tau)_{ij}^n\, \eta_y, (hu_\tau)_{ij}^n\, \eta_y + (hu_\tau)_{ij}^n\, \eta_x,\right)^T.$$

with

$$Q_i^n = -g\frac{h_i}{A_i}\begin{pmatrix} 0 \\ \\ \sum_{j \in N_i} Z_{ij} \cdot (\eta_{ij})_x |e_{ij}| \\ \\ \sum_{j \in N_i} Z_{ij} \cdot (\eta_{ij})_y |e_{ij}| \end{pmatrix}, \quad \text{where } \; Z_{ij} = \frac{Z_{f\,i} A_i + Z_{f\,j} A_j}{A_i + A_j}$$



$2D$ dam break over a step, water level t=1.2 s



$2D$ dam break over a step, water level, Cross $1D$, t=1.2s

8 The convergence stagnation problem

Let us consider the scalar equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -u\frac{dz}{dx} \tag{19}$$

with $a > 0$ and the following source function:

$$z(x) = \begin{cases} z_L \text{ if } x < 0 \\ z_R \text{ if } x > 0 \end{cases} \tag{20}$$

In the following we will call:

$$\Delta z = z_R - z_L \tag{21}$$

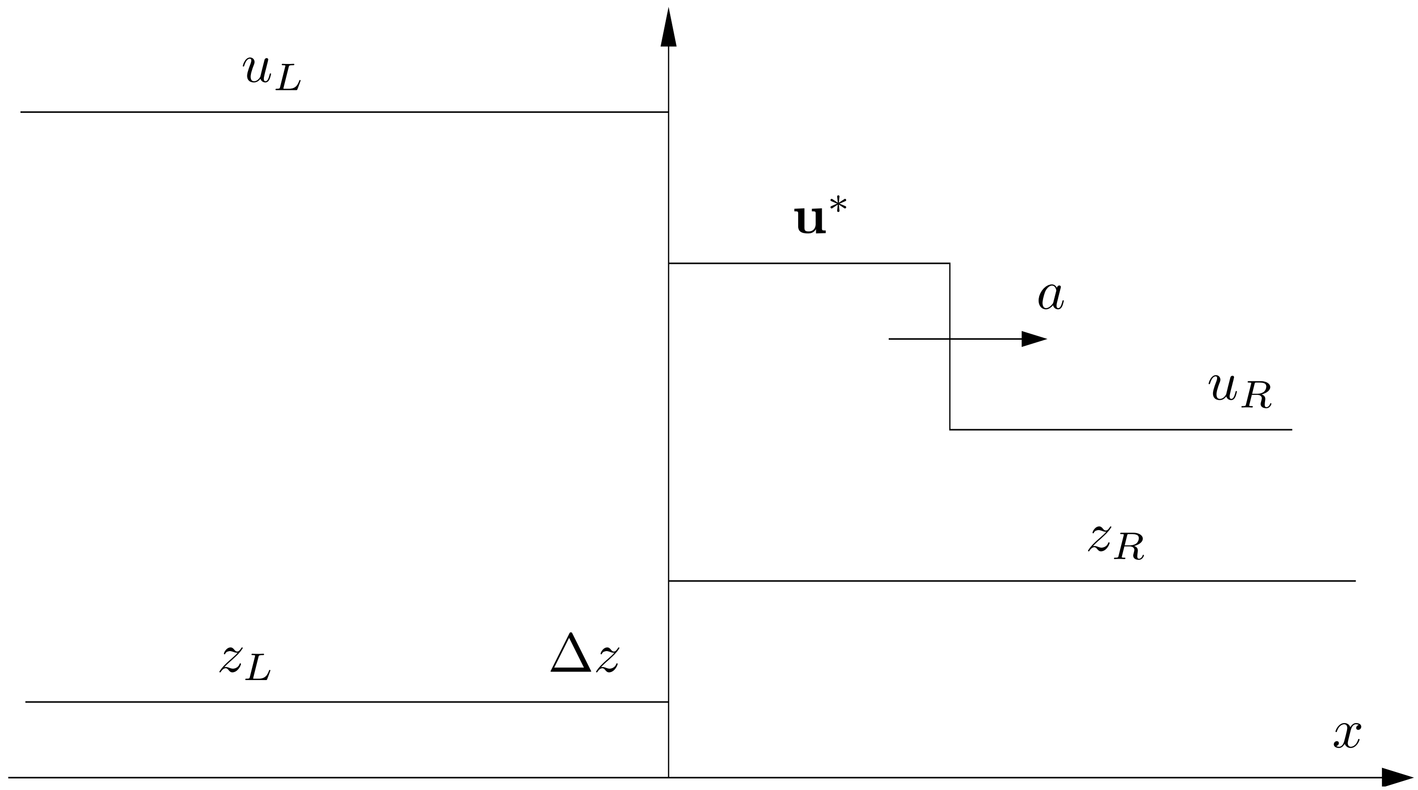


Figure 8: The Riemann solution for the linear equation.

The exact value u^* in terms of u_L and the problem parameters is:

$$u^*_{exact} = u_L \cdot e^{-\Delta z/a} = u_L(1 - (\Delta z/a) + \frac{1}{2}(\Delta z/a)^2 - \frac{1}{6}(\Delta z/a)^3 + ...)$$

Application of the *SRNHS* scheme to linear equation (19) with $a > 0$ leads to:

$$
\begin{aligned}
u_j^{n+1} &= u_j^n - ar\left(u_j^n - u_{j-1}^n\right) \\
&+ \frac{r}{4}\left[(u_{j+1}^n + u_j^n)(z_{j+1} - z_j) - (u_j^n + u_{j-1}^n)(z_j - z_{j-1})\right] \\
&- \frac{r}{8}(u_{j+1}^n + 2\,u_j^n + u_{j-1}^n)(z_{j+1} - z_{j-1})
\end{aligned}
\tag{22}
$$

which converges to: $u_{num}^* = u_L \frac{1 - \Delta z/2a + 3\Delta z^2/64a^2}{1 + \Delta z/2a + 3\Delta z^2/64a^2}$

hence: $u_{num}^* = u_L(1 - \frac{\Delta z}{a} + \frac{1}{2}\left(\frac{\Delta z}{a}\right)^2 - \frac{13}{64}\left(\frac{\Delta z}{a}\right)^3 + ...)$

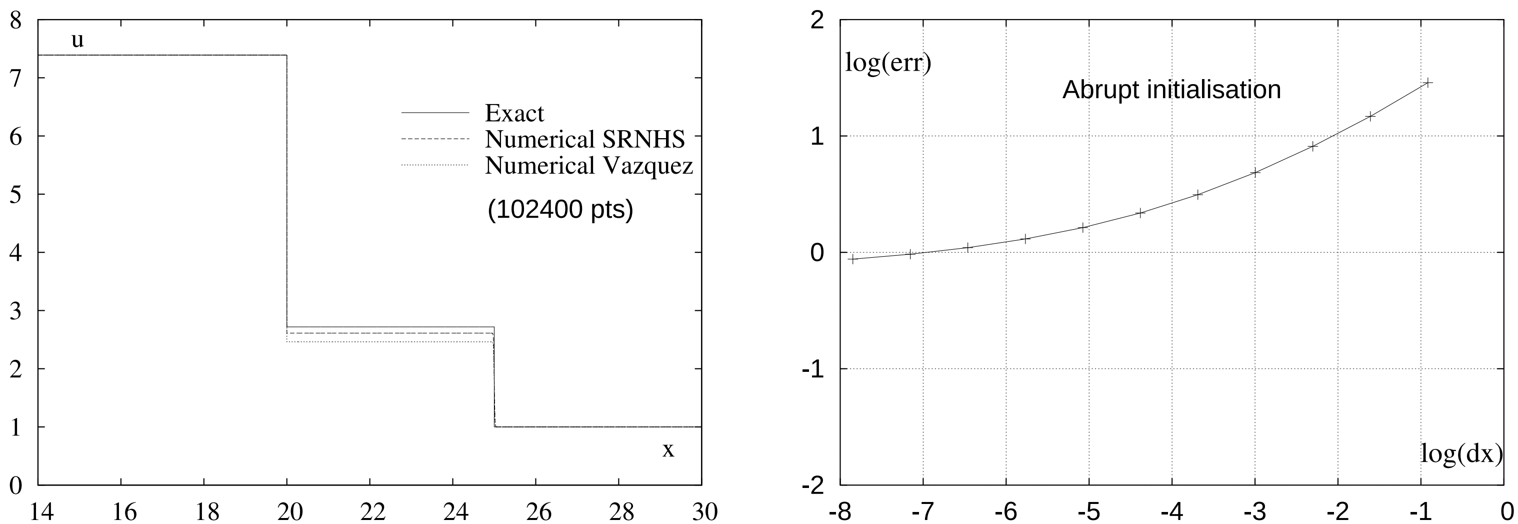


Figure 9: Riemann problem for linear scalar equation with $\Delta z/a = 1$. Initial discontinuity at $x = 20$. Exact versus numerical solution with 102400 nodes (left). Error convervenge rate (right).

Another way of solving this problem avoiding the use of an exact Godunov method is to regularize the source term discretization (and correspondingly the initial data) to ensure that parameter $\Delta z/a$ is small *at each cell interface*. This can be accomplished for instance by taking:

$$\hat{z}(x) = \frac{z_R + z_L}{2} + \frac{z_R - z_L}{2} \cdot \tanh\left(\frac{x}{C\Delta x^p}\right) \tag{23}$$

and

$$\hat{u}_0(x) = \frac{u_R + u_L}{2} + \frac{u_R - u_L}{2} \cdot \tanh\left(\frac{x}{C\Delta x^p}\right) \tag{24}$$

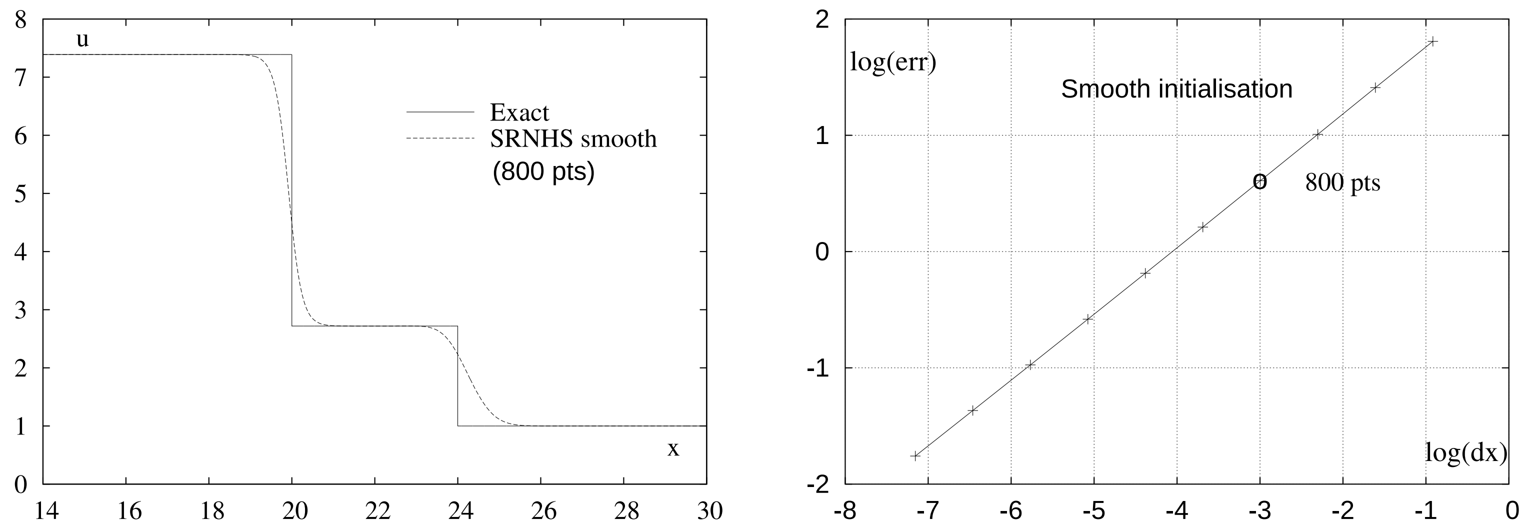
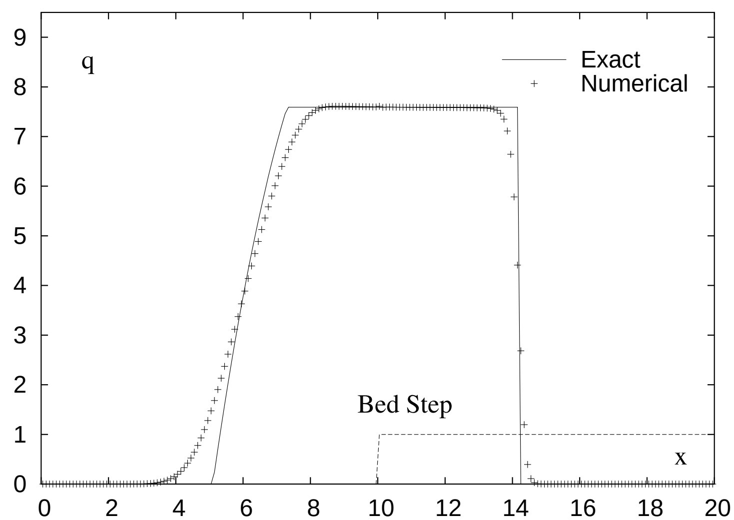


Figure 10: Riemann problem for linear scalar equation with $\Delta z/a = 1$. SRNHS scheme with smooth initialization. Initial discontinuity at $x = 20$. Exact versus numerical solution with 800 nodes (left). Error convervenge rate (right).



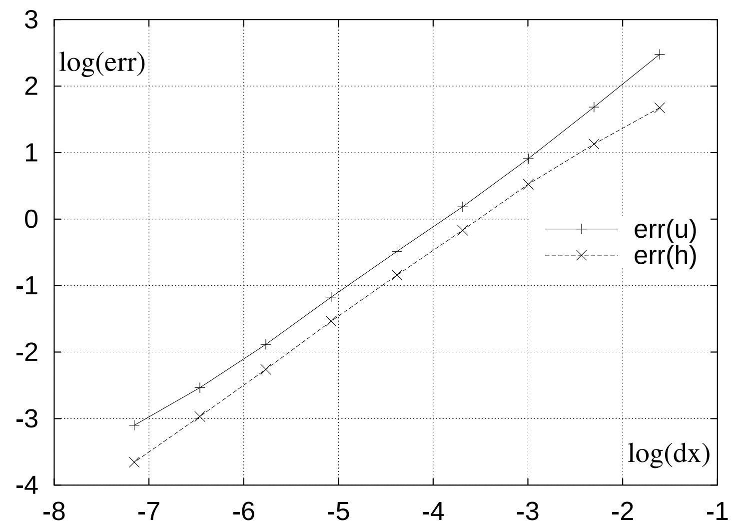


Figure 11: Smoothed dam break problem over a smoothed step. Flow rate (left) and L_1 Convergence plot of the velocity and the depth (right).

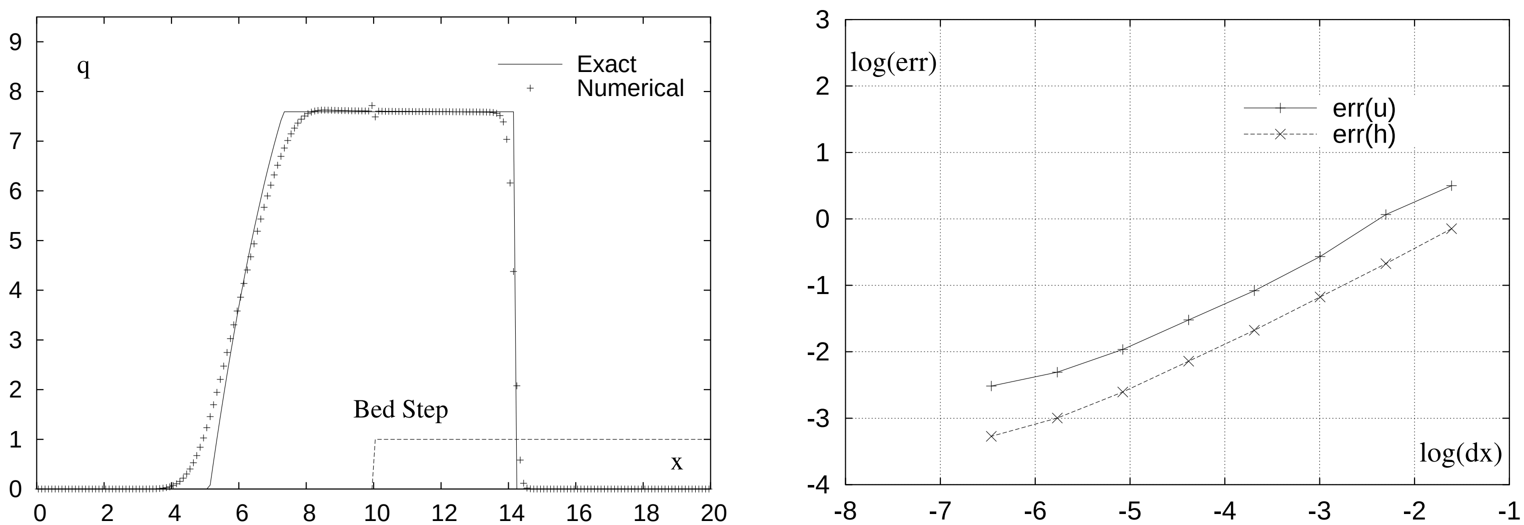


Figure 12: Abrupt initialisation dam break problem over a step. Flow rate (left) and L_1 Convergence plot of the velocity and the depth (right).

9 Pollutant Transport in the Strait of Gibraltar

For simplicity in presentation we write the equations in a conservative form as:

$$\partial_t \mathbf{W} + \partial_x \left(\mathbf{F}(\mathbf{W}) - \tilde{\mathbf{F}}(\mathbf{W}) \right) + \partial_y \left(\mathbf{G}(\mathbf{W}) - \tilde{\mathbf{G}}(\mathbf{W}) \right) = \mathbf{Q}(\mathbf{W}), \quad (25)$$

where $\mathbf{W}$ and $\mathbf{Q}$ are the vectors of conserved variables and source terms, $\mathbf{F}$ and $\mathbf{G}$ are the convection tensor fluxes, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{G}}$ are the diffusion tensor fluxes

$$\mathbf{W} = \begin{pmatrix} h \\ hu \\ hv \\ hC \end{pmatrix}, \qquad \mathbf{Q}(\mathbf{W}) = \begin{pmatrix} 0 \\ -gh(S_{0x} + S_{fx}) \\ -gh(S_{0y} + S_{fy}) \\ hQ \end{pmatrix},$$

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \\ huC \end{pmatrix}, \qquad \mathbf{G}(\mathbf{W}) = \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \\ hvC \end{pmatrix},$$

$$\tilde{\mathbf{F}}(\mathbf{W}) = \left(0, 0, 0, D_{xx}\partial_x\left(hC\right) + D_{xy}\partial_y\left(hC\right)\right)^T$$

$$\tilde{\mathbf{G}}(\mathbf{W}) = \left(0, 0, 0, D_{yx}\partial_x\left(hC\right) + D_{yy}\partial_y\left(hC\right)\right)^T$$

where D_{xx}, D_{xy}, D_{yx} and D_{yy} are entries of the diffusion matrix $\mathbf{D}$ assumed to be nonnegative. $S_{0x} = \partial_x Z,\ S_{0y} = \partial_y Z$, with $Z(x, y)$ denotes the bottom topography, while S_{fx} and S_{fy} are the friction losses along the x- and y-direction, and are defined by $S_{fx} = \eta^2 \dfrac{u\sqrt{u^2+v^2}}{h^{4/3}},\ S_{fy} = \eta^2 \dfrac{v\sqrt{u^2+v^2}}{h^{4/3}}$, where η is the Manning roughness coefficient.

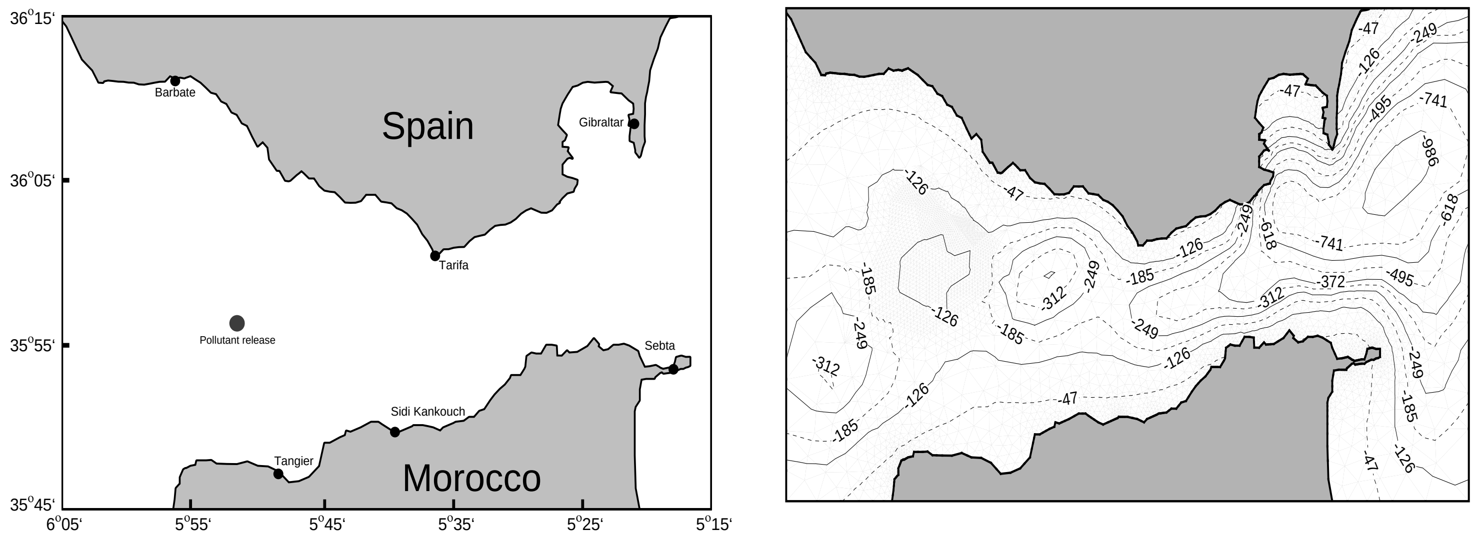


Figure 13: Definition of the strait of Gibraltar (left) and its bathymetry (right).

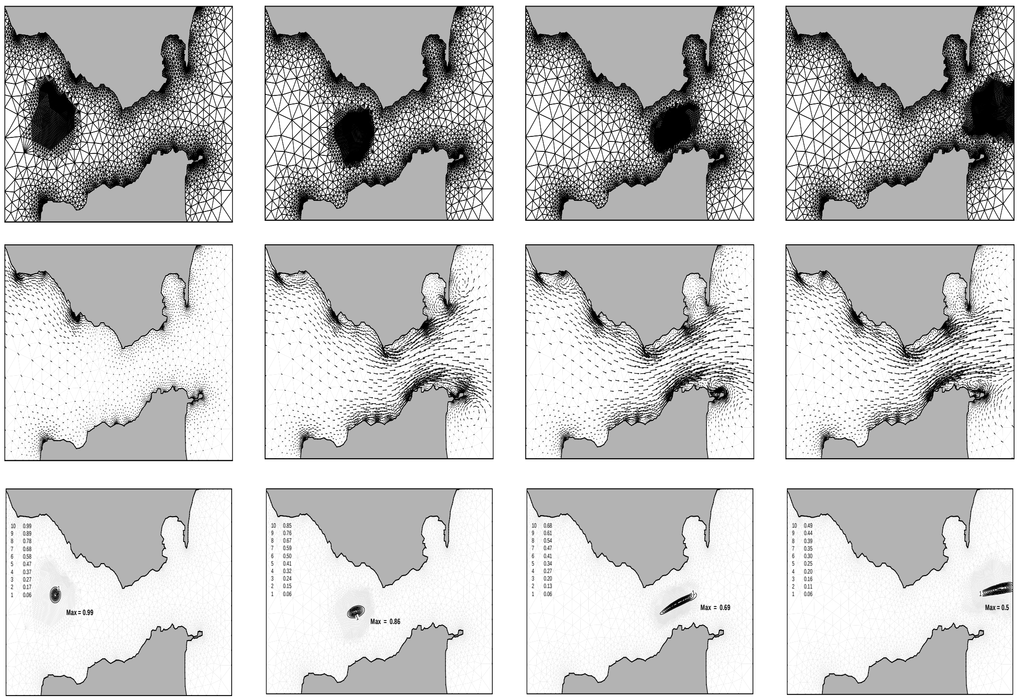


Figure 14: Adapted meshes (first row), velocity vectors (second row) and pollutant concentration (third row) at different simulation times. From left to right $t = 1$, 2, 3 and 4.5 *hours*.

10 Conclusions and future

* Construction of a new finite volume scheme designed for non homogeneous systems

* The approximate intermediate state is upwind instead of the numerical flux

* Both homogeneous and non homogeneous part of the system are upwind

* Equilibrium for steady states is respected

* New applications were considered (Flow in a duct, problems of pollutant transport)

* More complex problems (Water on a moving bed, realistic pollutant problems) are under study