Mesh adaptation for numerical calculation

Performance and Efficiency of a posteriori error estimates

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Introduction

FE-FV method is commonly used to approach complex problems in several physical phenomena. In many of these applications, adaptive techniques using a posteriori error estimators have become very usefull. These estimators measure the quality of the computed solution and provide information to control the mesh adaptation algorithms.

Permanent flow in porous media

We consider the diffusion problem model : Find $u \in \mathbf{H}_0^1(\Omega)$ such that:

$$-div(a(x)\nabla u) = f \ \ in\ \Omega \quad (1)$$

$$(2)$$

$$u = 0 \ \ on\ \partial\Omega \quad (3)$$

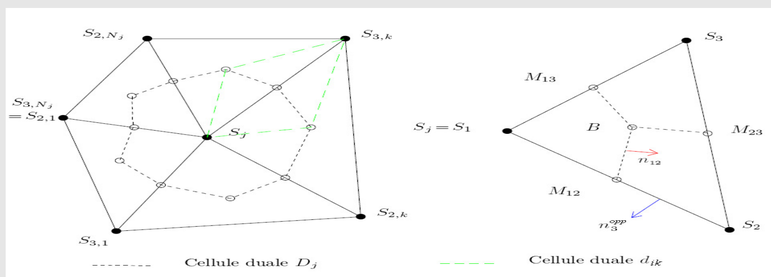
Where $\Omega \subset \mathbb{R}^2$, $f \in \mathbb{L}^2(\Omega)$, $a = a_i$ in Ω_i

Objectives

The goal of this work is to consider a new method of mesh refinement ensuring both accuracy and locality. We focus on the study of a posteriori error estimates applied to pure diffusion problems , and we analyze carefully the robustness of the estimators in particular when there is a singularity. We conduct numerical experiments to prove the efficiency of our method.

Finite Elements -Finite Volumes scheme :

$$\begin{cases} \text{Find} \quad u_h \in \mathbf{C}(\overline{\Omega}) \cap \mathbf{P}_1(\mathcal{T})_h \subset \mathbf{H}_0^1(\Omega), \\ -< a\nabla u_h, 1 >_{\partial D} = (f,1), \qquad \forall D\ in \mathcal{D}_h^{int} \end{cases}$$



$$-\int_{D_h} div(a(x)\nabla u)dx \quad = \quad \int_{D_h} f(x)dx$$

The finite volume scheme for the transport equation writes:

$$\sum_{T\in(\mathcal{D})_h} a_{12}^{flow}(T)(u2-u1) + a_{13}^{flow}(T)(u3-u1) = \int_{D_h} f(x)dx$$

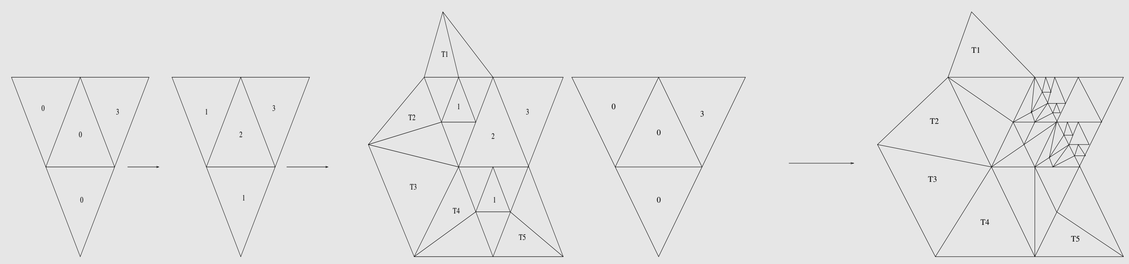
Flux reconstruction

- The flux $-a\nabla u \in \mathbf{H}(div,\Omega)$, but $-a\nabla u_h \notin \mathbf{H}(div,\Omega)$
- $t_h \in \mathbf{RTN}_0(\mathcal{S}_h) \subset \mathbf{H}(div,\Omega)$ and $(div\ t_h, 1)_D = (f,1)_D$, $\forall D\ in \mathcal{D}_h^{int}$

A guaranteed energy norm a posteriori error estimate

$$|||u-u_h|||_\Omega \leqslant \left\{ \sum_{D\in D_h} \left(\underbrace{m_D||f-\nabla\cdot t_h||_D}_{\text{Residual estimator}} + \underbrace{||a^{\frac{1}{2}}\nabla u_h + a^{\frac{-1}{2}} t_h||_D}_{\text{Diffusive flux estimator}} \right)^2 \right\}^{\frac{1}{2}}$$

(Left) Mesh refinement and conformity with ADAPT: propagation of refined elements



(Right) Mesh refinement with ADAPT and conformity with NEWEST

- There is no more propagation of the refinement on triangles T_1, T_2 et T_4.

CPU time performance on adaptive meshes versus uniform meshes

Regular mesh :

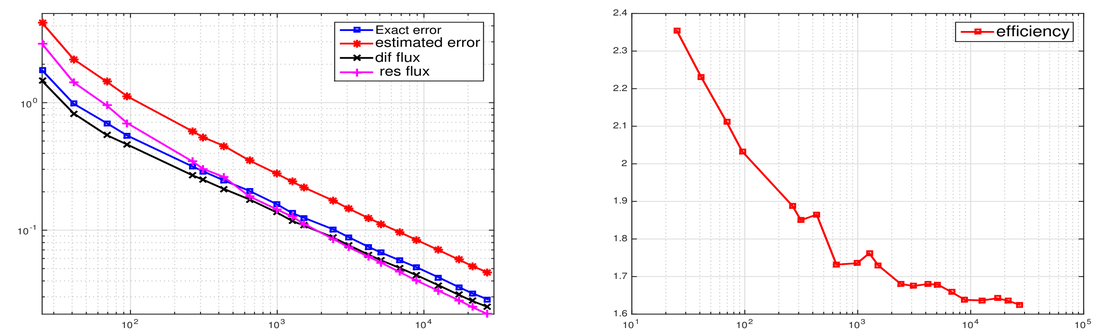
Num. elemts (adap.mesh)	Error (adap. mesh)	CPU time (adap. mesh)	Num. elemts (unif. mesh)	Error (unif. mesh)	CPU time (unif. mesh)
254	0,47	0,88	512	0,4530	3,4
1210	0,19	2,64	2048	0,22	14,18
3030	0,12	6,98	8192	0,11	84,82
12954	0,057	41,46	32768	0,056	588,79

Irregular mesh :

Num. elemts (adap.mesh)	Error (adap. mesh)	CPU time (adap. mesh)	Num. elemts (unif. mesh)	Error (unif. mesh)	CPU time (unif. mesh)
328	0,3628	0,94	448	0,3750	3,18
1249	0,19	2,81	1792	0,1884	11,45
5086	0,0982	11,86	7168	0,0942	66,6
16263	0,0559	54,83	28672	0,0471	550

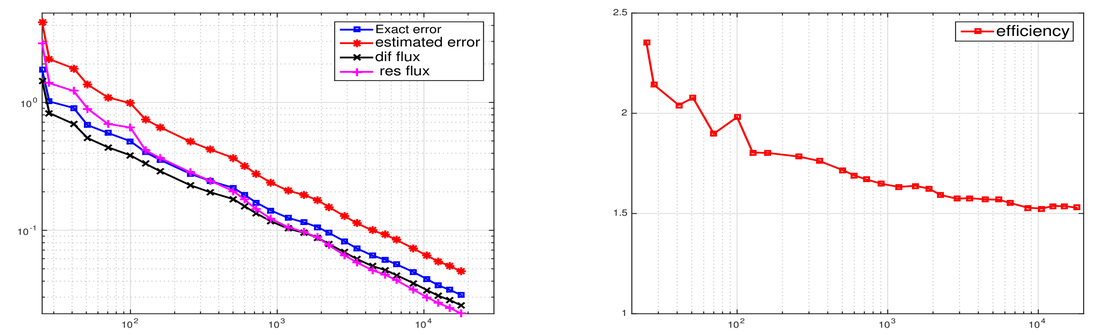
Test case 1: $a(x) = 1$, $u(x,y) = x(1-x)y(1-y)$

- **Refinement with Adapt-Newest: regular mesh**



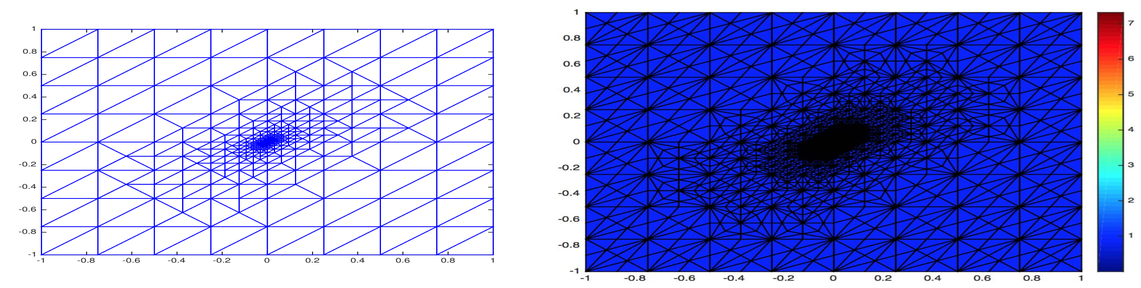
Exact energy error , Estimated energy error, Residual estimator , Diffusive flux estimator(left) , Efficiency (right)

- **Refinement with Newest-Vertex-Bisection, regular mesh**

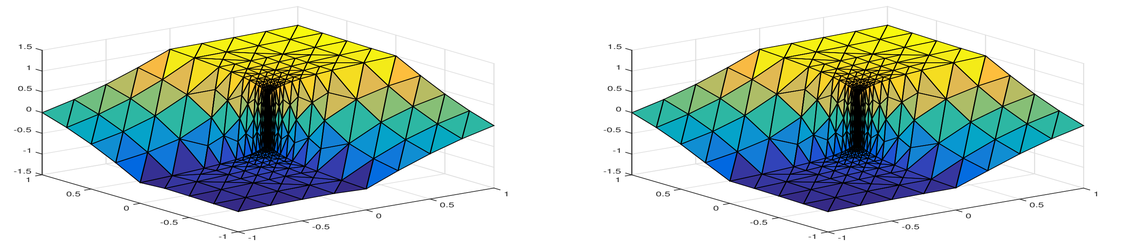


Exact energy error , Estimated energy error, Residual estimator , Diffusive flux estimator(left) , Efficiency (right)

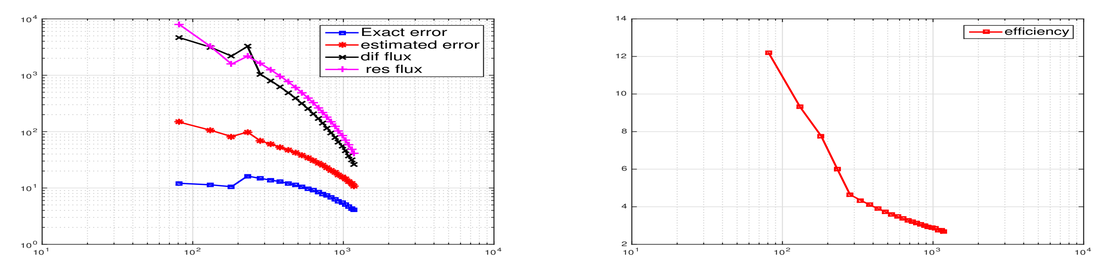
Test case 2: $a_1 = a_3 = 100, a_2 = a_4 = 1$, $\alpha = 0.127$ **and** $u(r,\theta)_{|\Omega_i} = r^\alpha(\alpha_i \sin(\alpha\theta) + \beta_i \cos(\alpha\theta))$



Refined mesh(left), Estimated energy error distribution (right)



Exact solution(gauche), Numerical(droite)



Exact energy error , Estimated energy error(left) , Efficiency (right)

Perspectives

- Study of Unsaturated problems and error estimates for non linear diffusion.
- Consider an appropriate porous media flow model ,
- Integration of this model within the Finite Volumes Object Oriented code ADAPT ,
- Integration of the error estimators in ADAPT
- Conducting new experiments
- Application to real porous media flow problems .