Exact Solutions for Shallow Water Equations
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Non Linear Systems Considered

We are interested by fluid flow problems described by such non linear systems:

\[
\frac{\partial W}{\partial t} + \frac{\partial F(W)}{\partial x} + \frac{\partial G(W)}{\partial y} + \frac{\partial H(W)}{\partial z} = 0
\]  

(1)

Examples:
Euler equations in one space dimension

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} &= 0 \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + P)}{\partial x} &= 0 \\
\frac{\partial E}{\partial t} + \frac{\partial [u (E + P) ]}{\partial x} &= 0
\end{align*}
\]

(2)

with ideal gas equation of state : \( p = (\gamma - 1) \left( E - \frac{1}{2} \rho u^2 \right) \), where \( \rho \) is fluid density, \( u \) the velocity, \( E \) the energy, and \( p \) : the pressure.
Shallow Water Flow

We consider water flow in a configuration where the water depth is negligible when compared to the characteristic length of the domain. [3]). If the bottom is flat, and the friction negligible, the problem is described by the following system:

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} &= 0 \\
\frac{\partial (hu)}{\partial x} + \frac{\partial}{\partial x} \left( hu^2 + \frac{1}{2} gh^2 \right) &= 0
\end{align*}
\]

(3)

$h$ being the water depth, $u$ the velocity, and $g$ the gravity constant.
Introduction

Consider the scalar problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= 0 \quad \text{in } \mathbb{R} \times ]0, T]\n
\{ u = u(x, t) \in \mathbb{R} \\
    u(x, 0) = u_0(x)
\end{align*}
\]

In the sequel, note \( X = \mathbb{R} \times [0, T] \).

Example, Burger’s equation:

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0
\]
Weak solution and jump condition

Smooth solution

If $u \in C^1(X)$, one has: (4) $\implies \frac{\partial u}{\partial t} + f'(u) \frac{\partial u}{\partial x} = 0$

then in the frame $(x, t)$, $u$ is constant on the characteristic curve given by:

$$\begin{align*}
\frac{dx(t)}{dt} &= f'[u(x(t), t)] \\
x(t = 0) &= x_0
\end{align*} \tag{5}$$

One deduce the solution $u$:

$$u(x(t), t) = u(x(0), 0) = u(x_0, 0) = u_0(x_0)$$
Weak solution and jump condition

Smooth solution

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\begin{cases}
\frac{dx(t)}{dt} = f'(u(x(t), t)) \\
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\[ u(x(t), t) = u(x(0), 0) = u(x_0, 0) = u_0(x_0) \]
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One deduce the solution $u$:

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u(x(t), t) = u(x(0), 0) = u(x_0, 0) = u_0(x_0)$$
Note \( f'(u) = a(u) \), the characteristic system considered is then:

\[
\begin{align*}
\frac{dx}{dt} &= a\left(u_0(x_0)\right) \\
u(x, t) &= u_0(x_0)
\end{align*}
\]

which gives:

\[
\begin{align*}
x_0 &= x - ta\left(u_0(x_0)\right) \\
u(x, t) &= u_0(x_0)
\end{align*}
\]

Applications:
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(6)

Applications:
1. Linear case

\[ f(u) = cu : \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \implies a(u) = c \]

characteristic curve : \( x = x_0 + tc \iff x_0 = x - ct \)

Solution : \( u(x, t) = u_0(x - ct) \)
1. Linear case

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Bibliography

F. Benkhaldoun - MHYCOF - MENHYDRO 2010
2. Burger’s equation

Consider the non linear equation:
\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0,
\]

here \( f(u) = \frac{u^2}{2} \), then \( a(u) = f'(u) = u \)

The characteristic curve is given by: \( x = x_0 + tu_0(x_0) \)
Consider the different initial conditions:

**case 1**

\[ u_0(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } x \geq 0 
\end{cases} \]

case \( x_0 < 0 \), \( x = x_0 \), \( u_1(x, t) = u_0(x) \)

case \( x_0 \geq 0 \), \( x = x_0 + tx_0 \), \( u_1(x, t) = u_0(x_0) = u_0 \left( \frac{x}{1 + t} \right) = \frac{x}{1 + t} \)

**case 2**

\[ u_0(x) = \begin{cases} 
1 & \text{if } x < 0 \\
0 & \text{if } x \geq 0 
\end{cases} \]

case \( x_0 < 0 \), \( x = x_0 + t \), \( u_2(x, t) = u_0(x - t) = u_0(x_0) = 1 \)

case \( x_0 \geq 0 \), \( x = x_0 \), \( u_2(x, t) = u_0(x) = 0 \)
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Bibliography

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Discontinuous solution and jump condition:

**Theorem**

The 3 following assertions are equivalent:

i) $u$ is a weak solution of problem (4), i.e:

$$
\int_0^\infty \int_{\mathbb{R}} \left( u \frac{\partial \varphi}{\partial t} + f(u) \frac{\partial \varphi}{\partial x} \right) \, dx \, dt + \int_{\mathbb{R}} u_0(x) \varphi(x, 0) \, dx = 0,
$$

$\forall \varphi \in D(\mathbb{R} \times [0, +\infty[)$

ii) $\forall R = [x_1, x_2] \times [t_1, t_2] \subset \Omega = \mathbb{R} \times [0, T]$, $\forall R = [x_1, x_2] \times [t_1, t_2] \subset \Omega = \mathbb{R} \times [0, T]$

$$
\int_{\partial R} [u \cdot n_t + f(u) \cdot n_x] \, d\sigma = 0
$$
iii) If \( u \) is \( C^1 \), \( u \) is classical solution of \( \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \), and on a shock curve \( \Gamma (u_l, u_r) \), the solution is governed by the jump condition: \( [f(u)] = s[u] \).

One defines the jump \( [u] = u_r - u_l \), and the curve \( \Gamma (u_l, u_r) \), which equation is: \( \frac{dx}{dt} = s \), separates the left and right states \( u_l \) and \( u_r \).

The jump condition is called the Rankime-Hugoniot condition in gas dynamics.
Example: Consider the Burger’s equation:

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \frac{u^2}{2} = 0
\]  \hspace{1cm} (7)

and the initial condition: \( u_0(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \)

First possibility: a weak solution with the shock \( \Gamma(0, 1) \).

The jump condition gives:

\[
[f(u)] = s[u] \implies \left[ \frac{u_r^2}{2} - \frac{u_l^2}{2} \right] = s[u_r - u_l] \implies s = \frac{1}{2} \implies
\]

\[
u(x, t) = \begin{cases} 0 & \text{if } \frac{x}{t} < \frac{1}{2} \\ 1 & \text{if } \frac{x}{t} \geq \frac{1}{2} \end{cases}
\]
Example: Consider the Burger’s equation:

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\[
u(x, t) = \begin{cases} 
0 & \text{if } x < \frac{1}{2t} \\
1 & \text{if } x \geq \frac{1}{2t} 
\end{cases}
\]
Second possibility : a continuous weak solution.

\[
\begin{align*}
    u(x, t) &= \begin{cases} 
        0 & \text{si } \frac{x}{t} < 0 \\
        \frac{x}{t} & \text{si } 0 \leq \frac{x}{t} < 1 \\
        1 & \text{si } \frac{x}{t} \geq 1
    \end{cases}
\end{align*}
\]

We come to the fact that one needs a specific criterium to select, among the above two weak solutions, the unique real solution.
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We come to the fact that one needs a specific criterium to select, among the above two weak solutions, the unique real solution.
Physical validation of the solution : the entropy condition

The entropy solution

**Definition**

A smooth convex function $U$, is said to be an entropy of the problem, if there exists an entropy flux $F$ such that:

$$U'(u)f'(u) = F'(u).$$

**Definition**

A weak solution $u$ of (4) is said entropy solution if

$$\forall \varphi \in D(\mathbb{R} \times [0, T]) : \int_0^T \int_{\mathbb{R}} \left( U(u) \frac{\partial \varphi}{\partial t} + F(u) \frac{\partial \varphi}{\partial x} \right) dx dt \geq 0,$$

where $U$ is an entropy of the problem, and $F$ its entropy flux.
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where $U$ is an entropy of the problem, and $F$ its entropy flux.
Remark 1: An entropy solution respects the entropy condition with the convex (though non derivable) function, $U(u) = |u - k|$, and the associated entropy flux: $F(u) = \text{sgn}(u - k)(f(u) - f(k))$, where $k \in \mathbb{R}$.

Remark 2: Reciprocally, since every convex function belongs to the convex hull of all affine functions, and functions of the form $x \rightarrow |x - k|$, a weak solution which respects the entropy condition with the convex function $U(u) = |u - k|$, is an entropy solution.

Theorem

(Kruzkov 1970) Under some regularity assumptions on $u_0$, there exists a unique entropy weak solution of problem (4).
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Theorem

(Kruzkov 1970) Under some regularity assumptions on $u_0$, there exists a unique entropy weak solution of problem (4).
About Entropy

Lemme : There exists a function $U$ which is transported in regions where $u$ is $C^1$. i.e. $\frac{\partial}{\partial t} U(u) + \frac{\partial}{\partial x} F(u) = 0$

proof : If $u$ is $C^1$ : $U'(u) \left( \frac{\partial u}{\partial t} + f'(u) \frac{\partial u}{\partial x} \right) = 0$, if there exists $F$ such that $U'(u)f'(u) = F'(u)$, then $\frac{\partial}{\partial t} U(u) + \frac{\partial}{\partial x} F(u) = 0$
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Consider the regularized problem:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= \varepsilon \frac{\partial^2 u}{\partial x^2} \\
\quad u(x, 0) &= u_0(x)
\end{aligned}
\]  \quad (8)

**Proposition**

There exists a unique smooth solution \( u^\varepsilon \) of the problem (8)

**Proposition**

The solution of problem (4) is the limit in the distribution sens of the solution of problem (8), as \( \varepsilon \) tends to 0.
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A piecewise $C^1$ function $u$, is an entropy weak solution of (4) if and only if:

i) $u$ is a classical solution in $(x, t)$ regions where $u$ is $C^1$

ii) On a shock curve $\Gamma$, $u$ satisfies $[F(u)] \leq s [U(u)]$, $\forall (U, F)$ a couple of entropy and antropy flux.

**Corollaire**

1) If $f$ is strictly convex, then a shock is entropic if and only if:

$f'(u_r) < s < f'(u_l)$

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2) If $f$ is strictly convex, then a shock is entropic if and only if:

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Application: the first weak solution in example (7) is non entropic, and hence non admissible.
