Schwarz, waveform relaxation, and Tchebychev

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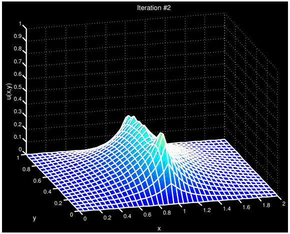

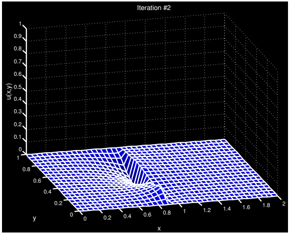

Outline

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1. Introduction

2. History and fundamentals

3. Schwarz waveform relaxation → The complex case

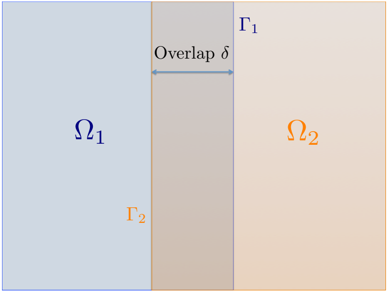
Outline

Introduction

What is “optimized” Schwarz method

$$\mathcal{L}u := -\Delta u + c^2 u = f \text{ in } \mathbb{R}^2$$

I. Take a Schwarz method, overlap δ.



$$-\Delta u_1^n = f \quad \text{in } \Omega_1$$

$$u_1^n = u_2^{n-1}, \text{ on } \Gamma_1$$

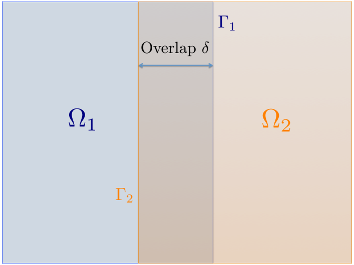
$$-\Delta u_2^n = f, \text{ in } \Omega_2$$

$$u_2^n = u_1^n \text{ on } \Gamma_2$$

Principle

II. Replace Dirichlet transmission condition by Robin or Ventcell transmission condition

$$\mathcal{L}u := -\Delta u + c^2 u = f \text{ in } \Omega$$



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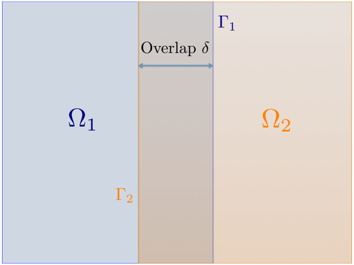


$$(-\Delta + c^2)u_1^n = f \quad \text{in } \Omega_1$$

$$(\partial_{n_1} + P_1)u_1^n = (\partial_{n_1} + P_1)u_2^{n-1}, \text{ on } \Gamma_1$$

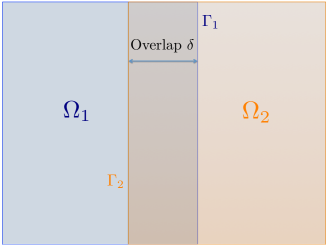
$$(-\Delta + c^2)u_2^n = f, \text{ in } \Omega_2$$

$$(\partial_{n_2} + P_2)u_2^n = (\partial_{n_2} + P_2)u_1^n \text{ on } \Gamma_2$$

Principle

III. Fourier transform in the transverse direction ($y \leftrightarrow k$)

$$\mathcal{L}u := -\Delta u + c^2 u = f \text{ in } \Omega$$



Algorithm for the error

$$u_1^n - u \to u_1^n \to \hat{u}_1^n$$

Principle

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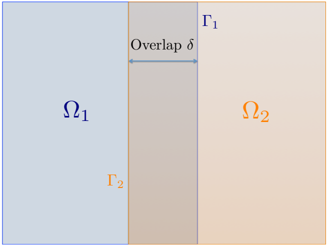

Algorithm for the error

$$u_1^n - u \to u_1^n \to \hat{u}_1^n$$

$$(-\partial_{xx} + (k^2 + c^2))\hat{u}_1^n = 0 \text{ in } (-\infty, \delta) \times \mathbb{R}$$

$$(\partial_x + P_1(ik))\hat{u}_1^n = (\partial_x + P_1(ik))\hat{u}_2^{n-1}, \text{ on } \Gamma_1$$

$$(-\partial_{xx} + (k^2 + c^2))\hat{u}_2^n = 0 \text{ in } (0, +\infty) \times \mathbb{R}$$

$$(-\partial_x + P_2(ik))\hat{u}_2^n = (-\partial_x + P_2(ik))\hat{u}_1^n \text{ on } \Gamma_2$$

Principle

III. Define the convergence factor

$$\hat{u}_1^n = a_1^n(k)e^{\sqrt{(k^2+c^2)}x} \quad \text{in } (-\infty,\delta)\times\mathbb{R}$$

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$$(-\partial_x + P_2(ik))\hat{u}_2^n = (-\partial_x + P_2(ik))\hat{u}_1^n \quad \text{on } \Gamma_2$$

Concentrate on $P_1 = P_2 = P$, $\hat{u}_1^n = \rho\hat{u}_1^{n-1}$,

$$\rho(k,P,\delta) = \left(\frac{P(ik)-\sqrt{(k^2+c^2)}}{P(ik)+\sqrt{(k^2+c^2)}}\right)^2 e^{-2\sqrt{(k^2+c^2)}\delta}$$

Exact operator : $P(\partial_y) = \sqrt{(-\partial_y^2 + c^2)}$

Principle

III. Define the convergence factor

$$a_2^n(k) = \left(\frac{P_1(ik) - \sqrt{(k^2+c^2)}}{P_1(ik) + \sqrt{(k^2+c^2)}} \, \frac{P_2(ik) - \sqrt{(k^2+c^2)}}{P_2(ik) + \sqrt{(k^2+c^2)}} \right) e^{-2\sqrt{(k^2+c^2)}\delta} a_2^{n-1}(k)$$

Concentrate on $P_1 = P_2 = P$, $\hat{u}_1^n = \rho \hat{u}_1^{n-1}$,

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Principle

IV. Optimization

$$\rho(k,P,\delta)=\left(\frac{P(ik)-\sqrt{(k^2+c^2)}}{P(ik)+\sqrt{(k^2+c^2)}}\right)^2 e^{-2\sqrt{(k^2+c^2)}\delta}$$

Exact operator: $P(\partial_y)=\sqrt{(-\partial_y^2+c^2)}$

Approximate operator: P polynomial of degree $\leq k$

$$\|u_2^n-u\|_{L^2(\Omega_2)}=\|\rho(k,P,\delta)(\hat{u}_2^0-\hat{u})\|_{L^2((0,+\infty)\times\mathbb{R})}$$

$$\inf_{P\in\mathbf{P}_m}\sup_{k\in K}\left|\frac{P(ik)-\sqrt{(k^2+c^2)}}{P(ik)+\sqrt{(k^2+c^2)}}\right| e^{-2\sqrt{(k^2+c^2)}\delta}$$

Transmission operator

$$\partial_x+P(\partial_y)=\begin{cases}\partial_x+a_0 & \text{Robin WP}\\ \partial_x+a_0+a_2\partial_y^2 & \text{Ventcel WP}\\ \text{etc.} & \text{IP}\end{cases}$$

And here comes Tchebytcheff

$$\inf_{P\in\mathbf{P}_m}\sup_{k\in K}\left|\frac{P(k^2)-\sqrt{(k^2+c^2)}}{P(k^2)+\sqrt{(k^2+c^2)}}\right|e^{-2\sqrt{(k^2+c^2)}\delta}$$

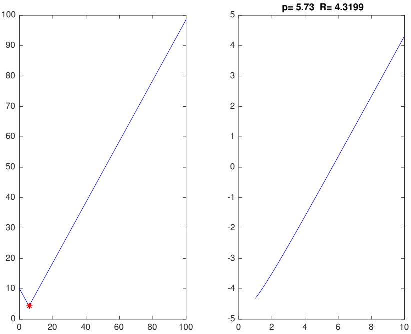
Chebyshev approximation

$$\inf_{P\in\mathbf{P}_m}\sup_{k\in K}\left|P(k^2)-\sqrt{(k^2+c^2)}\right|$$

Commun behavior

$$n = 0,\ f(k) = \sqrt{k^2 + c^2}$$

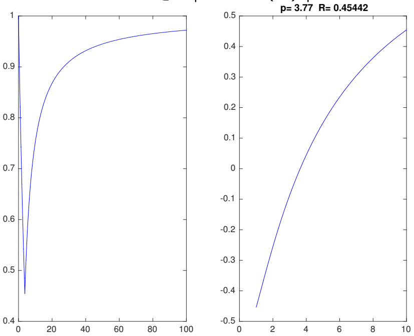
$$\inf_{p\in\mathbb{R}} \sup_{k\in K} |p - f(k)|$$



$$p \to \|p - f\|_\infty, \quad k \to p^* - f(k)$$

$$p^* = \frac{f(k_{min}) + f(k_{max})}{2}$$

$$\inf_{p\in\mathbb{R}} \sup_{k\in K} \left| \frac{p - f(k)}{p + f(k)} \right|$$



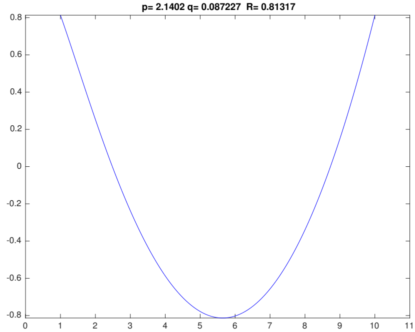
$$p \to \left\| \frac{p - f}{p + f} \right\|_\infty, \quad k \to \frac{p^* - f(k)}{p^* + f(k)}$$

$$p^* = \sqrt{f(k_{min}) f(k_{max})}$$

Equioscillation

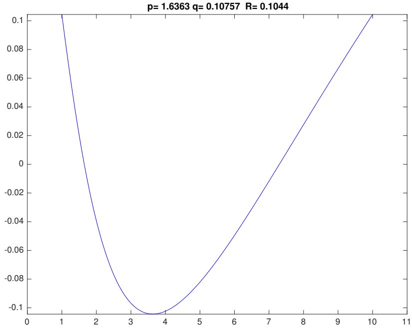
$$f(k) = \sqrt{k^2 + c^2},$$

$$\inf_{p \in \mathbb{R}} \sup_{k \in K} \left| p + qk^2 - f(k) \right|$$



$$k \to p^* + q^* k^2 - f(k)$$

$$\inf_{p \in \mathbb{R}} \sup_{k \in K} \left| \frac{p + qk^2 - f(k)}{p + qk^2 + f(k)} \right|$$



$$k \to \frac{p^* + q^* k^2 - f(k)}{p^* + q^* k^2 + f(k)}$$

History and fundamentals

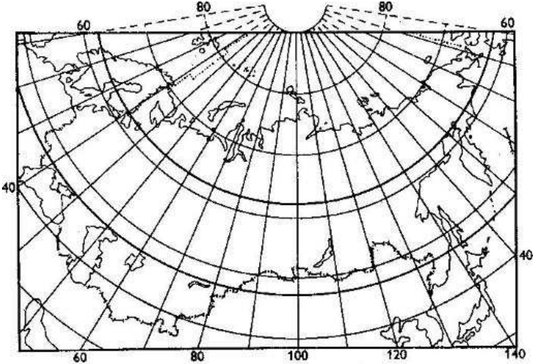
Precursors

See Steffens, and Anastassiou, *History of Approximation Theory From Euler to Bernstein*

Euler approximates the proportion of longitudes and latitudes of the conical projection (Delisle) to the real proportion of the terrestrial globe *De proiectione geographica De Lisliana in mappa generali imperii Russici usitata, 1777*

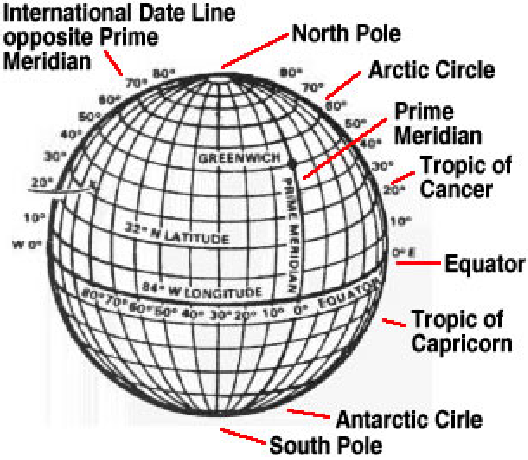
$$\min_{P \in \mathbf{P}_1} \max_{x \in [a,b]} |p(x) - \delta \cos(x)|$$

δ is the length of one degree of latitude of the globe.



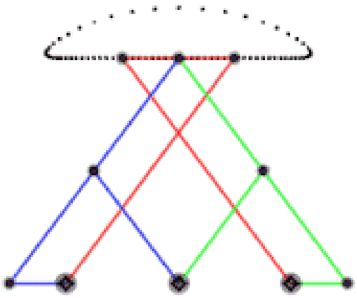
Precursors

Laplace in *Celestial mechanics* around 1830, wants to approximate the earth with an ellipsoid by minimizing the error between the meridians of the ellipsoid and the measured meridians of the earth.



Precursors

Tchebycheff gets interested through the theory of mechanisms in 1854, with no proof yet, using alternations.



Precursors

Tchebycheff gets interested through the theory of mechanisms in 1854, with no proof yet, using alternations. Then gives the first analysis of a general problem.

Théorème 2.

Les quantités $p_1, p_2, \ldots p_{n-1}, p_n$ *étant choisies de manière à ce que la fonction*

$$F(x) = p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n - Y$$

s'écarte le moins possible de zéro depuis $x = -h$ *jusqu'à* $x = +h$, *les équations*

$$F^2(x) - L^2 = 0, \ (x^2 - h^2)\, F'(x) = 0$$

ont au moins $n + 1$ *solutions communes, différentes entre elles et comprises entre* $x = -h$ *et* $x = +h$. *La quantité* L *désigne la limite des écarts de* $F(x)$ *de zéro entre* $x = -h$ *et* $x = +h$.

Schwarz waveform relaxation $\rightarrow$
The complex case

Optimized Schwarz Waveform relaxation

1. Decomposition in time windows.

$$\partial_t U - \Delta U = F \quad \text{in } [T_i, T_{i+1}]$$

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2. Decomposition in subdomains Ω_j, without or with overlap.

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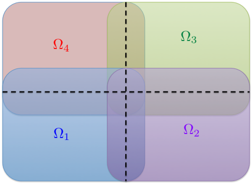

Decomposition in space

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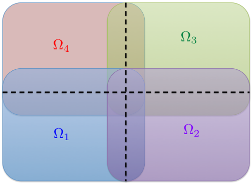
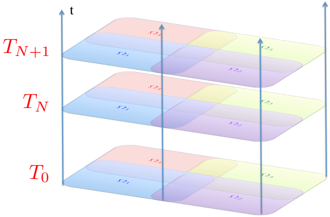


Decomposition in space



Space-time decomposition

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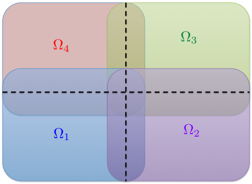
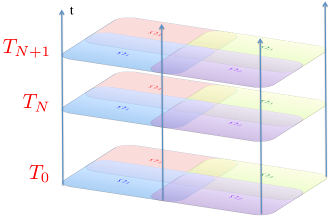


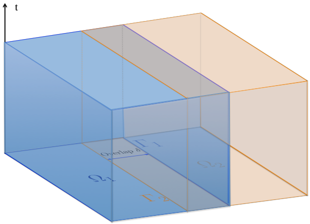
Decomposition in space



Space-time decomposition

Iteration process + optimized transmission conditions.

$$\mathcal{L}u := \partial_t u + (\boldsymbol{a}\cdot\boldsymbol{\nabla})u - \nu\Delta u + cu \text{ in } \Omega\times(0,T)$$



$$\begin{cases} \mathcal{L}u_1^{k+1} & = & f & \text{in } \Omega_1\times(0,T)\\ u_1^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_1\\ \color{blue}{\mathcal{B}_1} u_1^{k+1} & = & \color{blue}{\mathcal{B}_1} u_2^k & \text{on } \color{blue}{\Gamma_1}\times(0,T) \end{cases}$$

$$\begin{cases} \mathcal{L}u_2^{k+1} & = & f & \text{in } \Omega_2\times(0,T)\\ u_2^{k+1}(\cdot,0) & = & u_0 & \text{in } \Omega_2\\ \color{red}{\mathcal{B}_2} u_2^{k+1} & = & \color{red}{\mathcal{B}_2} u_1^{k+1} & \text{on } \color{red}{\Gamma_2}\times(0,T) \end{cases}$$

$$\mathcal{B}_j u := (\nu\nabla u - \boldsymbol{a})\cdot\boldsymbol{n}_j + \color{red}{\mathbf{p}}u + \color{red}{\mathbf{q}}(\partial_t + \boldsymbol{a}\cdot\boldsymbol{\nabla}u - \nu\Delta_S u + \color{magenta}{c}u)$$

Convergence factor

The case of half-spaces and constant coefficients: Fourier transform in time and transverse space

$$\phi(z) = a_1^2 + 4\nu c + 4\nu z,\ z = i(\omega + \boldsymbol{a_S} \cdot \boldsymbol{k}) + \nu|\boldsymbol{k}|^2,$$

Convergence factor

$$\rho(\omega, k, P, L) = \left(\frac{P - \phi^{1/2}}{P + \phi^{1/2}}\right)^2 e^{-2\phi^{1/2}L/\nu}, \quad P(z) = p + qz.$$

The homographic best approximation problem

Hypothesis: K compact in $\mathbb{C}$, $\operatorname{Re} f(K) > 0$, f continuous.

$$\delta_n = \inf_{P \in \mathbf{P}_n} \sup_{z \in K} |F(z,P)|, \quad F(z,P) = \frac{P(z) - f(z)}{\overline{P(z)} - \overline{f(z)}}, \quad h(P) = \|F(z,P)\|_{L^\infty(K)}.$$

I. $\delta_n < 1$,

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II. Existence: minimizing sequence P_n^k is bounded

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III. P_n^* solution $\implies$ alternation in at least $n+2$ points

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Real case $K = [a,b], (x-a)(x-b)\partial_z F(z,P) = 0, \; F^2(z,P) - \delta_n^2 = 0.$

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Real case $K = [a,b], (x-a)(x-b)\partial_z F(z,P) = 0,\ F^2(z,P) - \delta_n^2 = 0.$

- The case $n = 0$ is the simplest, but carries surprising behavior (see B. Delourme's talk, MS 11)
- This general theorem applies to many various problems.
- How to compute the best solution?

Existence

Linear Chebychev in $\mathbb{R}$

$(P_n^k)_k$ minimizing sequence $\|P_n^k - f\|_{L^\infty(K)} \searrow \delta_n \implies (P_n^k)_k$ bounded

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Homographic Chebychev in $\mathbb{C}$

$$(P_n^k)_k \text{ minimizing sequence } \left\| \frac{P_n^k - f}{P_n^k + f} \right\|_{L^\infty(K)} \searrow \delta_n \underset{?}{\implies} (P_n^k)_k \text{ bounded}$$

Existence

Linear Chebychev in $\mathbb{R}$

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Homographic Chebychev in $\mathbb{C}$

$(P_n^k)_k$ minimizing sequence $\left\| \dfrac{P_n^k - f}{P_n^k + f} \right\|_{L^\infty(K)} \searrow \delta_n \underset{?}{\implies} (P_n^k)_k$ bounded

$$\forall \epsilon,\ \forall z,\ n \geq N,\ Z = \frac{P_n^k(z)}{f(z)},\ \left\| \frac{Z-1}{Z+1} \right\| \leq \delta_n + \epsilon = \delta < 1$$

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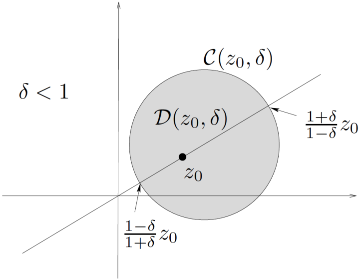

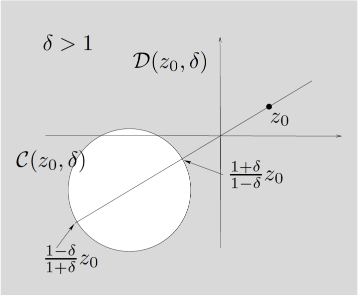

Existence

Linear Chebychev in $\mathbb{R}$

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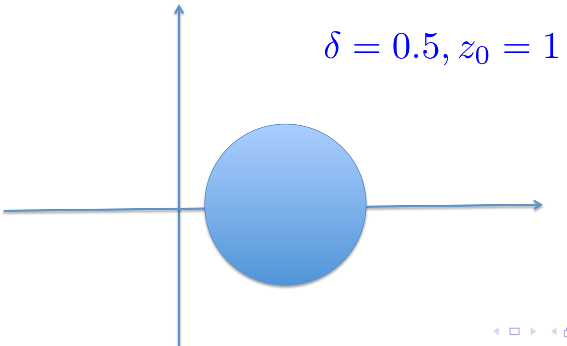
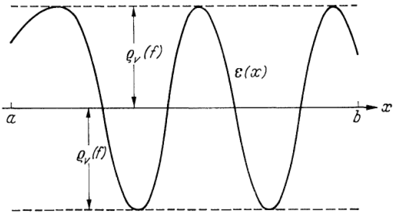


Equioscillation

By contradiction: suppose $z_1, \cdots, z_m$ points with $m \leq n+1$.



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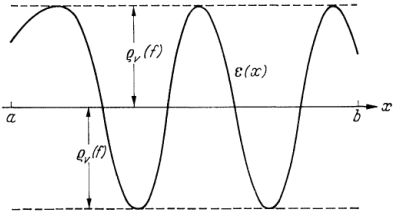

Linear Chebychev in $\mathbb{R}$ $F(z, P) = P(z) - f(z)$

$$F(z_i, P^*(z_i) + t\delta P(z_i)) = F(z_i, P^*(z_i)) + t\delta P(z_i)$$

$$\exists \delta P \in \mathbf{P}_n, \quad \delta P(z_i) = -F(z_i, P^*(z_i))$$

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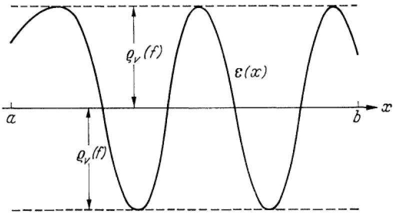

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Homographic Chebychev in $\mathbb{C}$ $F(z,P) = \frac{P(z)-f(z)}{P(z)+f(z)}$

$$F(z_i, P^*(z_i) + t\delta P(z_i)) = F(z_i, P^*(z_i)) + \theta \partial_2 F(z_i, P^*(z_i)) \cdot \delta P(z_i)$$

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Uniqueness: convexity

The set of best approximations is convex.

Linear Chebychev in $\mathbb{R}$

$$R = (1-t)P_n^* + tQ_n^*, \quad \|R-f\|_\infty = \|(1-t)(P_n^*-f)+t(Q_n^*-f)\|_\infty \ \leq \delta_n$$

$$R = \frac{1}{2}(P_n^* + Q_n^*), \ (x_i)_{1\leq i\leq n+2} \text{ alternation points.}$$

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Homographic Chebychev in $\mathbb{C}$

$$\forall z \in K,\ (\frac{P_n^*}{f}(z), \frac{Q_n^*}{f}(z)) \in D(1, \delta_n) \implies t\frac{P_n^*}{f}(z) + (1-t)\frac{Q_n^*}{f}(z) \in D(1, \delta_n)$$

$$\frac{R}{f}(z_i) \in C(1, \delta_n)(\frac{P_n^*}{f}(z_i), \frac{Q_n^*}{f}(z_i)) \in \bar{D}(1, \delta_n) \implies \frac{P_n^*}{f}(z_i) = \frac{Q_n^*}{f}(z_i) = \frac{R}{f}(z_i).$$

Local Minimum

Linear Chebychev in $\mathbb{R}$: De la Vallée Poussin's theorem.

3° La propriété 1° caractérise le polynome P d'approxima-
tion minimum. En effet, soit Q un polynome de degré $\leqq n$; si
l'écart $f - Q$ atteint son maximum absolu ρ' avec alternance
le signes en $n + 2$ points consécutifs de l'intervalle (a, b),
alors $\rho' = \rho$ et, en vertu de 2°, Q est identique à P.

Homographic Chebychev in $\mathbb{C}$

A strict local minimum for $P \mapsto \left\| \dfrac{P - f}{P + f} \right\|_\infty$ is THE global minimum.

Formulas for the coefficients in relation with DD.
Case $n = 0$ or 1

Real case.

▸ ♣ Case $n = 0$

Formulas for $n = 1$

$$F(k_{min}, P) = F(k_{max}, P) = -F(\bar{k}, P), \quad \partial_k F(\bar{k}, P) = 0.$$

Linear Tchebycheff

$$q^* = \frac{f(k_{max}) - f(k_{min})}{k_{max} - k_{min}} = f'(\bar{k});, \quad p^* = \frac{1}{2}\left(f(\bar{k}) + f(k_{min}) - f'(\bar{k})(\bar{k} - k_{min})\right)$$

Homographic Tchebycheff

$$p^* = \frac{k_{max} f(k_{min}) - k_{min} f(k_{max})}{\sqrt{2(k_{max}^2 - k_{min}^2)\left(f(k_{max}) - f(k_{min})\right)}\left(f(k_{max}) f(k_{min})\right)^{\frac{1}{4}}}$$

$$q^* = \frac{\sqrt{f(k_{max}) - f(k_{min})}}{\sqrt{2(k_{max}^2 - k_{min}^2)}\left(f(k_{max}) f(k_{min})\right)^{\frac{1}{4}}}$$

What if I cannot find formulas in closed form ?

$$f(z) = a^2 + 4\nu c + 4\nu i\omega.$$

- $n = 0$, $p \in \mathbb{R}_+^*$, and closed form p^*.

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4. Only asymptotic formulas can be obtained.

More difficult, higher dimension

$$f(z) = a^2 + 4\nu c + 4\nu z, z = i(\omega + b \cdot k) + \nu k^2.$$

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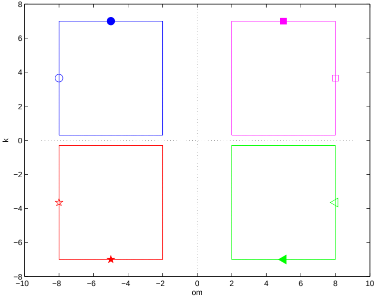
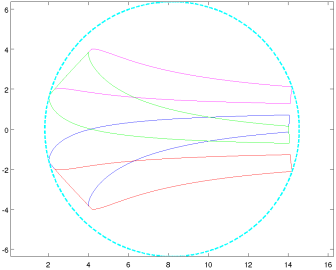


K



$\tilde{K} = z(K)$

It is worth the enormous work. OPTIMISM (Loic Gouarin)

$T = 1$, $\nu = 1, \mathbf{a} = (1,1)$ and $c = 0$.
Implicit scheme, $h = 0.01$ and $\Delta t = \frac{h}{4}$.
Number of iterations to reach residual 10^{-6}.
No overlap

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Comparison Taylor/Optimized

	Iterative				GMRES			
Domains	T0	O0	T2	O2	T0	O0	T2	O2
2x1	8538	156	840	17	41	43	71	13
4x1	9021	154	871	25	127	51	81	19

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Comparison Robin/Ventcel

		Iterative					GMRES				
h		0.04	0.02	0.01	0.005	0.0025	0.04	0.02	0.01	0.005	0.0025
	2x1	49	71	97	144	198	23	29	36	45	55
Robin	2x2	53	74	101	145	202	30	38	48	59	73
	4x1	52	72	101	140	204	30	40	50	63	78
$1/\sqrt{h}$	4x4	81	116	160	219	303	47	64	84	107	133
	2x1	13	15	18	21	24	10	12	14	16	18
Ventcel	2x2	23	29	39	48	63	16	19	22	25	29
	4x1	18	21	25	29	35	14	17	20	24	27
$1/\sqrt[4]{h}$	4x4	30	37	44	54	65	22	28	34	40	46

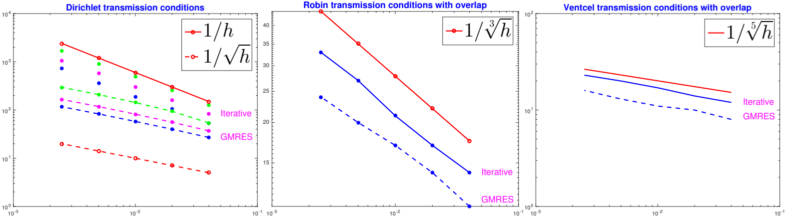
Numerical results, continue

Algorithms with overlap $2h$

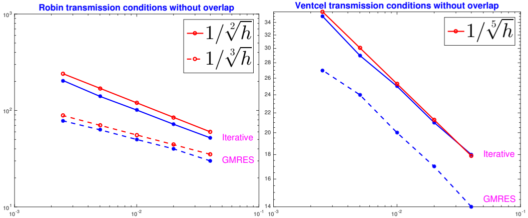
		Iterative					*GMRES*				
h		0.04	0.02	0.01	0.005	0.0025	0.04	0.02	0.01	0.005	0.0025
Dirichlet	2x1	54	106	189	360	733	27	40	58	83	117
	2x2	84	159	303	570	1058	37	56	82	118	166
	4x1	73	145	282	553	969	38	60	89	127	179
$1/h$	4x4	127	258	487	912	1706	54	94	143	209	296
Robin	2x1	12	14	16	19	23	8	10	12	14	17
	2x2	14	17	21	27	33	11	14	17	20	24
	4x1	14	15	18	23	29	11	13	16	20	24
$1/\sqrt[3]{h}$	4x4	19	24	32	41	52	14	20	26	32	40
Ventcel	2x1	9	10	11	12	13	6	7	8	9	10
	2x2	12	14	17	20	23	8	10	11	13	16
	4x1	12	11	11	14	16	10	9	9	11	13
$1/\sqrt[5]{h}$	4x4	16	17	19	24	29	13	13	14	18	22

Asymptotic behavior

Overlapping algorithms

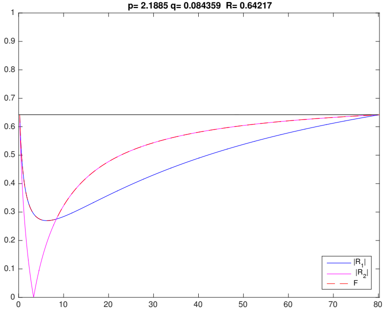


Nonoverlapping algorithms



What else ?

1. system of reaction-diffusion equations can be analysed as well.
2. Complex Helmholtz $\Delta + z^2$: p is not real anymore and equioscillation is not the end of the story. See Bérangère Delourme's talk in MS11.
3. System of equations like discrete duality finite volumes: there might be no uniqueness.



$$\delta_n = \inf_{P \in \mathbf{P}_n} \sup_{z \in K} \max_{1 \leq i \leq I} \left| \frac{P(z) - f_i(z)}{P(z) + f_i(z)} \right|$$

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