Master Vietnam-France in HCMC
High Performance computing

TP 1 : You can't Always Hear the Shape of a Drum. Multigrid algorithm

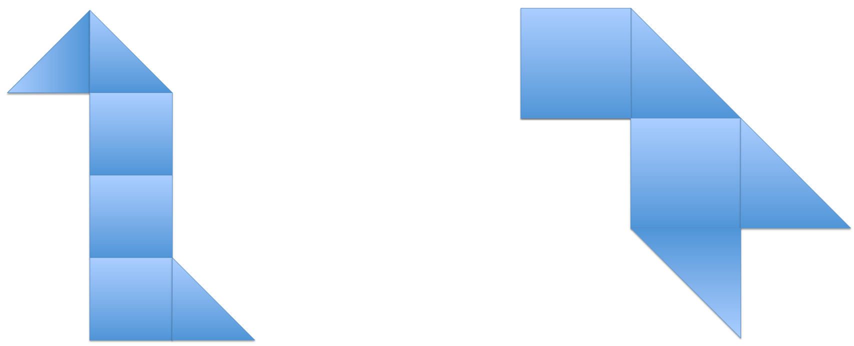

FIGURE 1 – Cocotte (left) and arrow (right)

http://www.ams.org/samplings/feature-column/fcarc-199706

This question has a mathematical counterpart that we are going to investigate. It is based on the problem defined in a domain $D \subset \mathbb{R}^2$,

$$\begin{cases} \partial_{tt} u - \Delta u = 0 & \text{in } D \ \times (0, +\infty) \\ u = 0 & \text{on } \partial D \ \times (0, +\infty) \\ u = u_0 & \text{on } D \ \times (0) \\ \partial_t u = u_1 & \text{on } D \ \times (0) \end{cases} \tag{1}$$

We suppose the initial conditions to be smooth enough ($u_0 \in H_0^1(D) \cap H^2(D)$, $u_1 \in H^1(D)$) so that the theory ensures a unique solution in $C(]0, \infty[; H_0^1(D) \ H^2(D)) \ C^1(]0, \infty[; H_0^1(D)) \ C^2(]0, \infty 1[; L^2(D))$. We suppose the initial conditions to be smooth enough ($u_0 \in H_0^1(D) \cap H^2(D)$, $u_1 \in H^1(D)$) so that the theory ensures a unique solution in $C(]0; 1[; H_0^1(D) \ H^2(D)) \ C^1(]0; 1[; H_0^1(D)) \ C^2(]0; 1[; L^2(D))$.

1 Radial solutions

We suppose that the membrane is exactly the disc of center O and radius 1.

1. 1. Prove that if the initial data are radial, the solution is radial as well.

1. 2. Using the formula for the laplacian in polar coordinates, find the equation $(*)$ that $v(r, t) = u(x, t)$ satisfies on $(0, 1) \times (0, +\infty)$.

2 Solution of the radial equation by finite differences

2. 1. Write an explicit finite differences scheme to solve $(*)$ with initial data $u_0 = 0$ and $u_1(x; y) = -((\sinh\sqrt{x^2+y^2})^2 - 1)^2$. Test the stability and the precision of the scheme.

2. 2. Write an implicit finite difference scheme for the same problem. Same questions. Compare with the explicit scheme

3 Solution of the $2 - D$ problem by finite elements

3. 1. Write the variational formulation and discretize in space-time with a θ-scheme and P_1 finite elements. Use the matlab script delivered. Analyze the stability and the precision with respect to θ, h and dt.

3. 2. Mass lumping. Is there a stage where the system becomes so large that the solution comes too slowly ? In that case one uses either "mass-lumping", or preconditioning. Write a script and compare the two options.

4 Multigrid solution

Each step of the resolution needs the resolution of a big linear system. The tools in the script given to you uses the "backslash" \ of `matlab`.

3. 3. Apply multigrid at each time step.

5 You can't hear the form of the drum

Consider to the two domains on the first page.

3. 4. Design the geometric mesh (each domain is a gathering of 7 unit triangles) Apply the previous study to compute the solution of the wave equation with oscillatory initial data.

6 Further documents

— *One cannot hear the shape of a drum*, Authors : Carolyn Gordon, David L. Webb and Scott Wolpert, Journal : Bull. Amer. Math. Soc. 27 (1992), 134-138. `http://www.ams.org/journals/bull/1992-27-01/S0273-0979-1992-00289-6/S0273-0979-1992-00289-6 pdf`

— More on this problem `https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/MarkKac.pdf`
`https://www.math.ucdavis.edu/~saito/courses/LapEig/lecpdf/lecture15.pdf`