Master Vietnam-France in HCMC
High Performance computing

TP 2 : You can't Always Hear the Shape of a Drum.
Substructuring

FIGURE 1 – Cocotte (left) and arrow (right)

http://www.ams.org/samplings/feature-column/fcarc-199706

This question has a mathematical counterpart that we are going to investigate. It is based on the problem defined in a domain $D \subset \mathbb{R}^2$,

$$\begin{cases} \partial_{tt} u - \Delta u = 0 & \text{in } D \ \times (0, +\infty) \\ u = 0 & \text{on } \partial D \ \times (0, +\infty) \\ u = u_0 & \text{on } D \ \times (0) \\ \partial_t u = u_1 & \text{on } D \ \times (0) \end{cases} \tag{1}$$

We suppose the initial conditions to be smooth enough ($u_0 \in H^1_0(D) \cap H^2(D)$, $u_1 \in H^1(D)$) so that the theory ensures a unique solution in $C(]0, \infty[; H^1_0(D)\ H^2(D))\ C^1(]0, \infty[; H^1_0(D))\ C^2(]0, \infty 1[; L^2(D))$.

1 Radial solutions

We suppose that the membrane is exactly the disc D_1 of center O and radius 1.

1. 1. Prove that if the initial data are radial, the solution is radial as well.

1. 2. Using the formula for the laplacian in polar coordinates, find the equation $(*)$ that $v(r,t) = u(x,t)$ satisfies on $(0,1) \times (0,+\infty)$.

2 Radial oscillatory solution of the radial equation by finite differences

2. 1. By separation of variables $v(r,t) = \phi(x)\psi(t)$, show that the solution oscillates in time, and that ϕ is solution of an eigenvalue problem denoted by $(**)$.

2. 2. Write a finite differences scheme to solve $(**)$ and deduce an approximate value of the five first frequencies. Compute and draw the eigenmodes.

3 Oscillatory Solutions of the $2-D$ problem by finite elements

3. 1. By separation of variables in (1), : $u(x;t) = \Phi(x)\Psi(t)$, show that the solution oscillates in time, and that Φ is solution of an eigenvalue problem denoted by $(***)$.

3. 2. Write the variational formulation and obtain a matrix eigenvalue problem when using P_1 finite elements.

3. 3. Use the matlab script delivered to solve the eigenvalue problem in case of the disk D_1.

3. 4. Compute the first five modes, and compare with the radial modes.

4 You can't hear the difference

Consider to the two domains on the previous page.

4. 1. Design the geometric mesh (each domain is a gathering of 7 unit triangles)

4. 2. Compute the first 10 modes for these domains and compare the results.

4. 3. Apply a substructuring process to compute the eigenmodes. Write the eigenmode problem, and solve it by finite elements.

5 Further documents

— A `matlab` library for finite elements graciously provided by Martin Gander (Genève University) and Felix Kwok (Hong-Kong Baptiste University).

— *One cannot hear the shape of a drum*, Authors : Carolyn Gordon, David L. Webb and Scott Wolpert, Journal : Bull. Amer. Math. Soc. 27 (1992), 134-138. `http://www.ams.org/journals/bull/1992-27-01/S0273-0979-1992-00289-6/S0273-0979-1992-00289-6 pdf`