Master Vietnam-France in HCMC
High Performance computing

TP 4 : The best way to solve to compute acoustics : domain decomposition

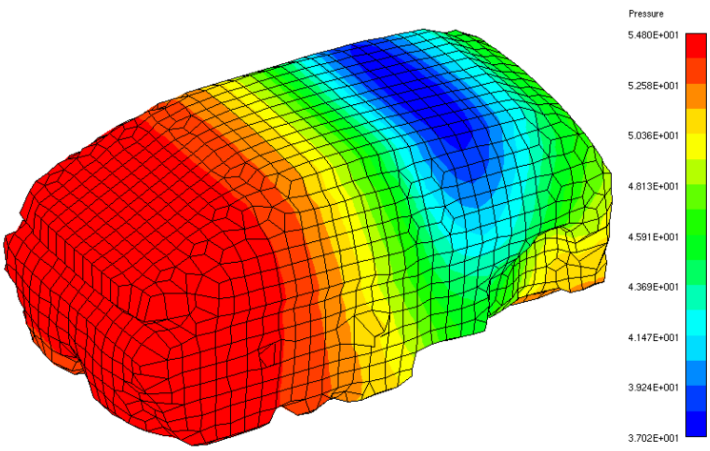

FIGURE 1 – Acoustic computation in a twingo

As seen in TP3, the computation of acoustics in a structure relies on the resolution of the Helmholtz equation

$$\Delta u + k^2 u = f \text{ in } D \tag{1}$$

1 Dimension 1

Consider the interval $D = (-1, 1)$.

1. 1. Show that the problem , with boundary condition $u = 0$ at $x = \pm 1$. as a unique solution $u \in \mathcal{C}^2(D)$ for $f \in \mathcal{C}(D)$, unless k takes the value $n\pi/2$, for $n \in \mathbb{Z}$.

1. 2. Show that the problem , with boundary condition $u' = iku$ at $x = -1$ and $u' = -iku$ at $x = 1$. as a unique solution $u \in \boldsymbol{\mathcal{C}}^2(D)$ for $f \in \mathcal{C}(D)$, for any $k \in \mathbb{R}$. (Attention, f is a real function, but the solution u takes complex values) .

1. 3. Build a $\mathbb{P}_1$ finite element approximation of the two problems, computing $u_n \in V_N = \mathbb{P}_1(x_0, \ldots, x_{n+1})$. Theoretical results say that if $hk^2 \leq C$, the finite approximation is quasi-optimal, that is

$$\|u - u_n\|_1 \leq C' \min_{v \in V_n} \|u - v\|_1$$

Estimate the right hand side (to be found in so many textbooks) and check numerically the result, using $f = 1$ on $(-0.5, 0.5)$, 0 elsewhere.

2 Domain decomposition in one dimension

An approximation of the solution is given by dividing the interval into two subdomains $D_1 = (-1, L)$ and $(D_2 = (0, 1)$ with $L > 0$.. A sequence of problems is introduced, u_j^n is the solution at step n in D_j,

$$\begin{array}{c|c} d_{xx}u_1^n + k^2 u_1^n = f \text{ in } D_1 & d_{xx}u_2^n + k^2 u_2^n = f \text{ in } D_2 \\ u_1^n(-1) = 0 \quad u_1^n(L) = u_2^{n-1}(L) & u_2^n(0) = u_1^{n-1}(0) \qquad u_2^n(1) = 0 \end{array}$$

With data at step $n = 0$, $u_2^0(L) \equiv g_1$ and $u_1^0(0) \equiv g_2$.

2. 1. Show that the error $u_j^n - u$ is solution of the same algorithm with data $f \equiv 0$, $u_2^0(L) \equiv g_1 - u(L)$ and $u_1^0(0) \equiv g_2 - u(0)$. Show that $u_1^n = a_n \sin k(x+1)$ and $u_2^n = b_n \sin k(1-x)$ and find the recursion relation between the a_n and the b_n.

2. 2. Analyze the convergence factor and show that for any k it is convergent except for an numerable set of values of L. What happens when k is large ? Draw the curve of convergence of the a_n for various relevant values of k.

2. 3. Compute the sequence by finite elements, and compare the curve of convergence of $\|u - u_j^n\|_{L_2(D_j)}$ to the curve of the a_n for various relevant values of k, h and h.

3 Improvement for domain decomposition in one dimension

We replace the transmission conditions above by

$$(d_x + ik)u_1^n(L) = (d_x + ik)u_2^{n-1}(L), \quad (d_x - ik)u_2^n(0) = (d_x - ik)u_1^{n-1}(0).$$

With data at step $n = 0$, $u_2^0(L) \equiv g_1$ and $u_1^0(0) \equiv g_2$.

3. 1. Show that the error $u_j^n - u$ is solution of the same algorithm with data $f \equiv 0$, $u_2^0(L) \equiv g_1 - u(L)$ and $u_1^0(0) \equiv g_2 - u(0)$. Show that $u_1^n = a_n \sin k(x+1)$ and $u_2^n = b_n \sin k(1-x)$ and find the recursion relation between the a_n and the b_n.

3. 2. Analyze the convergence factor and show that for any k it is convergent except for an numerable set of values of L. What happens when k is large ? Draw the curve of convergence of the a_n for various relevant values of k.

3. 3. Compute the sequence by finite elements, and compare the curve of convergence of $\|u - u_j^n\|_{L_2(D_j)}$ to the curve of the a_n for various relevant values of k, h and h.

4 Further documents

— *When all else fails, integrate by parts, an overview of new and old variational formulations for linear elliptic PDEs.* E.A. Spence. http ://people.bath.ac.uk/eas25/ibps.pdf

— *Une méthode de décomposition de domaine pour le problème de Helmholtz*, Bruno Després, In french.

— *Why it is Difficult to Solve Helmholtz Problems with Classical Iterative Methods*, Oliver G. Ernst and Martin J. Gander. `https://www.unige.ch/~gander/Preprints/HelmholtzReview.pdf`