Master Vietnam-France in HCMC
High Performance computing

TP 7 : Understanding and computing corner singularities for a vibrating plate

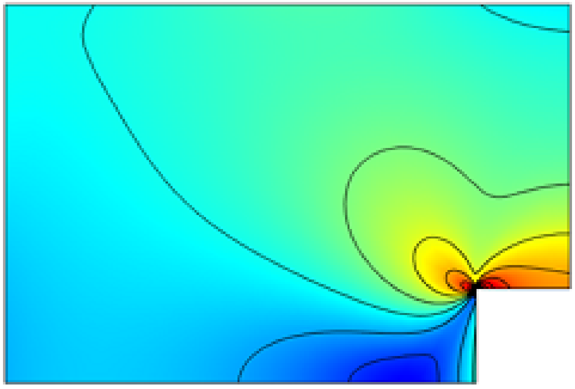

FIGURE 1 – DIfffraction by a corner

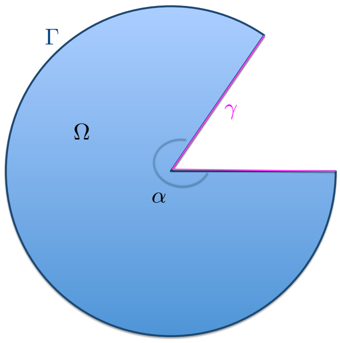

FIGURE 2 – Computational object

This subject aims at exploring the finite element method for elliptic problems, when the geometry is singular and not convex. In that case, the solution is not in $H^2(D)$, and therefore optimal error estimates are not available, for regular finite element methods.

Consider a metallic uniform plane plate, represented by the disk of center O and radius 1, cropped with an angle $\alpha > \pi$, see Figure . The object is fixed on the internal boundary γ, and free to vibrate on the remaining boundary Γ. The vertical vibrations are described by the wave equation

$$\begin{array}{rcl} \partial_t^2 u - \Delta u &=& f \quad \text{in } \Omega \ \times \mathbb{R}_+ \\ u &=& 0 \quad \text{on } \gamma \times \mathbb{R}_+ \\ \partial_n u &=& 0 \quad \text{on } \Gamma \times \mathbb{R}_+ \end{array} \qquad (1)$$

and initial conditions u and $\partial_t u$ at initial time.

1 The steady problem

We first consider the steady problem , that is solving

$$\begin{array}{rcl} -\Delta u &=& f \quad \text{in } \Omega \\ u &=& 0 \quad \text{on } \gamma \\ \partial_n u &=& 0 \quad \text{on } \Gamma \end{array} \qquad (2)$$

1. 1. Let ϕ decreasing and $\mathcal{C}^2$ on $(0, 1)$, such that $\phi(1) = 0$, $\phi'(0) = \phi(1) = \phi'(1) = 0$ (give an example). Define in polar coordinates

$$u_0(r, \theta) = \phi(r)\, r^{\frac{\pi}{\alpha}} \sin(\frac{\pi}{\alpha}\theta)$$

Show that $u_0 \in H^1(\Omega)$, $u_0 = 0$ on γ and $\partial_n u = 0$ on Γ (remember that $\partial_n u = \nabla u \cdot n$, and use the form of gradient and laplacian in polar coordinates

$$\nabla u = \partial_r u e_r + \frac{1}{r}\partial_\theta u e_\theta, \quad \Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u$$

Compute Δu_0 and verify that it belongs to $L^2(\Omega)$. Therefore it is solution of (2).

1. 2. Write the variational formulation of (2) in the space $H^1_\gamma = \{ v \in H^1(\Omega), v = 0 \text{ on } \gamma\}$.

1. 3. Compute the solution with $\mathbb{P}_1$ finite elements, using the script PDE provided, using $f = \Delta u_0$ as a source. Choose the function ϕ you want.

1. 4. Analysis : What is the numerical convergence order of the approximation : if p is the order, that means that $\|u - u_h\|_{L^2} \sim C(u)h^p$. If you draw the error for three values of h on the same plot in loglog scale, you should recover a straight line, whose slope is p. It can be calculated by

$$p \sim \log_2 \frac{\|u - u_h\|}{\|u - u_{h/2}\|}$$

How the numerical order does it depend on $\alpha \in (0, 2\pi)$. ?

1. 5. For a regular domain, a theoretical result says that $p = 2$. It relies on the following result : if $\Delta u \in L^2(\Omega)$, then $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Does u_0 belong to $H^2(\Omega)$?

2 Singularity coefficient

From now on $\alpha \in]\pi, 2\pi[$. We give below some theory in functional analysis. Let L be the operator

$$\begin{array}{rccc} L : & L^2(\Omega) & \to & H^1_\gamma, \\ & f & \mapsto & u \text{ solution of (2)} \end{array}$$

Now define $V = Range(L)$, then we have

$$H^2(\Omega) \cap V = \{ v \in H^2(\Omega) \cap H^1_\gamma(\Omega),\ \partial_n v = 0 \text{ on } \Gamma\}$$

2. 1. What do you think the convergence rate should be if $u \in H^2(\Omega) \cap V$? Check this numerically.

2. 2. Note $H = L^{-1}(H^2(\Omega) \cap V)$, and D its orthogonal in $L^2(\Omega)$

$$H = \{ f \in L^2(\Omega),\ \exists v \in H^2(\Omega) \cap V,\ -\Delta v = f\}, \quad D = \{ f \in L^2(\Omega),\ \forall v \in H^2(\Omega) \cap V,\ \int_\Omega f \Delta v = 0\}.$$

Define the function p in polar coordinates by

$$p(r, \theta) = c_\alpha (r^{\frac{\pi}{\alpha}} + r^{-\frac{\pi}{\alpha}}) \sin \frac{\pi}{\alpha}\theta, \quad c_\alpha = \sqrt{\frac{2}{\alpha} \frac{1 - (\pi/\alpha)^2}{2 - (\pi/\alpha)^2}}$$

Show that $p \in D$ and $\|p\|_{L^2} = 1$. We admit that D is of dimension 1 and H is closed in $L^2(\Omega)$. For $f \in L^2(\Omega)$, we note $c(f)$ and call singular coefficient the quantity $c(f) = (p, f)_{L^2(\Omega)}$.

2. 3. For any $f \in L^2(\Omega)$, define $\tilde{f} = f - c(f)p$. Show that $\tilde{f} \in H$.

2. 4. Write a `matlab` script to compute an approximation of $c(f)$. Verify that the order of approximation is at least 1 (test on $f = p$).

3 Approximation of the singular part

3. 1. Show that $-\Delta u_0 \notin H$. Deduce that for any f in $L^2(\Omega)$, there exists a real number $\beta(f)$ such that $L(f) - \beta(f)u_0 \in H^2(\Omega) \cap V$.

3. 2. (Optional) Note $\beta_0 = \beta(p)$. Then $\beta_0 = 1/(\pi c_\alpha)$.

3. 3. Show that that for any f in $L^2(\Omega)$, $f + c(f)\beta_0 \Delta u_0 \in H$. What do you think of the finite element approximation of $L(f + c(f)\beta_0 \Delta u_0 \in H)$? ‘

3. 4. Propose and realize an approximation $\hat{u}_h$ of $L(f)$ which is of order two, that is

$$\|u - \hat{u}_h\|_{L^2(\Omega)} \leq Ch^2 \|f\|_{L^2(\Omega)}$$

Verify numerically the order of convergence for $f \equiv 1$.

4 Eigenvalue problem

We admit that there exist a complete set of eigenfunctions of L, that is a sequence $(\phi_n) \subset H^1(\Omega)$ orthonormal in $L^2(\Omega)$, and a sequence $(\lambda_n) \subset \mathbb{R}_+$ with

$$L\phi_n = \phi_n, \quad -\Delta\phi_n = \lambda_n \phi_n, \quad \phi_n = 0 \text{ on } \gamma, \quad \partial_n \phi_n = 0 \text{ on } \Gamma.$$

We want to compute the fundamental mode of the structure, that is the largest eigenvalue of L $1/\lambda_1$, together with the fundamental mode ϕ_1. We will use the method of power or Von Mises iteration which computes the largest eigenvalue of an operator when it is simple. The initialization is with $v_0 \in L^2(\Omega)$. Then solve

$$\tilde{v}_{k+1} = L(v_k) \text{ or equivalently} - \Delta\tilde{v}_{k+1} = v_k, \quad \tilde{v}_{k+1} = 0 \text{ on } \gamma, \quad \partial_n \tilde{v}_{k+1} = 0 \text{ on } \Gamma.$$

and normalize by $\mu_{k+1} = \|\tilde{v}_{k+1}\|$, $v_{k+1} = \tilde{v}_{k+1}/\mu_{k+1}\|$. It can be shown that if the eigenvalue λ_1 is simple (which is the case) and v_0 has a non-void component on ϕ_1, then the sequence v_k converges to $\pm\phi_1$ and the sequence μ_k converges to λ_1.

4. 4. Write a solver for the method of power iteration, for the problem above with regular solution, that is $\alpha \in (0, \pi)$.

4. 5. Extend the solver to the problem above with singular solution, that is $\alpha \in (\pi, 2\pi)$.