Problem n°1 : matrices

Exercice 0 : From the lecture notes.

Study the exercise on p 16 on tridiagonal matrices

Exercice 1 : Special Matrices.

a) Show that the product of two lower (resp. upper) triangular matrices is a lower (resp. upper) triangular matrix.
b) Shwo that the inverse of the lower triangular (invertible) matrix $\mathbb{L}$, triangulaire is lower triangular. Furthermore $\left(\mathbb{L}^{-1}\right)_{ii} = \frac{1}{\mathbb{L}_{ii}}$.
c) Show that the product of two banded matrices is a banded matrix, and evaluate it bandwidth in terms of the bandwidth of the two matrices.

Exercice 2 : Block Matrices.

a) Show that the product of two block lower (resp. upper) triangular matrices is a block lower (resp. upper) triangular matrix. **b)** We want to calculate the determinant of the matrix $\mathbb{A} = \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix}$ split in blocks. The blocks $\mathbb{A}_{11}$ and $\mathbb{A}_{22}$ are square.

i) Calculate the determinant of matrices

$$\mathbb{A}_1 = \begin{pmatrix} \mathbb{A}_{11} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad \mathbb{A}_2 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{A}_{22} \end{pmatrix}$$

Deduce the determinant of

$$\mathbb{A}_3 = \begin{pmatrix} \mathbb{A}_{11} & 0 \\ 0 & \mathbb{A}_{22} \end{pmatrix}$$

ii) Calculate the determinant of

$$\mathbb{A}_4 = \begin{pmatrix} \mathbb{I} & \mathbb{A}_{12} \\ 0 & \mathbb{I} \end{pmatrix}$$

and the product of the two block matrices

$$\begin{pmatrix} \mathbb{A}_{11} & 0 \\ 0 & \mathbb{A}_{22} \end{pmatrix} \text{ et } \begin{pmatrix} \mathbb{I} & \mathbb{A}_{11}^{-1}\mathbb{A}_{12} \\ 0 & \mathbb{I} \end{pmatrix}.$$

Deduce the determinant of

$$\mathbb{A}_5 = \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ 0 & \mathbb{A}_{22} \end{pmatrix}$$

iii) Calculate the product of block matrices

$$\begin{pmatrix} \mathbb{I} & 0 \\ \mathbb{A}_{21}\mathbb{A}_{11}^{-1} & \mathbb{I} \end{pmatrix} \text{ et } \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ 0 & \mathbb{A}_{22} - \mathbb{A}_{21}\mathbb{A}_{11}^{-1}\mathbb{A}_{12} \end{pmatrix}.$$

Deduce the determinant of $\mathbb{A}$.

c) Calculate the determinant of the block triangular matrix

$$\begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} & & \mathbb{A}_{1n} \\ 0 & \mathbb{A}_{22} & & \mathbb{A}_{2n} \\ 0 & 0 & \ddots & \\ 0 & 0 & & \mathbb{A}_{nn} \end{pmatrix}$$

Exercice 3 : Irréducible Matrices .

A is a square matrix of size n, denoted $A = (a_{ij})_{1\le i,j\le n}$. We say that A is reducible if there exists a permutation matrix P such that

$$^tPAP = B = \begin{bmatrix} B^{(11)} & B^{(12)} \\ 0 & B^{(22)} \end{bmatrix}$$

where $B^{(11)}$ and $B^{(22)}$ are square matrices of size p et $n-p$ respectively. Recall that a permutation matrix is defined by $P_{ij} = \delta_{i\sigma(j)}$ where σ is a permutation of the set $\{1, .., n\}$.

a) Show that A est reducible if and only if there exists a partition of $\{1, .., n\}$ in two (disjoint) sets I and J such that $a_{ij} = 0$ for i in I and j in J.

We define the *graph* associated to A as the set of points X_i, for $1 \le i \le n$. The points X_i and X_j are linked by an *arch* if $a_{ij} \ne 0$. A *path* is a sequence of archs. We say that the arch is strongly *connected* if 2 points can always be related (in order) by a path.

b) Show that a matrix is irreducible if and only if its graph is strongly connected.

Exercice 4 : Diagonally Dominant Matrices.

a) Show the Gerschgörin-Hadamard theorem : any eigenvalue λ of A belongs to the union of discs D_k defined by

$$|z - a_{kk}| \le \Lambda_k = \sum_{\substack{1\le j\le n \\ j\ne k}} |a_{kj}|$$

b) Show that if A est irreducible, and if an eigenvalue λ is on the boundary of the union of discs D_k, then all the circles pass through λ.

We say that A est *diagonally dominant* if

$$\forall i, 1 \le i \le n, |a_{ii}| \ge \Lambda_i$$

We say that A is *strictly diagonally dominant* if

$$\forall i, 1 \leq i \leq n, |a_{ii}| > \Lambda_i$$

We say that A est à *strongly diagonally dominant* if it is diagonally dominant and if furthermore

$$\exists i, 1 \leq i \leq n, |a_{ii}| > \Lambda_i$$

c) Prove that if A is strictly diagonally dominant, it is invertible.

d) Prove that if A is strongly diagonally dominant and irreducible, it is invertible.

e) Prove that if A is, either strictly diagonally dominant,or strongly diagonally dominant and irreducible, and if the diagonal entries are strictly positive, then the real part of the eigenvalues is strictly positive.

Exercice 5 : Discretisation of laplacian in dimension 1.

Consider the boundary value problem on $]a, b[$

$$\begin{cases} -u'' = f \text{ sur }]a, b[, \\ u(a) = 0, \\ u(b) = 0. \end{cases} \tag{1}$$

where f is a continuous function on $]a, b[$.

This problem has a unique solution we want to compute by finite differences. **a)** Show that if u is $\mathcal{C}^2$,

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2) \tag{2}$$

We split the segment into n intervals of length $h = (b-a)/n$.

b) Write by using (2) the linear system issued from (1) whose unknowns u_i are approximations of $u(a + ih)$ for $1 \leq i \leq n-1$. Note A the matrix of the system.

c) Show by exercice II that A is symmetric definite positive.

d) Show the maximum principle : If all f_i are ≤ 0, then the u_i are ≤ 0 and the maximum is reached for $i = 1$ ou $n-1$.

e) Let a and b two real numbers. For $n \geq 0$, note Δ_n the tridiagonal le determinant

$$\Delta_n = \begin{vmatrix} a & -b & 0 & & & \\ -b & a & & & & \\ \ddots & & \ddots & & & \\ & & -b & a & -b \\ & & 0 & -b & a \end{vmatrix}$$

Write a two levels recursion relation on the Δ_n.

f) Note $P_n(\lambda)$ the characteristic polynomial of A. Using the change of variable

$$\lambda + 2 = -2\cos\theta,$$

prove that $P_n(\lambda) = \frac{\sin(n+1)\theta}{\sin\theta}$. Deduce that the eigenvalues of A are $\lambda_k = \frac{4}{h^2} sin^2(\frac{k\pi}{2n})$ and the associated eigenvectors $u^{(k)}$ given by $u_j^{(k)} = sin(\frac{k\pi j}{n})$.

f) Deduce the condition number of A.

Exercice 6 : Discretisation of laplacien in dimension 2.

Consider the boundary value problem on $]0,1[\times]0,1[$

$$\begin{cases} -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f \text{ sur }]0,1[\times]0,1[, \\ u = 0 \text{ on the boundary} \end{cases} \tag{3}$$

Divide the interval $[0,1]$ horizontally in $M+1$ intervals $[x_i, x_{i+1}]$, $x_i = a + ih$, $0 \le i \le M+1$, with $h = 1/(M+1)$. Divide the interval $[0,1]$ vertically en $M+1$ intervalles $[y_j, y_{j+1}]$, $y_j = c + jh$, $0 \le j \le M+1$. We then obtain a meshing in x, y. A point in the mesh is (x_i, y_j). An approximation of $u(x_i, y_j)$ is noted $u_{i,j}$.

The Poisson equation (3) is then discretized by $(f_{i,j} = f(x_i, y_j))$

$$-(\Delta_h u)_{i,j} = -\frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - \frac{1}{h^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = f_{i,j}, \quad 1 \le i \le M, 1 \le j \le M \tag{4}$$

The nodes of the mesh must be stored as a vector. They can be numbered by increasing j and for any j by increasing i (see Figure 1).

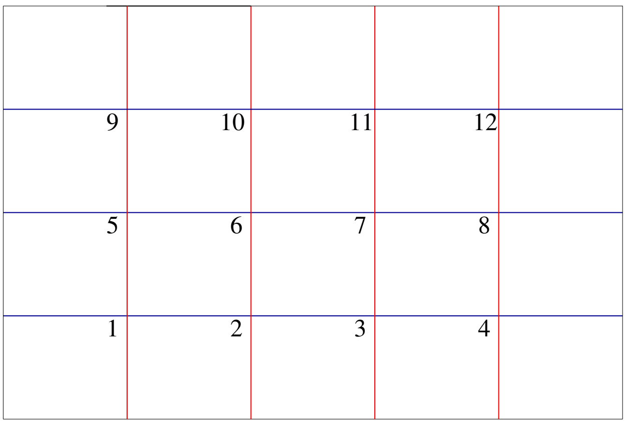

Figure 1 – numérotation by rows

Suppose only internal degrees of freedom are stored. Then the point (x_i, y_j) is numbered $i + (j-1)M$. We create a vector Z of all unknowns

$$Z = (u_{1,1}, u_{2,1}, u_{M,1}), (u_{1,2}, u_{2,2}, u_{M,2}), \cdots (u_{1,M}, u_{2,M}, u_{M,M})$$

with $Z_{i+(j-1)*M} = u_{i,j}$.

a) If the equations are numbered accordingly (the k-th equation is equation at point k, show that the matrix is block tridiagonal :

$$A = \frac{1}{h^2}\begin{pmatrix} B & -C & & 0_M & \\ -C & B & -C & & \\ & \ddots & \ddots & \ddots & \\ & & -C & B & -C \\ & 0_M & & -C & B \end{pmatrix} \tag{5}$$

with $C = I_M$, and B is the tridiagonal matrix

$$B = \begin{pmatrix} 4 & -1 & & 0 & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & 0 & & -1 & 4 \end{pmatrix}$$

The righthand side is then $b_{i+(j-1)*M} = f_{i,j}$, and the system is $AZ = b$.

b) Show that matrix A est invertible with strictly positive eigenvalues.

c) Show that the eigenvalues of A are $\lambda_{pq} = \frac{4}{h^2}(sin^2(\frac{p\pi}{2M}) + sin^2(\frac{q\pi}{2M}))$. Deduce the condition number of A.

d) Same study with the alternating numbering by columns.

Hints on eigenvalues and eigenvectors Consider the $1-D$ Laplace operator on $[0,1]$ with zero Dirichlet boundary values. The eigenvalues and (normalized) eigenmodes are defined by

$$-u'' = \lambda u$$

and given by

$$\lambda_k = k^2\pi^2, \quad u_k = \sqrt{2}\sin k\pi x, \quad k \in \mathbb{Z}$$

For the discretization matrix of size n,

$$A = \begin{pmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ \ddots & & \ddots & & \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 2 \end{pmatrix}$$

The eigenvalues are (cf exercice 5) $\frac{4}{h^2}sin^2(\frac{k\pi}{2n})$ and the associated eigenvectors $u^{(k)}$ given by $u_j^{(k)} = u_k(x_j) = \sqrt{2}sin(\frac{k\pi j}{n})$. They forme an orthonormal basis, therefore the matrix Q whos columns are the $u^{(k)}$ os orthogonale.

Consider now the $2-D$ Laplace operator on $[0,1]\times[0,1]$ with zero Dirichlet boundary values. The eigenvalues and (normalized) eigenmodes are defined by

$$-u_{xx} - u_{yy} = \lambda u$$

We search for eigenfunctions in separate variables : $u(x,y) = v(x)w(y)$.

$$-v''(x)w(y) - v(x)w''(y) = \lambda v(x)w(y)$$

$$-v(x)w''(y) = (\lambda v(x) + v''(x))w(y).$$

$$\frac{-w''(y)}{w(y)} = \frac{\lambda v(x) + v''(x)}{v(x)}$$

On the left we have a function of y, on the right a function of x, they must are constant : there is a a such that

$$\frac{-w''(y)}{w(y)} = \frac{\lambda v(x) + v''(x)}{v(x)} = a$$

which gives

$$w''(y) + aw(y) = 0, \quad w(0) = w(1) = 0.$$

This is a 1-D eigenequation, whose solutions are (modulo a multiplicative factor) with $a = p^2\pi^2$ for $p \in \mathbb{N}$

$$w_p = \sqrt{2}\sin p\pi y.$$

then for each p,

$$\lambda v(x) + v''(x) = p^2\pi^2 v(x)$$

$$\lambda v(x) + v''(x) = p^2\pi^2 v(x)$$

$$v''(x) + (\lambda - p^2\pi^2)v(x) = 0.$$

again this is a 1-D eigenmode problem, then

$$\lambda - p^2\pi^2 = q^2\pi^2, \quad v_q(x) = \sqrt{2}\sin q\pi x.$$

An orthonormal system is then

$$\lambda_{p,q} = (p^2 + q^2)\pi^2, \quad u_{p,q} = 2(\sin q\pi x)(\sin p\pi y)$$

For the $2 - D$ discrete system, we will do the same as for the continuous case : suppose that the eigenvalues are

$$\lambda_{p,q} = \frac{4}{h^2}(\sin^2(p\pi h) + \sin^2(q\pi h)), \quad u_{i,j}^{p,q} = 2(\sin q\pi x_i)(\sin p\pi y_j)$$

and check it directly on equation (4).