Master Vietnam-France in HCMC
High Performance computing

TP 3 : Why is it difficult to resolve the waves. Preconditioning

FIGURE 1 – Waves in a pool

Diving in a pool creates surface waves which can be modelized by the wave equation

$$\begin{cases} \partial_{tt}u - \Delta u = 0 & \text{in } D \ \times (0,+\infty) \\ u = 0 & \text{on } \partial D \ \times (0,+\infty) \\ u = u_0 & \text{on } D \ \times (0) \\ \partial_t u = u_1 & \text{on } D \ \times (0) \end{cases} \tag{1}$$

We suppose the initial conditions to be smooth enough ($u_0 \in H^1_0(D) \cap H^2(D)$, $u_1 \in H^1(D)$) so that the theory ensures a unique solution in $C(]0,\infty[; H^1_0(D)\ \ H^2(D))\ \ C^1(]0,\infty[; H^1_0(D))\ \ C^2(]0,\infty 1[; L^2(D))$. We suppose the initial conditions to be smooth enough ($u_0 \in H^1_0(D) \cap H^2(D)$, $u_1 \in H^1(D)$) so that the theory ensures a unique solution in $C(]0;1[; H^1_0(D)\ \ H^2(D))\ \ C^1(]0;1[; H^1_0(D))\ \ C^2(]0;1[; L^2(D))$.

1 Separation of variables

We suppose that the domaine is exactly the square of lenght 1. The pool is at rest at initial time, $u_1 = 0$. We search a solution in the form $u(x,t) = v(x)\psi(t)$.

1. 1. Show that u and ϕ are solutions of two problems, and give explicitely th e equations, the initial equations and the boundary data.

1. 2. Prove that v is solution of the **Helmholtz** equation in D, *i.e.* $\Delta v + \omega^2 v = 0$ in D with Dirichlet data on the boundary. Deduce the behavior of ψ.

1. 3. ω^2 is therefore an eigenvalues of $-\Delta$ in D. By separation of variables again, compute the eigenvalues and eigenfunctions.

2 Solution of the Helmholtz equation

We suppose that a disturbance is injected in the pool, with a frequency k which is NOT an eigenvalue of $-\Delta$ in D. We therefore have to solve

$$\begin{cases} \Delta u + k^2 u = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \tag{2}$$

2. 1. Write the variational formulation in $H^1(D)$. Does the Lax-Milgram lemma applies ? Why ? Prove that the bilinear form satisfy a Garding inequality, which ensures well-posedness for this problem (see ref 4).

3 Numerical solution of the Helmholtz equation

3. 1. Discretize in space with P_1 finite elements the Dirichlet problem. The matrix is symmetric but not definite positive !

3. 2. In the FEM script, replace the `matlab` backslash \ by a Jacobi, Gauss-Seidel algorithm, or Krylov algorithm (choose the last one carefully, the matrix is not definite). Run the algorithm with a source close to a Dirac on the north-east corner. Study the performance of the algorithm with respect to the mesh size(and hence the size of the matrix) and the frequency k. What happens when j is close to an eigenvalue ?

3. 3. Propose a preconditioner seen in the lecture for the problem, and study again the performance. Compare with the Laplace equation.

3. 4. Apply a multigrid solver with relaxed Jacobi smoother to the problem. What happens ?

4 Further documents

— *When all else fails, integrate by parts, an overview of new and old variational formulations for linear elliptic PDEs*. E.A. Spence. http ://people.bath.ac.uk/eas25/ibps.pdf

— *Why it is Difficult to Solve Helmholtz Problems with Classical Iterative Methods*, Oliver G. Ernst and Martin J. Gander. `https://www.unige.ch/~gander/Preprints/HelmholtzReview.pdf`