Master Vietnam-France in HCMC
High Performance computing

TP 5 : Effective coefficient of a composite material. Preconditioning

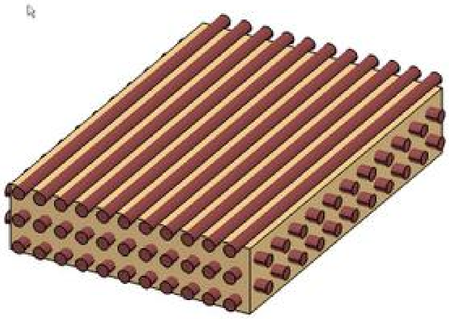

FIGURE 1 – Composite material

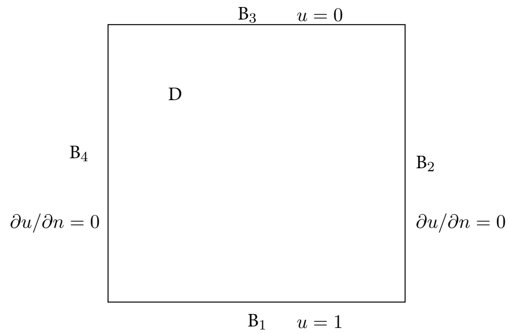

FIGURE 2 – Computational domain

Composite material are everywhere nowadays, in the building, aircrapfts, chairs, etc.. It is important to understand their mechanical and thermical properties.

Start with a heat conducting material, non homogeneous, occupying the space delimite by a square. $a(x)$ denotes the conductivity at point $\boldsymbol{x}$, and the heat flux is given by the Ficke law, $J = -a(\boldsymbol{x})\nabla u$, where u is the temperature. We are interested in computing the apparent conductivity of the material, defined as follows :

The boundaries Γ_1 and Γ_3 are heated to 0 and 1 respectively, and the heat flux across the lateral boundaries Γ_2 and Γ_4 are supposed to be zero. The system is

$$\begin{array}{rcll} -\nabla(a(x)\nabla u) &=& 0 & \text{in } D \\ \partial_n u &=& 0 & \text{on } B_2 \cup B_4 \\ u &=& 1 & \text{on } B_1 \\ u &=& 0 & \text{on } B_3 \end{array} \quad (1)$$

We are searching for a solution $u \in H^1(D)$. The apparent conductivity is the ratio between the mean flux and the temperature gradient, that is

$$a_{app} = -\int_{B_3} a\partial_y u(x,y)dx$$

We will suppose that the square is $(0,3) \times (0,3)$, that the conductivity depends on x only and

$$a(x) = 1 \text{ on } (0,1), \quad a(x) = \epsilon \text{ on } (1,2), \quad a(x) = 1 \text{ on } (2,3)$$

The aim of this work is to compute the apparent conductivity.

1 Closed form computations

1. 1. By separation of variables, if $u(x, y) = v(x)w(y)$, give the equations for v and w, together with their boundary conditions. Each of these functions is a solution of an eigenvalue problem for a second order operator in one dimension.

1. 2. Show that the equation in x amounts to solving with $k \in \mathbb{R}$,

$$(a(x)v')' + k^2 av = 0 \text{ in } (0, 3), \qquad \text{with } v'(0) = v'(3) = 0.$$

show that for $j = 0, \ldots 2$, v_j in $D_j = (j, j + 1)$ is solution of $u'' + k^2 u = 0$, and that the transmission conditions are enforced

$$\begin{aligned} v_1(1) = v_2(1), \quad v_2(2) = v_3(2), \\ v_1'(1) = \epsilon v_2'(1), \quad \epsilon v_2'(2) = v_3'(2). \end{aligned}$$

From this compute explicitely the v_j for fixed k up to a constant, and then use the boundary conditions to obtain an equation giving k. Determine k and then w.

1. 3. Therefore we can expand u as $u = \sum_k v^k(x)w^k(y)$. Deduce from that the apparent conductivity of the material with respect to ϵ when keeping N nodes in the sum above.

2 Numerical computations

2. 1. Write a variational formulation and show well-posedness through Lax-Milgram theorem.

2. 2. Use the `matlab` codes available, modify the `newmesh` to deal with the different coefficients.Modify the finite element code to take the Neumann condition into account.

2. 3. Write a routine computing the apparent conductivity.

2. 4. Fix $\epsilon = 0.1$. Compute the apparent conductivity for refined values of the step size, and compare with the solution obtained by keeping N modes in the expansion above.

2. 5. Choose $\epsilon = 10^{-4}$. Compute the apparent conductivity for refined values of the step size, draw the solution, and see what the mesh size must be to be accurate. What is the computational time in this case ?

2. 6. The FEM solver furnished are based on the use of the backslash $\backslash$ of `matlab`. Change it to a Krylov solver. Answer to the previous question.

2. 7. Add a Incomplete Cholewski preconditioner and compare the computational cost.