Master Vietnam-France in HCMC
High Performance computing

TP 9 Air conditioning and domain decomposition

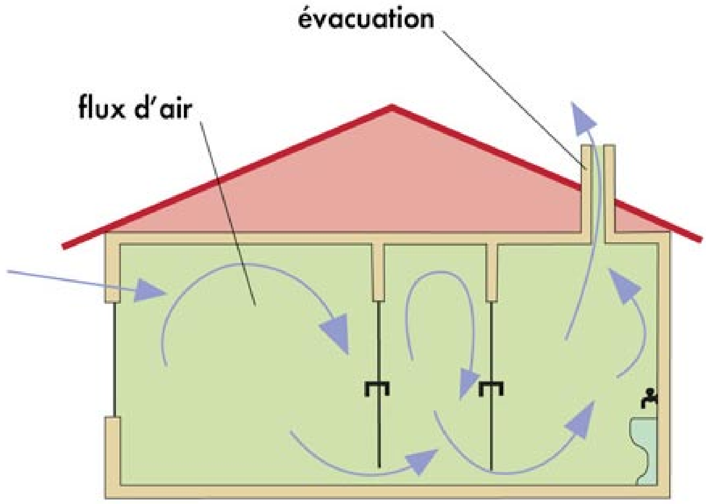

FIGURE 1 – Ventilation in a house

1 Modelisation

1. 1. Draw a scheme of your room with walls, windows, doors and main furnitures.

The temperature in the room occupying a geometrical domain Ω, is subject to variations due to source of heating (or cooling) and boundary conditions on the walls, windows and doors. The Fourier's law asserts that the equation modelizing the temperature in the room is subject to the **heat equation**

$$c(x)\rho(x)\partial_t u - \nabla(K\nabla u) = Q \tag{1}$$

where $c(x)$ is the specific heat, $\rho(x)$ is the density, and $K(x)$ the thermal conductivity of the material at point x. $Q(x,t)$ is used to represent any external sources or sinks (i.e. heat energy taken out of the system) of heat energy. If $Q(x,t) > 0$ then heat energy is being added to the system at that location and time and if $Q(x,t) < 0$ then heat energy is being removed from the system at that location and time.

The initial data u_0 tells us what the initial temperature distribution in the room is.

The **boundary data** tell us what happens on the walls and doors and windows. There are of four different sorts.

Prescribed temperature or Dirichlet boundary condition .

$$u = g$$

Prescribed heat flux or Neumann boundary conditions . Using Fourier's low it can be written

$$-K\partial_n u := -K\nabla u \cdot n = g$$

where n is the exterior unit normal to the boundary. If the boundary is insulated, *i.e.* there is no heat flow out of them then the boundary conditions reduces to $\partial_n u = 0$.

Newton's law of cooling or Robin boundary conditions . These are usually used when the material is in a moving fluid and note we can consider air to be a fluid for this purpose. In a room they can stand for wind coming from a window.

$$-K\partial_n u := H(u - g),$$

where H is a positive quantity that is experimentally determined, and g gives the temperature of the surrounding fluid at boundary.

Periodic boundary conditions They arise from particular geometries like a disk or an infinite channel

we can also mix and match these boundary conditions so to speak.

1. 2. Write a two dimensional model for the temperature in your room, and give precisely the boundary conditions on the wall, doors and windows. The source Q might be a heater (or a cooler !).

2 Insight on the heat equation in one dimension

Suppose the room is compressed into a nail Ω of length $L.$, with constant density, specific heat, and thermal conductivity. Suppose the nail with initial temperature u0(x), x ? (0,L) is placed between two ice cubes. We assume that the nail is thin enough so that we can consider it to be one dimensional. The ice cubes are naturally at temperature zero, and they touch the nail at both extremities. This defines the boundary conditions of the problem, and therefore we treat the heat equation

$$\partial_t u - \partial_{xx} u = 0 \text{ in } \Omega \times (0, T),$$

with given initial value u_0.

2. 1. Show that the solution tales the form of a series

$$u(x, t) = \sum_k \phi_k(x)\psi_k(t),$$

and give the explicit form for the basis functions $\phi_k(x)$ and $\psi_k(t)$. For each $j = 1, 510$, $u_0 = \sin(jpix/L)$, draw on the same plot the value of u at different times.

2. 2. Suppose now that Neumann conditions are imposed at each end of the nail. What physical meaning does it have ? Can you also get the solution explicitely. Draw some snapshots of the solution.

3 The steady state in two dimensions

Suppose again that the room has constant density, specific heat, and thermal conductivity. It seems from the 1-D analysis that a steady state can be reached. Suppose the heat furnished by the heater is $f(x)$ independent of time. At the moment the temperature u si stabilized in the room, it does not depend on time anymore, and is solution of the **Poisson equation**

$$-\Delta u = f$$

If there is no external force, we have the **Laplace equation** $-\Delta u = 0$.

3. 1. Write the the matlab script `NewMesh`into `Myroommesh` to insert the geometry, the material properties, and the boundaries of your room.

3. 2. When the heater is off, and the windows and doors are at $20°C$, draw the temperature repartition.

3. 3. Compute now a very warm day with 30 degrees outside, and the door are $20°C$.

3. 4. Turn the cooler on a temperature of $20°C$.

4 The heat equation in two dimensions

In the same geometry, we now want to solve the heat equation with a θ schéma. Let K be the stiffness matrix and M the mass matrix in the finite element approximation of $-\Delta$ with the boundary conditions designed in Section 3. The *semi-discrete* approximation of the problem (1) (with the simplification in the previous section) is $u_h(t) = \sum \xi_i(t)\phi_i$, where the ϕ are the basis functions in the finite element methods

$$\partial_t - \Delta u = f; M\partial_t\xi + K\xi = G_h$$

where G_h represents all sources contained in the volume and on the boundaries. This is now an ordinary differential equation we solve by the θ scheme. Let a time discretisation of the interval $(0,T)$ with step dt. The approximation of u_h at time t_n is denoted by u_h^n, and the scheme is :

$$M\frac{\xi^{n+1}-\xi^n}{dt} = \theta(G_h^{n+1} - K\xi^{n+1}) + (1-\theta)(G_h^n - K\xi^n) \tag{2}$$

When $\theta = 0$, it is called forward Euler, when $\theta = 1$, it is called backward Euler, and for $\theta = 1/2$, it is called Crank-Nicolson.

4. 1. Write a `matlab` program called `Heat` taking as entries θ, the data and the spacial geometry, and computing the sequence of values in time. The initial value is a gaussian centered at the center of the room, there is no source, the boundary data are zero everywhere, but on the window where the temperature it is equal to 0.5 for $t \geq 0.1$.

4. 2. It is well establish that for $\theta = 0$ the scheme is stable for θ sufficiently small and of order 1 in time, and inconditionnally stable and of order 1 for $\theta = 1/2$. Illustrate this result by numerical computation (find the relevant theory in reference 6.

5 Substructuring

Suppose the problem is first discretized in time by the implicit Euler scheme

$$\frac{u^{n+1}-u^n}{dt} - \Delta u^{n+1} = G^{n+1}.$$

Split the domain Ω into two rectangular subdomains Ω_j. Design one mesh in each subdomain, with the same interface mesh on I. The substructured problem can be designed as follows : take $g(t)$ on I, solve the problem **??** in each subdomain with data g on I, call $\mathcal{L}_j(f,g)$ the solution in Ω_j. Then the problem to be solved is

$$\mathcal{A}_1(f_1,g) + \mathcal{A}_2(f_2,g) = 0, \quad f_j = f_{|_{\Omega_j}}.$$

This is a linear problem, solved by iteration, like Richardson or Krylov method.

4. 3. Design the iteration process, using Gauss-Seidel algorithm.

4. 4. Plot the iterates, and the convergence history.

6 Further documents

— `http://tutorial.math.lamar.edu/Classes/DE/TheHeatEquation.aspx`
— Quarteroni Valli, *Numerical approximation of partial differential equations.*