An interpretation of Leibniz homology as functor homology

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Goal of the talk

Explain

Theorem (Hoffbeck and Vespa)

$$H_*^{Leib}(A, M) = Tor_*^{\Gamma_{sh}^{Lie}}(t, \mathcal{L}_{sh}^{Lie}(A, M))$$

where

- A a Lie algebra
- M an A-module
- H_*^{Leib} Leibniz homology
- Γ_{sh}^{Lie} a category (enriched over *Vect*)
- t and $\mathcal{L}_{sh}^{Lie}(A, M)$ functors.

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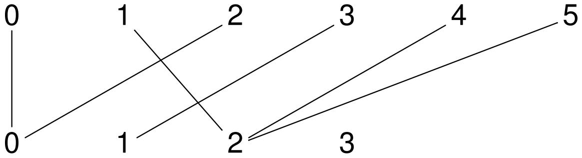
Recollections

The category Γ

Objects: $[n] = \{0, \ldots, n\}$ for $n \geq 0$ (with basepoint 0)
Morphisms: $\Gamma([n], [m])$ maps of pointed sets.

Example (n=5, m=3):



Recollections

The category Γ

Objects: $[n] = \{0, \dots, n\}$ for $n \geq 0$ (with basepoint 0)
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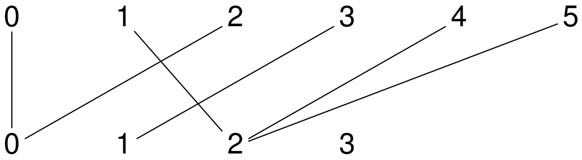
The Loday functor

For A unitary commutative algebra and M a A-module,

$$\begin{aligned} \mathcal{L}(A, M) : \quad & \Gamma \to \mathbb{k}\text{-Mod} \\ & [n] \mapsto M \otimes A^{\otimes n} \\ f : [n] \to [m] & \mapsto f_* : M \otimes A^{\otimes n} \to M \otimes A^{\otimes m} \end{aligned}$$

Recollections

Example for the morphism $f \in \Gamma([5],[3])$ depicted by



$$f_*(a_0 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5) = b_0 \otimes b_1 \otimes b_2 \otimes b_3$$

where

- $b_0 = a_0 . a_2$
- $b_1 = a_3$
- $b_2 = a_1 a_4 a_5$
- $b_3 = 1$

$$b_i = \prod_{j \in f^{-1}(i)} a_j$$

Recollections

Theorem (Pirashvili, Richter, Robinson, Whitehouse)

For A unitary commutative algebra and M a A-module,

$$H_*^{Harr}(A, M) = Tor_*^{\Gamma}(t, \mathcal{L}(A, M)) \quad \text{for a field } \Bbbk \text{ of char } 0$$

$$H_*^{E_\infty}(A, M) = Tor_*^{\Gamma}(t, \mathcal{L}(A, M)) \quad \text{in the general case}$$

Similar results for E_n-homology (Livernet, Richter) of E_n-algebras and Hochschild homology of associative algebras.

Question

Is there a similar theorem in other contexts?
For instance for algebras over an operad?

First case to try: Lie algebras.
One main problem is that the operad Lie is not a set operad, unlike the operads As and Com.

Last recollection

A Lie algebra A is a $\Bbbk$-vector space equipped with an anti-symmetric bracket $[-,-]$ satisfying $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$.

The Leibniz homology of a Lie algebra with coefficients in a A-module M is given by the homology of the complex

$$(C_n^{Leib}(A,M)=M\otimes A^{\otimes n},d)$$

where the differential is given by

$$d(x\otimes a_1\otimes\ldots\otimes a_n)=\sum_{1\leq i<j\leq n}\pm x\otimes a_1\otimes\ldots a_{i-1}\otimes[a_i,a_j]\otimes\ldots\otimes\widehat{a_j}\otimes\ldots\otimes a_n$$

$$+\sum_{1\leq j\leq n}\pm[x,a_j]\otimes a_1\otimes\ldots\otimes\widehat{a_j}\otimes\ldots\otimes a_n.$$

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Tensor product of functors

Given $\mathcal{C}$ a category

F a functor $\mathcal{C}^{op} \to \mathbb{k}$-Mod (called right $\mathcal{C}$-module)

G a functor $\mathcal{C} \to \mathbb{k}$-Mod (called left $\mathcal{C}$-module)

Definition

$F \otimes_{\mathcal{C}} G$ is the $\mathbb{k}$-module defined by

$$F \otimes_{\mathcal{C}} G = \bigoplus_{c \in \mathcal{C}} F(c) \otimes_{\mathbb{k}} G(c) / \sim$$

where $x \otimes_{\mathbb{k}} G(f)(y) \sim F(f)(x) \otimes_{\mathbb{k}} y$ for all $f : c \to c'$, $x \in F(c')$ and $y \in G(c)$.

Proposition

The tensor product of functors is right exact in both variables.

Tor functor of functors

There is a notion of projective resolutions for $\mathcal{C}$-modules.

Definition

$$Tor_*^{\mathcal{C}}(F, G) = H_*(P_\bullet \otimes_{\mathcal{C}} G)$$

where $P_\bullet$ is a projective resolution of F in the category of right $\mathcal{C}$-modules.

Moreover, the previous definitions of the tensor product of functors and of the Tor functor still hold when $\mathcal{C}$ is a category enriched over *Vect*.

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Where are we?

Remember our goal:

Theorem (Hoffbeck and Vespa)

$$H_*^{Leib}(A, M) = Tor_*^{\Gamma_{sh}^{Lie}}(t, \mathcal{L}_{sh}^{Lie}(A, M))$$

Want to define

- the category Γ_{sh}^{Lie}
- the functor $\mathcal{L}_{sh}^{Lie}(A, M)$ (from Γ_{sh}^{Lie} to $\mathbb{k}$-Mod).

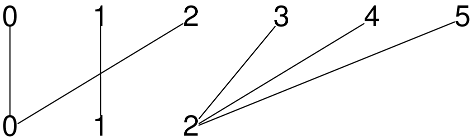
First: define a category Γ_{sh}, similar to Γ but with less symmetries.

The category Γ_{sh}

Definition: The category Γ_{sh}

Objects: $[n] = \{0, \ldots, n\}$ for $n \geq 0$ (with basepoint 0)
Morphisms: $\Gamma_{sh}([n], [m])$ surjective shuffle maps of pointed sets, that is maps α such that $min(\alpha^{-1}(i)) < min(\alpha^{-1}(j))$ whenever $i < j$.

Example of a morphism α in $\Gamma_{sh}([5], [2])$:



In this case, $0 < 1 < 3$.

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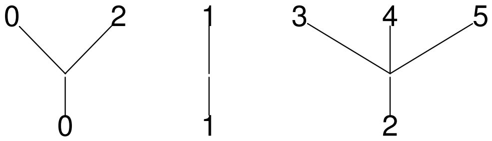


In this case, $0 < 1 < 3$.

The enriched category Γ^P_{sh} for a symmetric operad

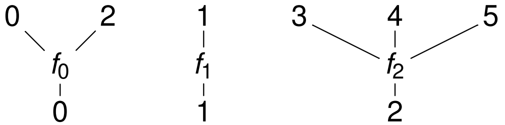
P symmetric reduced operad in *Vect* (reduced means $P(0) = 0$)

Definition: The category Γ^P_{sh} (Hoffbeck and Vespa)

Objects: $[n] = \{0, \dots, n\}$ for $n \geq 0$ (with basepoint 0)

Morphisms: $\Gamma^{\mathrm{P}}_{sh}([n],[m]) = \bigoplus_{\alpha \in \Gamma_{sh}([n],[m])} \mathrm{P}(\alpha^{-1}(0)) \otimes \ldots \otimes \mathrm{P}(\alpha^{-1}(m)).$

Example of a morphism f in $\Gamma^P_{sh}([5],[2])$:

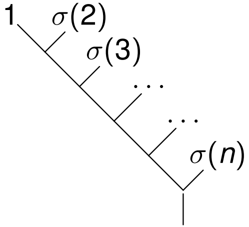


with $f_0 \in P(2), f_1 \in P(1), f_2 \in P(3)$.

The enriched category Γ_{sh}^{Lie}

Goal: make explicit the category Γ_{sh}^{P} for $P = Lie$.

We know a basis of the operad *Lie*, given in arity *n* by:



$$\left\{ \text{(tree)} \;\middle|\; n \geq 1, \sigma \text{ permutation of } \{2, \dots, n\} \right\}$$

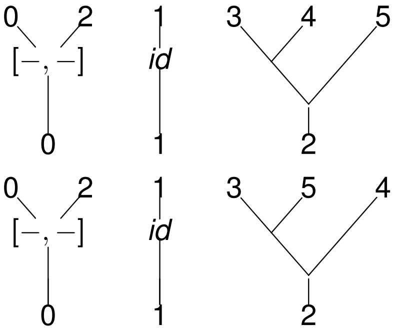
Note: this basis is related to Lie words of the form

$$[\dots[[x_1, x_{\sigma(2)}], x_{\sigma(3)}], \dots, x_{\sigma(n)}].$$

The enriched category Γ_{sh}^{Lie}

We obtain a linear basis of $\Gamma_{sh}^{Lie}([n],[m])$ by decorating the forests of $\Gamma_{sh}([n],[m])$ with elements of the basis of Lie.

In the previous example, 2 elements in the basis are associated to the shuffle map α.



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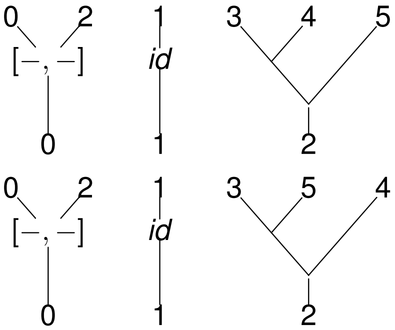

$\longleftrightarrow \quad (0,2|1|3,4,5)$

$\longleftrightarrow \quad (0,2|1|3,5,4)$

The functor $\mathcal{L}^{Lie}_{sh}(A, M) : \Gamma^{Lie}_{sh} \to \Bbbk\text{-Mod}$

Given a Lie algebra A and a A-module M

Definition (Hoffbeck and Vespa)

The functor $\mathcal{L}^{Lie}_{sh}(A, M) : \Gamma^{Lie}_{sh} \to \Bbbk$-Mod is defined on objects by

$$\mathcal{L}^{Lie}_{sh}(A, M)([n]) = M \otimes A^{\otimes n}$$

and for a morphism $f = (\alpha, f_0, \dots, f_m) \in \Gamma^{Lie}_{sh}([n], [m])$, the induced map $f_* : M \otimes A^{\otimes n} \to M \otimes A^{\otimes m}$ is given by

$$f_*(a_0 \otimes a_1 \otimes \dots \otimes a_n) = b_0 \otimes \dots \otimes b_m$$

where $b_i = \theta(f_i \otimes \bigotimes_{j \in \alpha^{-1}(i)} a_j)$ (with θ the evaluation map).

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Theorem (Hoffbeck and Vespa)

$$H_*^{Leib}(A,M) = Tor_*^{\Gamma_{sh}^{Lie}}(t, \mathcal{L}_{sh}^{Lie}(A,M))$$

Definition: Leibniz homology of a functor $\Gamma_{sh}^{Lie} \to \mathbb{k}$-Mod

The complex $C_*^{Leib}(T)$ is $T([n])$ in degree n with the differential $d : T([n]) \to T([n-1])$ defined by $T(\sum_{0\leq i<j\leq n} \pm d_{i,j})$.

For $T = \mathcal{L}_{sh}^{Lie}(A,M)$, we recover the definition of $C_*^{Leib}(A,M)$.

We are left to show $H_*^{Leib}(\mathcal{L}_{sh}^{Lie}(A,M)) = Tor_*^{\Gamma_{sh}^{Lie}}(t, \mathcal{L}_{sh}^{Lie}(A,M))$

How to obtain the theorem

We actually show that for any functor $T : \Gamma_{sh}^{Lie} \to \mathbb{k}$-Mod

$$H_*^{Leib}(T) = Tor_*^{\Gamma_{sh}^{Lie}}(t, T)$$

The idea is to use

Characterisation of a homological functor

If H_* is a functor from a category $\mathcal{C}$ to $\mathbb{k}$-grMod with

- $H_0(F)$ is isomorphic to $G \otimes_{\mathcal{C}} F$ for all $F \in \mathcal{C}$-mod
- $H_*(-)$ maps short exact sequences of $\mathcal{C}$-modules to long exact sequences
- $H_i(F) = 0$ for all projectives F and $i > 0$

then $H_i(F) = Tor_i^{\mathcal{C}}(G, F)$ for all F and all i.

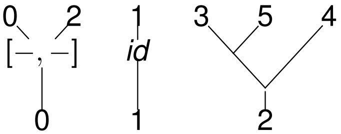
Idea of the rest of the proof

The proof of the third point relies on a filtration of the complex $C_*^{Leib}(\Gamma_{sh}^{Lie}([n],-))$ (for a fixed n).

Easy to get a filtration as a vector space, indexed by n-tuples.

Problem : show that this filtration is compatible with the differential.

Basis of $\Gamma_{sh}^{Lie}([n],[m])$ = forests of $m+1$ trees labelled by basis of *Lie*



$$\longleftrightarrow (0,2|1|3,5,4)$$

Split tuples can be sent to tuples by forgetting the vertical bars

$$proj : (0,2|1|3,5,4) \mapsto (0,2,1,3,5,4)$$

The n-tuples can be ordered lexicographically
$\Rightarrow$ partial order on the basis elements of $\bigoplus_m \Gamma_{sh}^{Lie}([n],[m])$

Example: $(0,2|1|3,5,4) > (0,2|1,3,4,5)$

Filtration

Obtain a filtration (as vector space) of the complex $C_*^{Leib}(\Gamma_{sh}^{Lie}([n], -))$, indexed by n-tuples :

$$F_u = \bigoplus_{proj(b) \geq u} \mathbb{K}.b$$

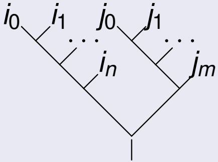
where b basis element of $\bigoplus_m \Gamma_{sh}^{Lie}([n], [m])$.

Problem

Compatibility with the differential?

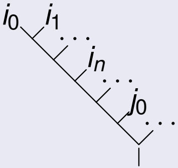
Recall : d = postcomposition with $\Sigma \pm d_{i,j}$.

Proposition

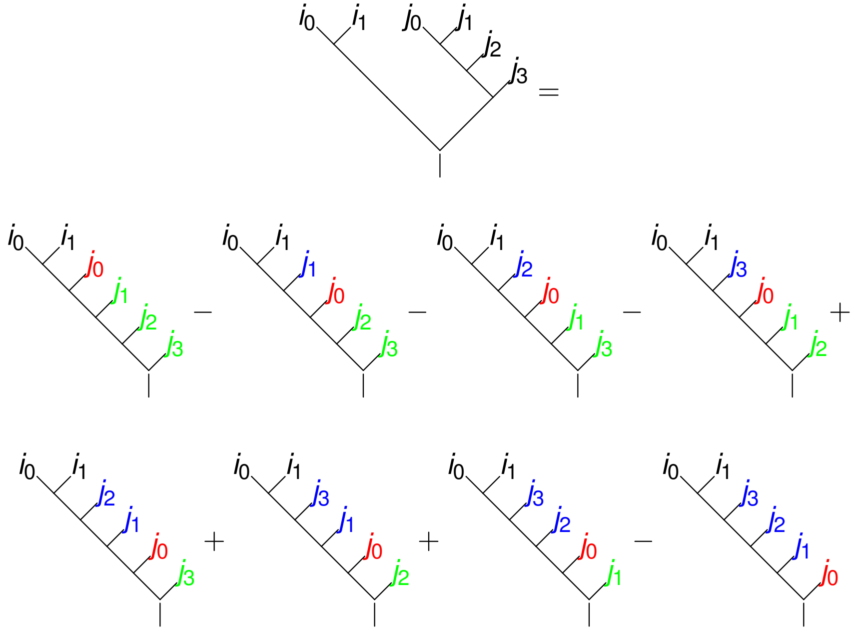


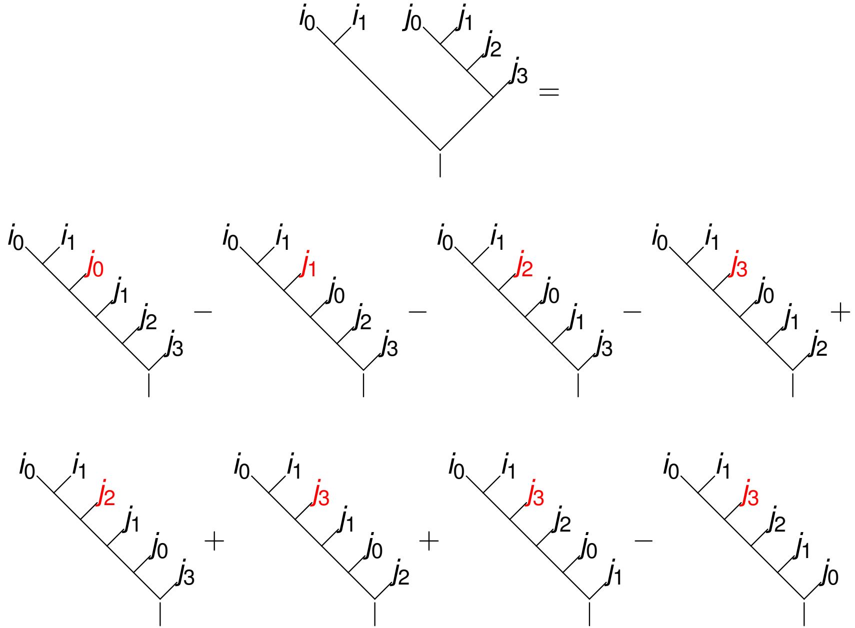
decomposes in the basis as the following sum:

$$\sum_{k=0}^{m} \sum_{S\subset\{1,\ldots,m\},|S|=k} (-1)^k$$



where everything is explicit.


i_0 i_1 j_0 j_1 j_2 j_3 =
i_0 i_1 j_0 j_1 j_2 j_3 −
i_0 i_1 j_1 j_0 j_2 j_3 −
i_0 i_1 j_2 j_0 j_1 j_3 −
i_0 i_1 j_3 j_0 j_1 j_2 +
i_0 i_1 j_2 j_1 j_0 j_3 +
i_0 i_1 j_3 j_1 j_0 j_2 +
i_0 i_1 j_3 j_2 j_0 j_1 −
i_0 i_1 j_3 j_2 j_1 j_0



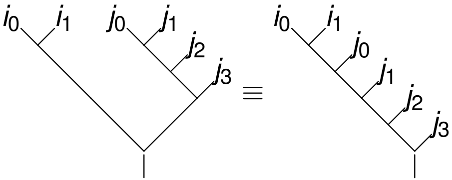
i_0 i_1 j_0 j_1 j_2 j_3 =
i_0 i_1 j_0 j_1 j_2 j_3 −
i_0 i_1 j_1 j_0 j_2 j_3 −
i_0 i_1 j_2 j_0 j_1 j_3 −
i_0 i_1 j_3 j_0 j_1 j_2 +
i_0 i_1 j_2 j_1 j_0 j_3 +
i_0 i_1 j_3 j_1 j_0 j_2 +
i_0 i_1 j_3 j_2 j_0 j_1 −
i_0 i_1 j_3 j_2 j_1 j_0

Corollary

The differential is compatible with the filtration:

$$d(F_u) \subset F_u.$$

We obtain in the associated graded complex



This means $(d_{0,1})_*(i_0, i_1 | j_0, j_1, j_2, j_3) \equiv (i_0, i_1, j_0, j_1, j_2, j_3)$.

Proposition

In the associated graded complex, the differential $d = \sum \pm (d_{i,j})_*$ removes vertical bars.

$$d(0,2|1|3,5,4) = (0,2,1|3,5,4) \pm (0,2|1,3,5,4)$$

Proposition

The associated graded complex splits as a sum of known acyclic complexes.

Corollary

The complex $C_*^{Leib}(\Gamma_{sh}^{Lie}([n],-))$ is acyclic.

This concludes the proof of the theorem.

Thank you for your attention.