OBSTRUCTION THEORY FOR ALGEBRAS OVER AN OPERAD

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Abstract. The goal of this paper is to set up an obstruction theory in the context of algebras over an operad and in the framework of differential graded modules over a field. Precisely, the problem we consider is the following: Suppose given two algebras A and B over an operad P and an algebra morphism from H_*A to H_*B. Can we realize this morphism as a morphism of P-algebras from A to B in the homotopy category? Also, if the realization exists, is it unique in the homotopy category?

We identify obstruction cocycles for this problem, and notice that they live in the first two groups of operadic Γ-cohomology.

In this paper we study a question of realizability of morphisms in a category of algebras over an operad.

In general, a realization problem takes the following form. We fix a category $\mathcal{C}$ equipped with a model structure (for instance: topological spaces, spectra, differential graded algebras over an operad). We have a homology (or homotopy) functor $H\colon \mathcal{C} \to \mathcal{A}$ with values in a purely algebraic category (for instance: graded modules, graded algebras). The usual questions are the existence of a realization of an object a in $\mathcal{A}$ by an object c in $\mathcal{C}$ such that $H(c) = a$ and the existence of a realization of a morphism $f\colon H(c_1) \to H(c_2)$ by a morphism $\phi\colon c_1 \to c_2$ such that $H(\phi) = f$.

Generally, the obstructions to these existences can be interpreted as classes in some (co)homology theory.

The most classical example goes back to Steenrod for $\mathcal{C}$ the category of topological spaces and $H = H^*_{\mathrm{sing}}$. A solution of this problem in the case of rationally nilpotent CW-complexes has been given by Halperin and Stasheff in [HS]. They apply rational homotopy theory to reduce this topological realization problem to a realization problem in the category of differential graded commutative algebras. The obstructions then live in some Harrison cohomology groups. The obstruction theory of Blanc, Dwyer and Goerss [BDG] for the realizability of Π-algebras by a space, the theories of Robinson [Rob] and of Goerss and Hopkins [GH] for the realizability of an algebra by an E_∞-spectra are other fundamental examples of obstruction theories in homotopy theory.

We are here interested in the case $\mathcal{C} = {}_{\mathsf{P}}\mathrm{dgMod}_{\mathbb{K}}$, the category of algebras over a fixed operad P in the framework of differential graded modules (for short dg-modules) over a field $\mathbb{K}$. The functor H is the homology of the underlying dg-module of an algebra over P. This homology inherits a $H_*\mathsf{P}$-algebra structure. The target category $\mathcal{A}$ consists of the graded $H_*\mathsf{P}$-algebras. The realization problem has been studied by Livernet [Liv, Section 3] in the setting of $\mathbb{N}$-graded dg-modules and when the ground field $\mathbb{K}$ has characteristic 0.

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The obstruction classes live in some cohomology groups of a natural cohomology theory associated to P, generalizing the Harrison cohomology for $\mathsf{P} = \mathsf{Com}$.

In this paper, we obtain an obstruction theory for the realization of morphisms in the setting of $\mathbb{Z}$-graded dg-modules and when the ground ring $\mathbb{K}$ is any field. We can identify a sequence of obstructions lying in some cohomology groups. Precisely, the Γ-cohomology of algebras over an operad (defined in [Hof], generalizing Robinson's and Whitehouse's Γ-homology [RW]) appears in our construction and we get the following theorems:

Theorem (Theorem 2.7)**.** *Let P be a connected graded operad and let $\tilde{\mathsf{P}}$ be an operadic cofibrant replacement of P. Let A and B be two algebras over $\tilde{\mathsf{P}}$. Suppose given a P-algebra morphism $f : H_*A \to H_*B$ (where H_*A and H_*B have the structure induced by the action of $\tilde{\mathsf{P}}$ in homology).*

The obstruction cocycles to the realizability of the morphism f lie in $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$. If $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$=0, then there automatically exists a morphism ϕ in the homotopy category of $\tilde{\mathsf{P}}$-algebras such that $H_*\phi = f$.

Theorem (Theorem 3.5)**.** *Let P be a connected graded operad and let $\tilde{\mathsf{P}}$ be an operadic cofibrant replacement of P. Let A and B be two algebras over $\tilde{\mathsf{P}}$. Suppose given a P-algebra morphism $f : H_*A \to H_*B$ and two homotopy morphisms ϕ_1, ϕ_2 such that $H_*\phi_1 = H_*\phi_2 = f$.*

The obstruction cocycles to the uniqueness of the realizations in the homotopy category lie in the group $H\Gamma^0_{\mathsf{P}}(H_*A, H_*B)$. If $H\Gamma^0_{\mathsf{P}}(H_*A, H_*B) = 0$, then $\phi_1 = \phi_2$ in the homotopy category of $\tilde{\mathsf{P}}$-algebras.

The groups of Γ-cohomology involved here come from a derived functor of the operadic derivations from H_*A to H_*B. In particular, the group $H\Gamma^0_{\mathsf{P}}(H_*A, H_*B) = 0$ is just P-derivations from H_*A to H_*B, and in the associative case, we can identify, for $* > 0$, $H\Gamma^*_{As}$ with HH^{*+1}, the shifted Hochschild cohomology.

Notice also that, as in the usual examples, the obstructions to uniqueness lie one degree lower than the obstructions to existence.

To obtain these theorems, the method is first to reduce our study to the case where the differentials of A and B are trivial. Then we use model category structures to make explicit cofibrant replacements of the algebras A and B. The crucial point of the proof is a natural filtration of the cooperad $B(\mathsf{P} \boxtimes \mathsf{E})$, which allows us to filter the generators of the cofibrant replacements. We construct step by step a map inducing the realization and identify where the obstructions to this construction live.

An important thing to notice in our theorems is that only the structures of P-algebra on H_*A and H_*B appear. So we do not need to know the complete $\tilde{\mathsf{P}}$-algebra structures on A and B, but only a part of it.

There are some immediate corollaries to the previous theorems. First, one defines the set of homotopy automorphisms $\mathrm{haut}_{\tilde{\mathsf{P}}}(A) := \{\phi : A \xrightarrow{\sim} A\}$ for A a cofibrant $\tilde{\mathsf{P}}$-algebra. We can consider its connected components for the following homotopy relation

$$\phi^0 \sim \phi^1 \;\Leftrightarrow\; \begin{tikzcd} A \arrow[d, "\sim"'] \arrow[dr, "\sim\ \phi^0"] & \\ A \otimes \Delta^1 \arrow[r, dotted, "\exists \phi^t"] & A. \\ A \arrow[u, "\sim"] \arrow[ur, "\sim\ \phi^1"'] & \end{tikzcd}$$

where $A \otimes \Delta^1$ denotes the cylinder object of A.

Consider the map $H_*(-) : \pi_0(\mathrm{haut}_{\tilde{\mathsf{P}}}(A)) \to \mathrm{aut}_{\mathsf{P}}(H_*A)$. Our obstruction theory implies the following results:

- If $H\Gamma^0_{\mathsf{P}}(H_*A, H_*A) = 0$ then $H_*(-)$ is injective.
- If $H\Gamma^1_{\mathsf{P}}(H_*A, H_*A) = 0$ then $H_*(-)$ is surjective.

Moreover, for a P-algebra H such that $H\Gamma^1_{\mathsf{P}}(H, H) = 0$, the first theorem implies that all $\tilde{\mathsf{P}}$-algebras A such that $H_*(A) = H$ are connected by weak equivalences.

In Section 1, we recall some results about operads, cooperads and operadic Γ-homology. In Section 2, we identify the obstructions to the realizability. In the last section, we study the obstructions to the uniqueness up to homotopy of the realizations.

Convention. We work in the differential graded setting. We take as ground category the category of differential $\mathbb{Z}$-graded modules (for short dg-modules) over a fixed field $\mathbb{K}$.

Our dg-modules M are equipped with an internal differential $d_M : M \to M$, decreasing the degree by 1. We sometimes twist it by a cochain $\partial \in Hom(M, M)$ of degree -1 in order to get a new differential $\partial + d_M$. The relation $\partial^2 + d_M \circ \partial + \partial \circ d_M = 0$ is assumed, so that $\partial + d_M$ satisfies $(\partial + d_M)^2 = 0$. We omit the internal differential d_M in the notation: We write M to denote the module M with differential d_M and write (M, ∂) for the module M with differential $\partial + d_M$.

All operads P will be assumed to be connected in the sense that $\mathsf{P}(0) = 0$ and $\mathsf{P}(1) = \mathbb{K}$.

1. Recollections

1.1. **Model structures.** We give references for the model structures of the categories which are used in this paper. For general references on the subject, we refer the reader to the survey of Dwyer and Spalinksi [DS] and the books of Hirschhorn [Hir] and Hovey [Hov]. For model structures in the operadic context, we refer to the articles of Hinich

[Hin], of Berger and Moerdijk [BM1] and of Goerss and Hopkins [GH], and the book of Fresse [F1].

Just recall the following standard definitions:

(1) The category of dg-modules is equipped with the model structure such that a morphism is a fibration (resp. a weak equivalence) if it is an epimorphism (resp. induces an isomorphism in homology).
(2) The category of operads inherits a model structure where fibrations (resp. weak equivalences) are fibrations (resp. weak equivalences) of the underlying dg-modules.
(3) The category of algebras over a cofibrant operad inherits a model structure where fibrations (resp. weak equivalences) are fibrations (resp. weak equivalences) of the underlying dg-modules.

In all cases, cofibrations are given by the LLP with respect to acyclic fibrations.

We usually call Σ_*-module a collection of dg-modules $\{M(r)\}_{r\in\mathbb{N}}$ where each $M(r)$ is equipped with an action of the r-th symmetric group Σ_r. The category of Σ_*-modules also inherits a model structure such that fibrations (resp. weak equivalences) are fibrations (resp. weak equivalences) of the underlying dg-modules. Every operad has an underlying Σ_*-module and we say that an operad is Σ_*-cofibrant if the underlying Σ_*-module is cofibrant. The category of algebras over a Σ_*-cofibrant operad can also be equipped with a semi-model structure, but we will not need this refinement.

We will use a cofibrant replacement of operads given by the cobar-bar duality, which can be found in the paper of Getzler and Jones [GJ] in characteristic 0, and the paper of Berger and Moerdijk [BM2, Section 8.5] in our more general context. We denote by B the bar construction of an operad, introduced in [GK], and by B^c the cobar construction, introduced in [GJ]. Recall that an element of the bar (or cobar) construction $B(\mathsf{P})$ can be seen as a tree labelled by elements of P. Thus the bar (and cobar) construction is equipped with a weight, given by the number of vertices of the tree representing an element. The operad E denotes the Barratt-Eccles operad, whose definition is recalled later in Section 1.4, and $\boxtimes$ denotes the arity-wise tensor product of Σ_*-modules, i.e. $(\mathsf{P}\boxtimes\mathsf{E})(r) = \mathsf{P}(r)\otimes\mathsf{E}(r)$ for all $r\in\mathbb{N}$.

1.1.1. **Fact** ([BM2, Theorem 8.5.4]). *Let* P *be an operad.*
The operad $B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$ *is a cofibrant replacement of the operad* P.

If Q is a cofibrant replacement of an operad P, working with algebras over Q is equivalent to working with algebras over $B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$. In this paper, we always pick this particular cofibrant replacement, which will always be denoted by $\tilde{\mathsf{P}}$.

1.2. **General recollections on cooperads and coderivations.** Let D be a cooperad with $\mathsf{D}(0)=0$ and $\mathsf{D}(1)=\mathbb{K}$. These hypotheses are verified by the bar construction of an operad. The cooperad structure is given by a coproduct

$$\nu : \mathsf{D}(n) \to \bigoplus \mathsf{D}(r)\otimes\mathsf{D}(n_1)\otimes\ldots\otimes\mathsf{D}(n_r)$$

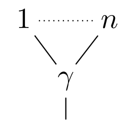
where the sum ranges over decompositions $n = n_1+\ldots+n_r$.

This total coproduct induces a quadratic coproduct

$$\nu_2 : \mathsf{D}(n) \to \bigoplus \mathsf{D}(n_1)\otimes\mathsf{D}(n_2)$$

where the sum ranges over decompositions $n = n_1 + n_2 - 1$.

It is convenient to work with a graphical representation of the elements of the cooperad and the coproducts.

We represent an element $\gamma \in \mathsf{D}(n)$ by a corolla with n inputs 

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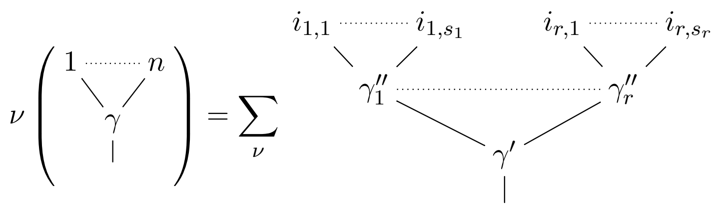
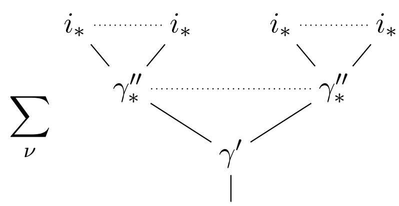
The coproduct maps an element $\gamma \in \mathsf{D}$ to a sum of formal composites of elements represented by

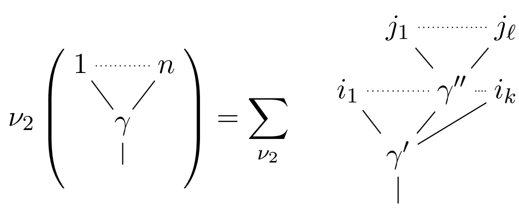


where $\gamma', \gamma''_1, \ldots, \gamma''_r$ are elements of D and the entries form a multi-shuffle of $\{1, \ldots, n\}$ (i.e. $i_{1,1} < i_{2,1} < \ldots < i_{r,1}$ and $i_{k,1} < i_{k,2} < \ldots < i_{k,s_k}$ for all $1 \leq k \leq r$).

To avoid too many indices, we will write such a sum in the following form:



The quadratic coproduct of an element $\gamma \in \mathsf{D}$ is represented by



where γ' and the γ'' are elements of D and the $\{i_1, \ldots, i_k\} \coprod \{j_1, \ldots, j_\ell\}$ run over partitions of $\{1, \ldots, n\}$.

Recall that for P an operad and A, B two P-algebras, a P-derivation between A and B with respect to a given morphism $f : A \to B$ is a linear map θ satisfying

$$\forall p \in \mathsf{P}(n), \forall (a_1, \ldots, a_n) \in A^{\otimes n}, \theta(p(a_1, \ldots, a_n)) = \sum_{i=1}^{n} p(f(a_1), \ldots, \theta(a_i), \ldots, f(a_n)).$$

In the special case $B = A$ and $f = id$, the condition becomes

$$\forall p \in \mathsf{P}(n), \forall (a_1, \ldots, a_n) \in A^{\otimes n}, \theta(p(a_1, \ldots, a_n)) = \sum_{i=1}^{n} p(a_1, \ldots, \theta(a_i), \ldots, a_n).$$

Dually, for D a cooperad, one can define the notion of a coderivation. Considering a D-coalgebra C, a D-coderivation $\theta : C \to C$ is a linear map satisfying

$$\forall c \in C, \nu(\theta(c)) = \sum_{\nu} \gamma(c_1, \ldots, \theta a_i, \ldots, c_n),$$

where $\nu(c) = \sum_\nu \gamma(c_1, \ldots, c_n)$ is the coproduct of c.

1.3. **Quasi-free coalgebras over cooperads.** Recall that $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$ is a cofibrant replacement of the operad P, and let D denote $B(\mathsf{P} \boxtimes \mathsf{E})$.

The first goal of this subsection is to provide an explicit cofibrant replacement of a $\tilde{\mathsf{P}}$-algebra A, of the form $(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial)$. The second goal is then to recall how one can reduce the study of morphisms from A to B in the homotopy category of $\tilde{\mathsf{P}}$-algebras to the study of some particular maps from $D(A)$ to B.

These results have been first given in the preprint of Getzler and Jones [GJ], using the notion of twisting cochain. But we use them in the wider context of $\mathbb{Z}$-graded modules and over a field of any characteristic, and we refer to the article of Fresse [F2] for the generalization in the latter setting. As we need precise formulas for our study, we recall some propositions in full details, but without proofs.

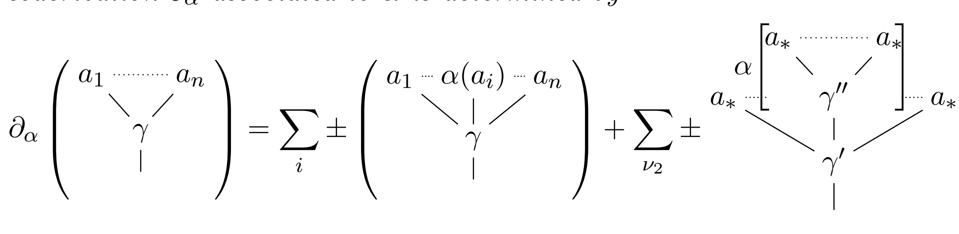
Let A be a dg-module, whose differential is denoted by d_A. Recall that $\mathsf{D}(A)$ is the cofree connected coalgebra given by

$$\mathsf{D}(A) = \bigoplus (\mathsf{D}(r) \otimes A^{\otimes r})_{\Sigma_r}.$$

The element $\gamma(a_1, \ldots, a_r) \in \mathsf{D}(A)$ is associated to the tensor $\gamma \otimes (a_1, \ldots, a_r)$. We represent such an element in $\mathsf{D}(A)$ by a corolla with inputs indexed by elements of A.

The next two propositions give the precise definition of ∂_α in $(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial)$ (which will be a cofibrant replacement of A) and one condition it must satisfy.

1.3.1. **Proposition** ([GJ, Proposition 2.14], [F2, Proposition 4.1.3])**.** *For a cofree coalgebra $\mathsf{D}(A)$, we have a bijective correspondance between D-coderivations $\partial : \mathsf{D}(A) \to \mathsf{D}(A)$ and homomorphisms $\alpha : \mathsf{D}(A) \to A$. The homomorphism α associated to a coderivation ∂ is given by the composition with the canonical projection. Conversely, the coderivation ∂_α associated to α is determined by*



for every $\gamma(a_1, \ldots, a_n)$ in $\mathsf{D}(A)$. The first term corresponds to α applied to $a_i \in A \subset \mathsf{D}(A)$. For the second term, we use the quadratic coproduct ν_2 and then apply α on the upper corolla which represents an element in $\mathsf{D}(A)$.

1.3.2. **Proposition** ([F2, Proposition 4.1.4]). *Let $\alpha : \mathsf{D}(A) \to A$ be a homomorphism of degree -1 such that $\alpha_{|A} = 0$.*

A D-coderivation $\partial_\alpha : \mathsf{D}(A) \to \mathsf{D}(A)$ of degree -1 defines a differential graded quasi-cofree coalgebra $(\mathsf{D}(A), \partial_\alpha)$ if and only if the homomorphism $\alpha : \mathsf{D}(A) \to A$ satisfies the relation

$$\delta(\alpha)\left(\begin{array}{ccc} a_1 & \cdots & a_n \\ \diagdown & & \diagup \\ & \gamma & \\ & | & \end{array}\right) + \sum_{\nu_2} \pm\alpha\left(\begin{array}{ccc} & \alpha\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma'' & \end{array}\right] & \\ a_* \cdots & & \cdots a_* \\ \diagdown & | & \diagup \\ & \gamma' & \\ & | & \end{array}\right) = 0 \tag{1}$$

for every element $\gamma(a_1, \ldots, a_n)$ in $\mathsf{D}(A)$, where $\delta(\alpha)$ denotes $d_A \circ \alpha \pm \alpha \circ (\partial_\alpha + d_{\mathsf{D}(A)})$.

The following proposition explains how one can encode a $B^c(\mathsf{D})$-algebra structure on A into a map $\mathsf{D}(A) \to A$.

1.3.3. **Proposition** ([GJ, proposition 2.15], [F2, Proposition 4.1.5]). *A $B^c(\mathsf{D})$-algebra structure on a dg-module A is equivalent to a map $\alpha : \mathsf{D}(A) \to A$ which satisfies Equation* (1) *and such that the restriction $\alpha_{|A}$ vanishes.*

When we are given an operad morphism $B^c(\mathsf{D}) \to \mathsf{Q}$, we have a functor which, to any D-coalgebra C, associates a quasi-free Q-algebra $R_{\mathsf{Q}}(C) = (\mathsf{Q}(C), \partial)$ for some twisting differential ∂ (cf. [GJ] or [F2, Section 4.2.1]).

We apply this construction to $\mathsf{D} = B(\mathsf{P} \boxtimes \mathsf{E})$, the morphism $id : B^c(\mathsf{D}) \to B^c(\mathsf{D}) = \tilde{\mathsf{P}}$ and the coalgebra $C = (\mathsf{D}(A), \partial_\alpha)$ associated to a $\tilde{\mathsf{P}}$-algebra A (the action is denoted by α). We get the following result:

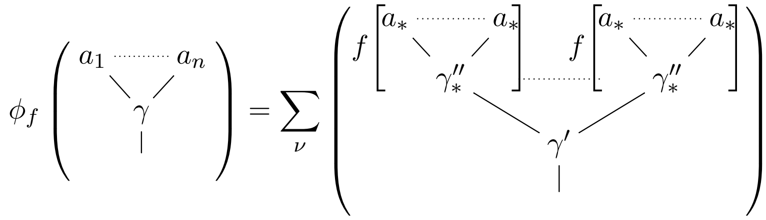
1.3.4. **Proposition** ([GJ, Theorem 2.19], [F2, Theorem 4.2.4]). *Let A be an algebra over $\tilde{\mathsf{P}}$ and let α denote the action. Let D denote $B(\mathsf{P} \boxtimes \mathsf{E})$. The augmentation ϵ : $R_{\tilde{\mathsf{P}}}(\mathsf{D}(A), \partial_\alpha) = (\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial) \to A$ defines a weak equivalence and $(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial)$ forms a cofibrant replacement of A in the category of $\tilde{\mathsf{P}}$-algebras.*

In this context, to study morphisms in the homotopy category of $\tilde{\mathsf{P}}$-algebras, we just have to study morphisms of quasi-cofree D-coalgebras. Indeed, a map from A to B in the homotopy category of $\tilde{\mathsf{P}}$-algebras is a class of morphisms of $\tilde{\mathsf{P}}$-algebras between cofibrant replacements of A and B. Such morphisms between $(\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial)$ and $(\tilde{\mathsf{P}}(\mathsf{D}(B), \partial_\beta), \partial)$ can be obtained using the functor $R_{\tilde{\mathsf{P}}}$ from D-coalgebras maps between $(\mathsf{D}(A), \partial_\alpha)$ and $(\mathsf{D}(B), \partial_\beta)$.

The following two propositions show how to reduce our study to the corestrictions of such morphisms.

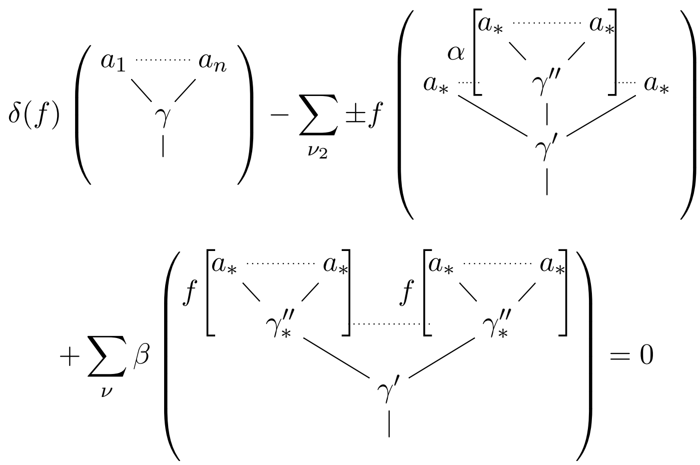
1.3.5. **Proposition** ([F2, Observation 4.1.7]). *The homomorphisms $\phi : \mathsf{D}(A) \to \mathsf{D}(B)$ of degree* 0 *and commuting with coalgebra structures are in bijection with homomorphisms of dg-modules $f : \mathsf{D}(A) \to B$. The homomorphism f associated to ϕ is given by the composite of ϕ with the projection. Conversely, the homomorphism $\phi = \phi_f$*

associated to f is determined by the formula



for every element $\gamma(a_1, \ldots, a_n)$ in $\mathsf{D}(A)$. We use the total coproduct and we apply f to all upper corrolas.

1.3.6. **Proposition** ([F2, Proposition 4.1.8])**.** *The homomorphism of cofree coalgebras $\phi_f : \mathsf{D}(A) \to \mathsf{D}(B)$ associated to a homomorphism $f : \mathsf{D}(A) \to B$ defines a morphism between quasi-cofree coalgebras $(\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B), \partial_\beta)$ if and only if we have the identity*



for every element $\gamma(a_1, \ldots, a_n)$ in $\mathsf{D}(A)$, where $\delta(f)$ denotes $d_B \circ f \pm f \circ (\partial_\alpha + d_{\mathsf{D}(A)})$.

In conclusion, a morphism from A to B in the homotopy category of $\tilde{\mathsf{P}}$-algebras can be obtained from a map $\mathsf{D}(A) \to B$ satisfying the identity of Proposition 1.3.6, where α encodes the algebra structure on A as specified in Proposition1.3.

1.4. **The Barratt-Eccles operad and its action on cochains.** Recall that an E_∞-operad is a Σ_*-cofibrant replacement of the commutative operad.

The Barratt-Eccles operad E is an example of E_∞-operad, defined by the normalized chain complex $\mathsf{E} = N_*(E\Sigma_\bullet)$, where $E\Sigma_n$ is the total space of the universal Σ_n-bundle in simplicial spaces. The chain complex $N_*(E\Sigma_n)$ is identified with the acyclic homogeneous bar construction of the symmetric group Σ_n, the module spanned in degree t by the $(t+1)$-tuples of permutations $\underline{w} = (w_0, \ldots, w_t)$ together with the differential δ such that $\delta(\underline{w}) = \sum_i (-1)^i (w_0, \ldots, \widehat{w_i}, \ldots, w_t)$. We consider the left action of the symmetric group on this chain complex.

The composition product of E is obtained using the composition product of permutations (which is just the insertion of a block). More precisely, for $\underline{w} = (w_0, \ldots, w_m) \in$

$\mathsf{E}(r)$ and $\underline{w}' = (w'_0, \ldots, w'_n) \in \mathsf{E}(s)$, the composite $\underline{w} \circ_i \underline{w}' \in \mathsf{E}(r+s-1)$ is defined by

$$\underline{w} \circ_i \underline{w}' = \sum_{x_*, y_*} \pm (w_{x_0} \circ_i w'_{y_0}, \ldots, w_{x_{m+n}} \circ_i w'_{y_{m+n}})$$

where the sum ranges over the monotonic paths from $(0,0)$ to (m,n) in $\mathbb{N} \times \mathbb{N}$.

The operad E acts on $N^*(\Delta^1)$, according to the paper by Berger and Fresse [BF]. We denote this action by σ. For our purposes, we simply recall the action of the component of degree 0 of E. We have the equality of dg-modules $N^*(\Delta^1) = \mathbb{K}.\underline{0}^\# \oplus \mathbb{K}.\underline{1}^\# \oplus \mathbb{K}.\underline{01}^\#$ where $\underline{0}^\#$, $\underline{1}^\#$ (both in degree 0) and $\underline{01}^\#$ (in degree -1) denote the dual of the basis of non-degenerate simplices. The differential ∂_N satisfies $\partial_N(\underline{01}^\#) = 0$, $\partial_N(\underline{0}^\#) = -\underline{01}^\#$ and $\partial_N(\underline{1}^\#) = \underline{01}^\#$. The r-th component in degree 0 of E is actually Σ_r, and the identity of Σ_r acts on $N^*(\Delta^1)$ as follows:

- $id.(\underline{0}^\#, \ldots, \underline{0}^\#, \underline{01}^\#, \underline{1}^\#, \ldots, \underline{1}^\#) = \underline{01}^\#$
- $id.(\underline{0}^\#, \ldots, \underline{0}^\#) = \underline{0}^\#$
- $id.(\underline{1}^\#, \ldots, \underline{1}^\#) = \underline{1}^\#$
- $id.(u_1, \ldots, u_r) = 0$ otherwise.

The equivariance gives the action of the other permutations of Σ_r. We will not need the formula for the action of E in higher degrees.

1.5. **The path object of an algebra over an operad.** Let Q be any cofibrant operad, for instance $\mathsf{Q} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$. Let B be a Q-algebra, with the structure given by β. We recall in this section the results we need from [BF, Section 3.1].

The path object of B in the category of Q-algebras is $B \otimes N^*(\Delta^1)$.

It is naturally endowed with the action $\beta \otimes \sigma$ of $\mathsf{Q} \boxtimes \mathsf{E}$:

$$(q \otimes \pi)(b_1 \otimes u_1, \ldots, b_r \otimes u_r) = q(b_1, \ldots, b_r) \otimes \pi(u_1, \ldots, u_r)$$

for $q \in \mathsf{Q}, \pi \in \mathsf{E}, (b_1, \ldots, b_r) \in B^r, (u_1, \ldots, u_r) \in N^*(\Delta^1)^r$. Fixing an operadic section $\rho : \mathsf{Q} \to \mathsf{Q} \boxtimes \mathsf{E}$ of the augmentation $\mathsf{Q} \boxtimes \mathsf{E} \to \mathsf{Q}$, we can see $B \otimes N^*(\Delta^1)$ as a Q-algebra. In Section 3.2, we will fix an explicit map ρ.

1.6. **Operadic Γ-cohomology.** In [Hof], we have defined a generalization of Robinson's and Whitehouse's Γ-(co)homology. The aim was to get a (co)homology theory of algebras over an operad when the objects in the underlying category are dg-modules over a field of positive characteristic (or over a ring). We recall here the definition of Γ-cohomology with coefficients in an algebra, which is enough for us in the context of the current paper.

Let A and B be P-algebras and $f : A \to B$ a morphism of P-algebras. The Γ-cohomology $H\Gamma^*_{\mathsf{P}}(A, B)$ of A with coefficients in B is defined by $H_*(\mathrm{Der}_{\tilde{\mathsf{P}}}(\tilde{A}, B))$ where $\tilde{\mathsf{P}}$ is a Σ_*-cofibrant replacement of P and $\tilde{A}$ a cofibrant replacement of A as $\tilde{\mathsf{P}}$-algebras. In this definition, the derivations are the $\tilde{\mathsf{P}}$-derivations relatively to the morphism $f \circ \psi$, where ψ denotes the morphism $\tilde{A} \xrightarrow{\sim} A$. The differential of this complex of derivations is the usual differential of a complex of morphisms. One can show that the definition of Γ-cohomology is independent of the choice of the cofibrant replacements.

An easy way to understand Γ-cohomology is the following: the Γ-cohomology of a P-algebra A is the usual André-Quillen cohomology of A seen as an algebra over a Σ_*-cofibrant replacement of P.

Note that the Γ-cohomology $H\Gamma^*_{\mathsf{P}}(A,B)$ depends on the morphism $f : A \to B$, but we do not specify it in the notation when there is no ambiguity.

2. Realizability of morphisms

Suppose given

- an operad P with the canonical operadic cofibrant replacement $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$;
- two algebras, A and B, over $\tilde{\mathsf{P}}$;
- a P-algebra morphism $f_0 : H_*A \to H_*B$ (where H_*A and H_*B have the structure induced in homology).

We want to understand the obstructions to the existence of a morphism $\phi : A \to B$ in the homotopy category of $\tilde{\mathsf{P}}$-algebras such that $H_*\phi = f_0$.

2.1. **Outline of the study.** We will proceed in the following way:

We first show in Section 2.2 that we can restrict our study to the case where the differentials of A and B are trivial, and we give some results concerning the structures induced in homology. We consider the cooperad $\mathsf{D} = B(\mathsf{P} \boxtimes \mathsf{E})$, and the explicit cofibrant replacements of A and B from Proposition 1.3.4. In Section 2.5, we want to construct a D-coalgebra map $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B), \partial_\beta)$ extending f_0 (it will lead to the expected morphism in the homotopy category). We introduce a filtration on $\mathsf{D}(A)$, to proceed by induction. We notice that the obstructions to the construction of ϕ_f lie in a certain cohomology group which can be identified with the first group of Γ-cohomology of H_*A with coefficients in H_*B. If ϕ_f can be constructed, then (as the construction $R_{\tilde{\mathsf{P}}}$ is functorial, see Proposition 1.3.4) we obtain $R_{\tilde{\mathsf{P}}}(\phi_f)$ which fits a diagram

$$\begin{tikzcd} (\tilde{\mathsf{P}}(\mathsf{D}(A), \partial_\alpha), \partial) \arrow[r, "R_{\tilde{\mathsf{P}}}(\phi_f)"] \arrow[d, "\sim"] & (\tilde{\mathsf{P}}(\mathsf{D}(B), \partial_\beta), \partial) \arrow[d, "\sim"] \\ A & B \end{tikzcd} .$$

and thus we obtain a morphism from A to B in the homotopy category of $\tilde{\mathsf{P}}$-algebras.

2.2. **Restriction of the hypotheses.** We show here that we can reduce our study to the case where the differentials of A and B are trivial.

First, recall the following result concerning the transfer of structures:

2.2.1. **Fact.** *Let $f : A \xrightarrow{\sim} B$ be a weak equivalence of dg-modules. Suppose that B has an action of a cofibrant operad Q.*

Then A inherits the structure of a Q-algebra such that

1. *$A \xleftarrow{\sim} \cdot \xrightarrow{\sim} B$ where the morphisms are weak equivalences of Q-algebras,*
2. *$H_*(A \xleftarrow{\sim} \cdot \xrightarrow{\sim} B) = H_*f$.*

This result in the A_∞ context was already in Kadeishvili's work [Kad]. In our context, we refer to the result stated by Fresse [F4, Theorem A]. The second assertion is not made explicit in the theorem of this reference but immediately follows from the proof.

Let $H = H_*A$ be the homology of a Q-algebra A. The graded module H can be seen as a dg-module equipped with a trivial differential, weakly equivalent to A as dg-modules. We fix a splitting $A_* = Z_*A \oplus B'_{*-1}A$, where Z_*A denote the cycles of A (and where $B'_{*-1}A$ is isomorphic to the boundaries $B_{*-1}A$). This yields a map $A \to Z_*A$, which induces a map $A \to H$ by composition with the projection $Z_*A \to H$. As we are working over a field, we can fix a section of dg-modules $s_A : H_*A \to Z_*A$ of the projection $Z_*A \twoheadrightarrow H_*A$, and thus a map $H \xrightarrow{\sim} A$.

The fact 2.2.1 implies that H inherits a structure of a Q-algebra such that $H \xleftarrow{\sim} \cdot \xrightarrow{\sim} A$, where the morphisms are weak equivalences of Q-algebras. This action of Q on H induces in homology an action of $H_*\mathsf{Q}$ on $H = H_*H$.

On the other hand, as H is the homology of the Q-algebra A, it inherits the structure of an algebra over $H_*\mathsf{Q}$.

2.2.2. **Lemma.** *These actions of $H_*\mathsf{Q}$ on H coincide.*

Proof. The zig-zag of Q-algebras $H \xleftarrow{\sim} \cdot \xrightarrow{\sim} A$ induces in homology the zig-zag of $H_*\mathsf{Q}$-algebras $H \xleftarrow{\cong} H_*(\cdot) \xrightarrow{\cong} H_*A$. By the second point of the Fact 2.2.1, H (with the first action) and H_*A (with the second action) are equal as $H_*\mathsf{Q}$-algebras. □

Let B be another Q-algebra and $K = H_*B$ its homology. Let $\tilde{H}$ and $\tilde{K}$ be cofibrant replacements of H and K in the category of Q-algebras. We get the following diagram of Q-algebras, where every vertical arrow is a quasi-isomorphism:

$$\begin{tikzcd}
\tilde{H} \arrow[r] \arrow[d, "\sim"'] & \tilde{K} \arrow[d, "\sim"] \\
H \arrow[r, dotted] & K \\
\cdot \arrow[u, "\sim"] \arrow[d, "\sim"'] & \cdot \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
A \arrow[r, dotted] & B
\end{tikzcd} .$$

It implies the identity

$$\mathrm{Hom}_{\mathrm{Ho}\,\mathsf{Q}-alg}(A, B) = \mathrm{Hom}_{\mathrm{Ho}\,\mathsf{Q}-alg}(H, K) = [\tilde{H}, \tilde{K}]_{\mathsf{Q}-alg}$$

where the notation $[-,-]$ refers to the homotopy classes, and we get

2.3. **Proposition.** *Our initial problem (of finding a lift of a P-algebra morphism from H_*A to H_*B to a $\tilde{\mathsf{P}}$-algebra homotopy morphism from A to B) is equivalent to finding a $\tilde{\mathsf{P}}$-algebra homotopy morphism from H_*A to H_*B (both equipped with the $\tilde{\mathsf{P}}$-algebra structure induced by the map from $\tilde{\mathsf{P}}$ to P).*

Therefore, in the rest of the paper, we only consider the case of trivial differentials on A and B.

2.4. **Description of the homology action.** Let α denote the action of the operad Q on the dg-module A. We now make explicit the action α_1 of $H_*\mathsf{Q}$ on H_*A, as it will be used in the next section.

Let $Z_*\mathsf{Q}$ denote the cycles of Q. As before, we can consider a section of the homology $s_\mathsf{Q} : H_*\mathsf{Q} \to Z_*\mathsf{Q}$.

2.4.1. **Observation.** *The action α_1 can be determined by the commutativity of the following diagram:*

$$\begin{tikzcd} H_*\mathsf{Q}(r) \otimes H_*A^{\otimes r} \arrow[r, "\alpha_1"] \arrow[d, "s_\mathsf{Q}\otimes(s_A)^{\otimes r}"] & H_*A\,, \\ Z_*\mathsf{Q}(r) \otimes Z_*A^{\otimes r} \arrow[r, dotted, "\alpha"] \arrow[d] & Z_*A \arrow[u] \arrow[d] \\ \mathsf{Q}(r) \otimes A^{\otimes r} \arrow[r, "\alpha"] & A. \end{tikzcd}$$

where the dotted map is the restriction. The image of this restriction is included in the cycles of A.

We now consider the case where A is an algebra over $\mathsf{Q} = \tilde{\mathsf{P}} := B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$, where P is a graded operad. We use the particular section $\mathsf{P} \hookrightarrow B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$ given by the composite of the inclusion $\mathsf{P} \to \mathsf{P} \boxtimes \mathsf{E}$ (sending $p \in \mathsf{P}(r)$ to $p \otimes id_{\Sigma_r}$), with the obvious inclusions $\mathsf{P}\boxtimes\mathsf{E}$ to $B(\mathsf{P}\boxtimes\mathsf{E})$ and $B(\mathsf{P}\boxtimes\mathsf{E}) \to B^c(B(\mathsf{P}\boxtimes\mathsf{E}))$. The above paragraphs give an action of P on H_*A. If $d_A = 0$, then we identify A and H_*A, and thus we obtain the action α_1 of P on A.

2.5. **Construction of the morphism of coalgebras.** We can now study our problem. We are given

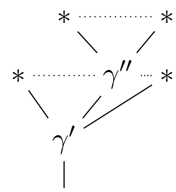
- a differential graded operad P such that d_P=0,
- two algebras, A and B, over $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$, with actions denoted by α and β, with trivial differentials,
- a P-algebra morphism $f_0 : (H_*A, \alpha_1) \to (H_*B, \beta_1)$.

In this section, we do not distinguish between A (resp. B) and H_*A (resp. H_*B) as they are equal as dg-modules. We specify the structure (α or α_1, β or β_1) when we consider them as algebras over $\tilde{\mathsf{P}}$ or P.

We want to define a morphism ϕ_f of D-coalgebras from $(\mathsf{D}(A), \partial_\alpha)$ to $(\mathsf{D}(B), \partial_\beta)$ such that the first component for a certain graduation is f_0. Recall from Section 2.1 that such a morphism ϕ_f will induce a morphism from A to B in the homotopy category. The morphism $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B), \partial_\beta)$ will be the morphism induced by $f : \mathsf{D}(A) \to B$, as defined in Proposition 1.3.5.

We use the graduation of $\mathsf{D} = B(\mathsf{P} \boxtimes \mathsf{E})$ given by the sum of the bar weight and the degree in E. Recall that the bar construction is given by a quasi-free object, and that the underlying free cooperad is equipped with a weight given by the number of tensors, as in the usual algebraic world. For instance, as dg-modules, $D_{[0]}$ is just $\mathbb{K}$ in arity 1, and $D_{[1]}$ in any arity r is $s\overline{\mathsf{P}}(r) \otimes \mathbb{K}[\Sigma_r]$, where s denotes the suspension of dg-modules and $\overline{\mathsf{P}}$ the augmentation ideal of P. This grading of D induces a splitting $\mathsf{D}(A) = \bigoplus_d \mathsf{D}_{[d]}(A)$ (we do not take into account any degree of A or weight in A).

The quadratic coproduct ν_2 on D sends $\gamma \in \mathsf{D}_{[d+1]}$ to composites



such that $\gamma' \in \mathsf{D}_{[p]}$, $\gamma'' \in \mathsf{D}_{[q]}$ and $p+q = d+1$. We will denote γ' by $\gamma'_{[p]}$ to keep in the notation in which degree it lies. In a similar way, for the map $f : \mathsf{D}(A) \to B$, we denote the component $\mathsf{D}(A)_{[d]} \to B$ by $f_{[d]}$ and the component $\mathsf{D}(A)_{[<d]} \to B$ by $f_{[<d]}$.

We want to construct the map f by induction on the degree, that is to construct $f_{[d]}$ supposing that $f_{[<d]}$ is known. We notice that in degree zero, $\mathsf{D}_{[0]}(A)$ is reduced to A and thus we define $f_{[0]} = f_0$ (remember we want ϕ_f to realize f_0).

The morphism ϕ_f must fit the following commutative diagram:

$$\begin{tikzcd}
\mathsf{D}(A) \arrow[r, "\phi_f"] \arrow[d, "\partial_\alpha + d_{\mathsf{D}}"'] & \mathsf{D}(B) \arrow[d, "\partial_\beta + d_{\mathsf{D}}"'] \arrow[ddr, "\beta"] & \\
\mathsf{D}(A) \arrow[r, "\phi_f"] \arrow[drr, "f"'] & \mathsf{D}(B) \arrow[dr, "proj"] & \\
& & B.
\end{tikzcd}$$

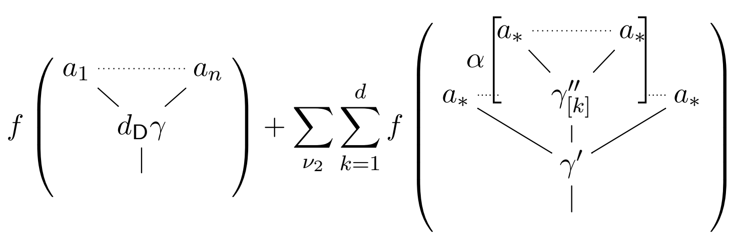
The triangle on the right obviously commutes. The commutativity of the triangle on the left defines f, the restriction of ϕ_f at the target. The commutativity of the exterior diagram is equivalent to the commutativity of the inner square.

The commutativity of this diagram is equivalent to the equation:

$$f \circ (d_{\mathsf{D}} + \partial_\alpha) = \beta \circ \phi_f, \tag{2}$$

that is the condition obtained in Proposition 1.3.6 in the case $d_A = 0$ and $d_B = 0$.

We now suppose that f is defined for degrees smaller than d and we consider an element $\gamma(a_1, \dots, a_n)$ where γ lies in $\mathsf{D}_{[d+1]}$. For this element, Equation (2) is equivalent to



$$= \sum_{\nu} \sum_{k=1}^{d+1} \beta \left(\begin{array}{ccc} f\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right] & \cdots\cdots & f\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right] \\ \diagdown & & \diagup \\ & \gamma'_{[k]} & \\ & | & \end{array} \right).$$

Specifying the degrees of f and taking the terms for $k=1$ out of the sums, we get:

$$f_{[d]} \left(\begin{array}{ccc} a_1 & \cdots & a_n \\ \diagdown & & \diagup \\ & d_{\mathsf{D}}\gamma & \\ & | & \end{array} \right) + \sum_{\nu_2} f_{[d]} \left(\begin{array}{ccc} & \alpha_1\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_{[1]} & \end{array}\right] & \\ a_* \cdots & | & \cdots a_* \\ \diagdown & \gamma' & \diagup \\ & | & \end{array} \right)$$

$$+ \sum_{\nu_2} \sum_{k=2}^{d} f_{[d+1-k]} \left(\begin{array}{ccc} & \alpha\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_{[k]} & \end{array}\right] & \\ a_* \cdots & | & \cdots a_* \\ \diagdown & \gamma' & \diagup \\ & | & \end{array} \right)$$

$$= \beta_1 \left(\begin{array}{ccc} & f_{[d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_{[d]} & \end{array}\right] & \\ f_0 a_* \cdots & | & \cdots f_0 a_* \\ \diagdown & \gamma'_{[1]} & \diagup \\ & | & \end{array} \right) + \sum_{\nu} \sum_{k=1}^{d+1} \beta \left(\begin{array}{ccc} f_{[<d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right] & \cdots\cdots & f_{[<d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right] \\ \diagdown & & \diagup \\ & \gamma'_{[k]} & \\ & | & \end{array} \right)$$

The last sum of the left hand side and the last sum of the right hand side involve f in degrees smaller than d, while the three other terms involve f only in degree exactly d. The second and fourth terms involve respectively α_1 and β_1, as only the restricted structure matters for elements in degree 1 (according to 2.2.2 and 2.4).

Thus we write the above equation in the following form:

$$f_{[d]} \left(\begin{array}{ccc} a_1 & \cdots & a_n \\ \diagdown & & \diagup \\ & d_{\mathsf{D}}\gamma & \\ & | & \end{array} \right) + \sum_{\nu_2} f_{[d]} \left(\begin{array}{ccc} & \alpha_1\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_{[1]} & \end{array}\right] & \\ a_* \cdots & | & \cdots a_* \\ \diagdown & \gamma' & \diagup \\ & | & \end{array} \right)$$

$$
-\sum_{\nu}\beta_1\left(\begin{array}{ccc}
 & f_{[d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \backslash & & / \\ & \gamma''_{[d]} & \end{array}\right] & \\
f_0 a_* \cdots & & \cdots f_0 a_* \\
\backslash & | & / \\
 & \gamma'_{[1]} & \\
 & | &
\end{array}\right)
$$

$$
= -\sum_{\nu_2}\sum_{k=2}^{d} f_{[d+1-k]}\left(\begin{array}{ccc}
 & \alpha\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \backslash & & / \\ & \gamma''_{[k]} & \end{array}\right] & \\
a_* \cdots & & \cdots a_* \\
\backslash & | & / \\
 & \gamma' & \\
 & | &
\end{array}\right)
$$

$$
+\sum_{\nu}\sum_{k=1}^{d+1}\beta\left(\begin{array}{ccc}
f_{[<d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \backslash & & / \\ & \gamma''_{*} & \end{array}\right] & \cdots & f_{[<d]}\left[\begin{array}{ccc} a_* & \cdots & a_* \\ \backslash & & / \\ & \gamma''_{*} & \end{array}\right] \\
\backslash & & / \\
 & \gamma'_{[k]} & \\
 & | &
\end{array}\right)
$$

with f in degree d grouped in the left hand side and f in degrees smaller than d grouped in the right hand side.

The left hand side can be identified with $\partial(f_{[d]})(\gamma(a_1,\ldots,a_n))$ where ∂ is the differential in $\mathrm{Hom}((\mathsf{D}(A),\partial_{\alpha_1}),(B,\beta_1))$. Note that this differential involves only the P-algebra structures α_1 and β_1 induced in homology, and not the whole $\tilde{\mathsf{P}}$-algebra structures α and β.

According to our induction hypothesis, the right hand side is known and is the obstruction.

2.6. **Proposition.** *If the cohomology group* $H^1\,\mathrm{Hom}((\mathsf{D}(A),\partial_{\alpha_1}),(B,\beta_1))$ *is equal to* 0*, we can construct a map* $f_{[d]}$ *(i.e. continue our induction), and hence a map* ϕ_f *answering the initial problem.*

Proof. We have one thing left to prove: Check that the obstruction is a cocycle.

This is just a direct computation, but quite hard to write down explicitely. The best way to understand the computation is first to do it in characteristic two (to avoid being lost because of signs), and in the special case $\mathsf{D}=As^{\mathsf{i}}$ and $\tilde{\mathsf{P}}=A_\infty$ (even if this case is not exactly included in our context) and then follow the same procedure for our general context.

In the special case, we denote by A and B the A_∞-algebras, with trivial differentials, and the action is denoted α and β. To respect the previous notation, α_i corresponds to the action on the operation μ_{i+1} of A_∞. The homologies H_*A and H_*B are associative algebras, and f_0 is an associative morphism, meaning that we have $\beta_1(f_0,f_0)=f_0\circ\alpha_1$. Let us also denote $f_{[i]}\mu_{i+1}$ by f_i, $\alpha_i\mu_{i+1}$ by α_i and $\beta_i\mu_{i+1}$ by β_i to make the notation lighter.

The construction of f_d is possible if the following equation is satisfied:

$$\beta_1(f_d, f_0) + \beta_1(f_0, f_d) + f_d \circ \alpha_1 = \underbrace{\sum \beta(f_{<d}, \ldots, f_{<d})}_{:=R_1} + \underbrace{\sum f_{<d} \circ \alpha_{>1}}_{:=R_2},$$

where the indices in the sums are chosen in such a way that the trees have arity $d+2$ and $\circ$ denotes the PreLie composition product

$$F \circ G = \sum_{\text{inputs of } F} F(id, \ldots, G, \ldots, id).$$

As before, we suppose that for $k < d$, the maps f_k are already defined, and we check what happens when we construct f_d.

Let us call $R = R_1 + R_2$ the right handside of this equation and compute $\partial(R)$ where ∂ is the Hochschild differential. First notice that, by definition, $\partial(R) = R \circ \alpha_1 + \beta_1(f_0, R) + \beta_1(R, f_0)$. We compute separately four terms:

The first term is:

$$\begin{aligned} R_1 \circ \alpha_1 &= \sum \beta(f_{<d}, \ldots, f_{<d}) \circ \alpha_1 \\ &= \sum \beta(f_{<d}, \ldots, f_{<d} \circ \alpha_1, \ldots, f_{<d}) \\ &= \sum \beta(f_{<d}, \ldots, f_{<d} \circ \alpha_{>1}, \ldots, f_{<d}) + \sum \beta(f_{<d}, \ldots, \sum \beta(f_{<d}, \ldots, f_{<d}), \ldots, f_{<d}) \\ &= \underbrace{\sum \beta(f_{<d}, \ldots, f_{<d}) \circ \alpha_{>1}}_{:=S_1} + \underbrace{\sum \beta(f_{<d}, \ldots, \sum \beta(f_{<d}, \ldots, f_{<d}), \ldots, f_{<d})}_{:=S_2} \end{aligned}$$

In the first sum S_1, the terms $\beta_1(f_0, \sum f_{<d} \circ \alpha_{>1})$ and $\beta_1(\sum f_{<d} \circ \alpha_{>1}, f_0)$ are missing, and in the second sum S_2, the terms $\beta_1(f_0, \sum \beta(f_{<d}, \ldots, f_{<d}))$ and $\beta_1(\sum \beta(f_{<d}, \ldots, f_{<d}), f_0)$ are missing.

We have used the induction hypothesis for $k < d$ to go from the second line to the third line.

The second term is:

$$\beta_1(f_0, R_1) + \beta_1(R_1, f_0) = \underbrace{\beta_1(f_0, \sum \beta(f_{<d}, \ldots, f_{<d}))}_{:=S_3} + \underbrace{\beta_1(\sum \beta(f_{<d}, \ldots, f_{<d}), f_0)}_{:=S_4}$$

The third term is:

$$\begin{aligned} R_2 \circ \alpha_1 &= (\sum f_{<d} \circ \alpha_{>1}) \circ \alpha_1 \\ &= \sum f_{<d} \circ (\alpha_{>1} \circ \alpha_1) + \sum f_{<d} \circ (\alpha_1 \circ \alpha_{>1}) + \sum (f_{<d} \circ \alpha_1) \circ \alpha_{>1} \\ &= \underbrace{\sum f_{<d} \circ (\alpha_{>1} \circ \alpha_1)}_{:=S_5} + \underbrace{\sum f_{<d} \circ (\alpha_1 \circ \alpha_{>1})}_{:=S_6} + \underbrace{\sum \beta(f_{<d}, \ldots, f_{<d}) \circ \alpha_{>1}}_{:=S_7} + \underbrace{\sum (f_{<d} \circ \alpha_{>1}) \circ \alpha_{>1}}_{:=S_8} \end{aligned}$$

We have first used the PreLie relation, and finally the induction hypothesis for the last sum.

The fourth and last term is:

$$\beta_1(f_0, R_2) + \beta_1(R_2, f_0) = \underbrace{\beta_1(f_0, \sum f_{<d} \circ \alpha_{>1})}_{:=S_9} + \underbrace{\beta_1(\sum f_{<d} \circ \alpha_{>1}, f_0)}_{:=S_{10}}$$

We can now gather these 10 terms as follows:

- $S_1 + S_9 + S_{10} = S_7$.
- $S_2 + S_3 + S_4 = \sum(\beta \circ \beta)(f_{<d}, \dots, f_{<d}) = 0$ using the Stasheff relation for B.
- $S_5 + S_6 + S_8 = \sum f_{<d} \circ (\alpha \circ \alpha) = 0$ using the Stasheff relation for A.

This concludes the special case.

For the general case, the computations are precisely the same. We get three families of terms which vanish as in the example with As. Though a few points differ:

- When computing for a given $\gamma \in \mathsf{D}$, the use of coproducts makes appear γ', γ'' and γ''' (which correspond to the μ_i of the special case), and the coassociativity of the coproduct allows us to identify some terms.
- Some additional terms involving the internal differential of D appear. They would have appeared in the special case in the Stasheff relation if we had a non-trivial differential on A or B. The Stasheff relation is now replaced by the relation which appears in Proposition 1.3.2.
- Signs are given by the Koszul rule.

□

We now relate this cohomology group with one group of Γ-homology:

2.7. **Theorem.** *The obstructions to the realizability of morphisms lie in $H\Gamma^1_{\mathsf{P}}(H_*A, H_*B)$.*

Proof. First, as a derivation is defined by the image of the generators, there is an isomorphism $\mathrm{Hom}((\mathsf{D}(A), \partial_{\alpha_1}), (B, \beta_1)) \simeq \mathrm{Der}_{\tilde{\mathsf{P}}}(R_{\tilde{\mathsf{P}}}(\mathsf{D}(A), \partial_{\alpha_1}), (B, \beta_1))$. The $\tilde{\mathsf{P}}$-algebra $R_{\tilde{\mathsf{P}}}(\mathsf{D}(A), \partial_{\alpha_1})$ is nothing but a cofibrant replacement of (A, α_1) (according to Proposition 1.3.4), so the cohomology $H^* \mathrm{Der}_{\tilde{\mathsf{P}}}(R_{\tilde{\mathsf{P}}}(\mathsf{D}(A), \partial_{\alpha_1}), (B, \beta_1))$ is the Γ-cohomology of the $\tilde{\mathsf{P}}$-algebra A with coefficients in B, for the actions α_1 and β_1. This cohomology is actually $H\Gamma^*_{\mathsf{P}}(H_*A, H_*B)$, cf. Section 1.6. □

2.8. **Remark.** It is possible to work over a ring $\mathbb{K}$ instead of a field, but some additional assumptions are then necessary. We need to assume that relevant dg-modules over $\mathbb{K}$ are projective and that we have sections of the maps: $H_*A \to A$ and $H_*B \to B$.

3. Realizability of homotopies

In this section, we consider the problem of uniqueness of realizations in the homotopy category. We are given

- a graded operad P with the canonical operadic cofibrant replacement $\tilde{\mathsf{P}} = B^c(B(\mathsf{P} \boxtimes \mathsf{E}))$;
- two algebras over $\tilde{\mathsf{P}}$, (A, α) and (B, β), with trivial differentials;

- two morphisms $f^0, f^1 : (\mathsf{D}(A), \partial_\alpha) \to B$ realizing the same P-algebra morphism $\psi : H_*A \to H_*B$.

The morphisms f^0 and f^1 induce morphisms $R_{\tilde{\mathsf{P}}}(\phi_{f^0})$ and $R_{\tilde{\mathsf{P}}}(\phi_{f^1})$ from $R_{\tilde{\mathsf{P}}}(\mathsf{D}(A), \partial_\alpha)$ to $R_{\tilde{\mathsf{P}}}(\mathsf{D}(B), \partial_\beta)$, and thus two morphisms of $\tilde{\mathsf{P}}$-algebras from A to B in the homotopy category. The question we want to study in this section is: what is the obstruction to the equality of these morphisms in the homotopy category? We show that the obstructions lie in a group of Γ-cohomology.

3.1. **Outline of the study.** We restrict our study to the case where the differentials of A and B are trivial. We consider the cooperad D defined by $B(\mathsf{P} \boxtimes \mathsf{E})$. We also consider the path object $B \otimes N^*(\Delta^1)$ of B in the category of $\tilde{\mathsf{P}}$-algebras, whose action is denoted $(\beta \otimes \sigma) \circ \rho$, cf. Section 1.5. For this matter, we define an explicit section $\rho : \tilde{\mathsf{P}} \to \tilde{\mathsf{P}} \boxtimes \mathsf{E}$ in Section 3.2.

In Section 3.3, we want to construct a D-coalgebra map $\phi_f : (\mathsf{D}(A), \partial_\alpha) \to (\mathsf{D}(B \otimes N^*(\Delta^1)), \partial_{(\beta\otimes\sigma)\circ\rho})$ giving a homotopy between ϕ_{f^0} and ϕ_{f^1}. Its restriction f must fit into the following commutative diagram:

$$\begin{tikzcd} & & B \\ \mathsf{D}(A) \arrow[urr, "f^0"] \arrow[rr, "f"] \arrow[drr, "f^1"'] & & B \otimes N^*(\Delta^1) \arrow[u, "d_0"'] \arrow[d, "d_1"] \\ & & B\ . \end{tikzcd}$$

As in the previous section, we will construct ϕ_f by induction, and see the obstructions to the construction. Such a map ϕ_f induces a homotopy between the morphisms $R_{\tilde{\mathsf{P}}}(\phi_{f^0})$ and $R_{\tilde{\mathsf{P}}}(\phi_{f^1})$ and thus their equality in the homotopy category. Our study is very similar to the previous one, except we have to consider the path object $B\otimes N^*(\Delta^1)$ instead of B itself.

3.2. **Construction of a section.** We define in this section an explicit operadic section $\rho : \tilde{\mathsf{P}} \to \tilde{\mathsf{P}} \boxtimes \mathsf{E}$.

Recall from [BM2] that the cobar-bar construction $B^c(B(-))$ can be identified with the cubical W-construction $W_\square(-)$. Markl and Shnider [MS] have constructed a diagonal on the W-construction: a map $W_\square(\mathsf{Q}) \xrightarrow{\Delta_\mathsf{Q}} W_\square(\mathsf{Q}) \boxtimes W_\square(\mathsf{Q})$ for any operad Q.

Moreover, for any operads P and Q, one can map $B^c(B(\mathsf{P} \boxtimes \mathsf{Q}))$ to $B^c(B(\mathsf{P})) \boxtimes B^c(B(\mathsf{Q}))$.

Combining these two facts, we can consider the composite :

$$\begin{array}{rcl} B^c(B(\mathsf{P}\boxtimes\mathsf{E})) & \to & B^c(B(\mathsf{P}))\boxtimes B^c(B(\mathsf{E})) \\ & \overset{id\boxtimes\Delta_{\mathsf{E}}}{\to} & B^c(B(\mathsf{P}))\boxtimes B^c(B(\mathsf{E}))\boxtimes B^c(B(\mathsf{E})) \\ & = & B^c(B(\mathsf{P}\boxtimes\mathsf{E}))\boxtimes B^c(B(\mathsf{E})) \\ & \overset{id\boxtimes aug}{\to} & B^c(B(\mathsf{P}\boxtimes\mathsf{E}))\boxtimes\mathsf{E} \end{array}$$

where aug denotes the augmentation $B^c(B(\mathsf{E}))\to\mathsf{E}$.

We denote this composite by $\rho:\tilde{\mathsf{P}}\to\tilde{\mathsf{P}}\boxtimes\mathsf{E}$.

3.3. **Construction of the morphism of coalgebras.** Suppose A and B are algebras over $\tilde{\mathsf{P}}$. The same argument as in Section 2.2 allows us to suppose their differentials are trivial. We use the same graduation as in Section 2.5.

The morphism ϕ_f must fit the following commutative diagram:

$$\begin{tikzcd} \mathsf{D}(A) \arrow[r, "\phi_f"] \arrow[d, "\partial_\alpha+d_{\mathsf{D}}"'] & \mathsf{D}(B\otimes N^*(\Delta^1)) \arrow[d, "\partial_N+\partial_{(\beta\otimes\sigma)\circ\rho}+d_{\mathsf{D}}"'] \arrow[ddr, "(\beta\otimes\sigma)\circ\rho+proj\circ\partial_N"] & \\ \mathsf{D}(A) \arrow[r, "\phi_f"] \arrow[drr, "f^{01}\otimes\underline{01}^{\#}"'] & \mathsf{D}(B\otimes N^*(\Delta^1)) \arrow[dr, "proj"] & \\ & & B\otimes\underline{01}^{\#}. \end{tikzcd}$$

The triangle on the right obviously commutes. The commutativity of the triangle on the left defines f^{01}, the restriction of f at the target in the component of $\underline{01}^{\#}$. The commutativity of the exterior diagram is equivalent to the commutativity of the inner square.

The commutativity of this diagram is equivalent to the equation:

$$(f^{01}\otimes\underline{01}^{\#})\circ(d_{\mathsf{D}}+\partial_\alpha)=(\beta\otimes\sigma)\circ\rho\circ\phi_f+(f^1-f^0)\otimes\underline{01}^{\#}. \tag{3}$$

We want to construct the map f^{01} by induction on the degree. We notice that in degree zero, $\mathsf{D}_{[0]}(A)$ is reduced to A and that $f^1_{[0]}-f^0_{[0]}=\psi-\psi=0$. Thus we define $f^{01}_{[0]}=0$.

We now suppose by induction that f^{01} is defined for degrees smaller than d and we consider an element $\gamma(a_1,\ldots,a_n)$ where γ lies in $\mathsf{D}_{[d+1]}$. For this element, Equation (3) is equivalent to

$$(f^{01}\otimes\underline{01}^{\#})\begin{pmatrix} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ d_{\mathsf{D}}\gamma \\ | \end{pmatrix} + \sum_{\nu_2}\sum_{k=1}^{d}(f^{01}\otimes\underline{01}^{\#})\begin{pmatrix} \alpha\begin{bmatrix} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \gamma''_{[k]} \end{bmatrix} \\ a_* \cdots \qquad\qquad \cdots a_* \\ \diagdown \quad | \quad \diagup \\ \gamma' \\ | \end{pmatrix}$$

$$= \sum_{\substack{\nu \\ \epsilon_* \in \{0,1,01\}}} \sum_{k=1}^{d+1} (\beta \otimes \sigma) \circ \rho \left(\begin{array}{c} f^{\epsilon_1} \begin{bmatrix} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_1^{\#} \quad \cdots\cdots \quad f^{\epsilon_r} \begin{bmatrix} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_r^{\#} \\ \diagdown \qquad \diagup \\ \gamma'_{[k]} \\ | \end{array} \right)$$

$$+((f^1 - f^0) \otimes \underline{01}^{\#}) \left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \ \diagup \\ \gamma \\ | \end{array} \right)$$

where $\gamma'_{[k]}$ and $\gamma''_{[k]}$ denote elements in $\mathsf{D}_{[k]}$.

The main difficulty in this equation (and the main difference with the study in Section 2.5) comes from the term

$$\sum_{\nu} \sum_{k=1}^{d+1} (\beta \otimes \sigma) \circ \rho \left(\begin{array}{c} f^{\epsilon_1} \begin{bmatrix} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_1^{\#} \cdots f^{\epsilon_r} \begin{bmatrix} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \otimes \underline{\epsilon}_r^{\#} \\ \diagdown \qquad \diagup \\ \gamma'_{[k]} \\ | \end{array} \right).$$

If γ' is an element of $\mathsf{D}_{[k]}, k \geq 2$, then the maps f^{01} appearing in this term are applied to elements $\gamma''_{[\ell]}$ with $\ell \leq d-k$. Thus these terms are already known, according to the induction hypothesis.

If $\gamma' = p \otimes \pi$ is an element of $\mathsf{D}_{[1]}$, we first notice that $\rho(p \otimes \pi) = (p \otimes \pi) \otimes \pi$ for $p \otimes \pi \in \mathsf{P} \boxtimes \mathsf{E} \subset \tilde{\mathsf{P}}$. Then we can rewrite the term for $k = 1$ as

$$\beta \left(\begin{array}{c} f^{\epsilon_1} \begin{bmatrix} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \quad \cdots\cdots \quad f^{\epsilon_r} \begin{bmatrix} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{bmatrix} \\ \diagdown \qquad \diagup \\ p \otimes \pi \\ | \end{array} \right) \otimes \sigma(\pi, \underline{\epsilon}_1^{\#}, \ldots, \underline{\epsilon}_r^{\#})$$

with p in P and π in E_0. Exactly one of the $\underline{\epsilon}^{\#}$ has to be $\underline{01}^{\#}$ so that this term ends up in $B \otimes \underline{01}^{\#}$ (cf. the description of the action of E_0 on $N^*(\Delta^1)$ in Section 1.4). Thus there is only one map f^{01} involved. If this map f^{01} is applied to an element $\gamma''_{[\ell]}$ with $\ell \leq d-1$, the term is known. If this map f^{01} is applied to an element $\gamma''_{[\ell]}$ with $\ell = d$, we know that all other γ'' must be in degree 0, and thus the f^{ϵ} applied to these γ'' are just ψ.

Thus we rewrite Equation (3) as

$$(f^{01}\otimes\underline{01}^{\#})\left(\begin{array}{ccc} a_1 & \cdots & a_n \\ \diagdown & & \diagup \\ & d_{\mathsf{D}}\gamma & \\ & | & \end{array}\right)+\sum_{\nu_2}(f^{01}_{[d]}\otimes\underline{01}^{\#})\left(\begin{array}{ccccc} & \alpha & \left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_{[1]} & \end{array}\right] & & \\ a_* & \cdots & | & \cdots & a_* \\ & \diagdown & & \diagup & \\ & & \gamma' & & \\ & & | & & \end{array}\right)$$

$$-\sum_{\substack{\nu \\ \epsilon_*\in\{0,1\}}}(\beta\otimes\sigma)\circ\rho\left(\begin{array}{ccccc} \psi\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{\epsilon_1}^{\#} & \cdots & f^{01}_{[d]}\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{01}^{\#} & \cdots & \psi\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{\epsilon_r}^{\#} \\ & \diagdown & | & \diagup & \\ & & \gamma'_{[1]} & & \\ & & | & & \end{array}\right)$$

$$=\sum_{\substack{\nu \\ \epsilon_*\in\{0,1\}}}\sum_{\ell=0}^{d-1}(\beta\otimes\sigma)\circ\rho\left(\begin{array}{ccccc} f^{\epsilon_1}\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{\epsilon_1}^{\#} & \cdots & f^{01}_{[\ell]}\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{01}^{\#} & \cdots & f^{\epsilon_r}\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{\epsilon_r}^{\#} \\ & \diagdown & | & \diagup & \\ & & \gamma'_{[1]} & & \\ & & | & & \end{array}\right)$$

$$+\sum_{\substack{\nu \\ \epsilon_*\in\{0,1,01\}}}\sum_{k=2}^{d+1}(\beta\otimes\sigma)\circ\rho\left(\begin{array}{ccc} f^{\epsilon_1}\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{\epsilon_1}^{\#} & \cdots & f^{\epsilon_r}\left[\begin{array}{ccc} \cdots & a_* & \cdots \\ \diagdown & & \diagup \\ & \gamma''_* & \end{array}\right]\otimes\underline{\epsilon_r}^{\#} \\ \diagdown & & \diagup \\ & \gamma'_{[k]} & \\ & | & \end{array}\right)$$

$$-\sum_{\nu_2}\sum_{k=2}^{d}(f^{01}_{[d+1-k]}\otimes\underline{01}^{\#})\left(\begin{array}{ccccc} & \alpha & \left[\begin{array}{ccc} a_* & \cdots & a_* \\ \diagdown & & \diagup \\ & \gamma''_{[k]} & \end{array}\right] & & \\ a_* & \cdots & | & \cdots & a_* \\ & \diagdown & & \diagup & \\ & & \gamma' & & \\ & & | & & \end{array}\right)+((f^1-f^0)\otimes\underline{01}^{\#})\left(\begin{array}{ccc} a_1 & \cdots & a_n \\ \diagdown & & \diagup \\ & \gamma & \\ & | & \end{array}\right).$$

The last sum of the left hand side can be simplified. Actually, for a given $\gamma'_{[1]}=p\otimes\pi$, we have

$$
(\beta\otimes\sigma)\circ\rho\left(\begin{array}{c}
\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\otimes\underline{\epsilon}_1^{\#}\cdots\cdots f^{01}_{[d]}\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\otimes\underline{01}^{\#}\cdots\cdots\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\otimes\underline{\epsilon}_r^{\#}\\
\diagdown\qquad\qquad | \qquad\qquad\diagup\\
\gamma'_{[1]}\\
|
\end{array}\right)
$$

$$
=\beta\left(\begin{array}{c}
\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\cdots\cdots f^{01}_{[d]}\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\cdots\cdots\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\\
\diagdown\qquad\qquad | \qquad\qquad\diagup\\
p\otimes\pi\\
|
\end{array}\right)\otimes\sigma(\pi,\epsilon_1^{\#},\dots,\underline{01}^{\#},\dots,\epsilon_r^{\#})
$$

$$
=\beta_1\left(\begin{array}{c}
\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\cdots\cdots f^{01}_{[d]}\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\cdots\cdots\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\\
\diagdown\qquad\qquad | \qquad\qquad\diagup\\
p\\
|
\end{array}\right)\otimes\sigma(\pi,\epsilon_1^{\#},\dots,\underline{01}^{\#},\dots,\epsilon_r^{\#}).
$$

Only one choice of ϵ's will give a non-zero term: the one where after composition with the permutation π, the sequence is $(\underline{0}^{\#},\dots,\underline{0}^{\#},\underline{01}^{\#},\underline{1}^{\#},\dots,\underline{1}^{\#})$, according to the action of E_0 on $N^*(\Delta^1)$.

Thus we finally get

$$
(f^{01}\otimes\underline{01}^{\#})\left(\begin{array}{c}
a_1\cdots\cdots a_n\\
\diagdown\quad\diagup\\
d_{\mathsf{D}}\gamma\\
|
\end{array}\right)+\sum_{\nu_2}(f^{01}_{[d]}\otimes\underline{01}^{\#})\left(\begin{array}{c}
\alpha_1\left[\begin{array}{c}a_*\cdots\cdots a_*\\ \diagdown\quad\diagup\\ \gamma''_{[1]}\end{array}\right]\\
a_*\cdots\qquad\qquad\cdots a_*\\
\diagdown\qquad | \qquad\diagup\\
\gamma'\\
|
\end{array}\right)
$$

$$
-\sum_{\nu}\beta_1\left(\begin{array}{c}
\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\cdots\cdots f^{01}_{[d]}\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\cdots\cdots\psi\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\\
\diagdown\qquad\qquad | \qquad\qquad\diagup\\
\gamma'_{[1]|\mathsf{P}}\\
|
\end{array}\right)\otimes\underline{01}^{\#}
$$

$$
=\sum_{\substack{\nu\\ \epsilon_*\in\{0,1\}}}\sum_{\ell=0}^{d-1}(\beta_1\otimes\sigma)\circ\rho\left(\begin{array}{c}
f^{\epsilon_1}\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\otimes\underline{\epsilon}_1^{\#}\cdots\cdots f^{01}_{[\ell]}\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\otimes\underline{01}^{\#}\cdots\cdots f^{\epsilon_r}\left[\begin{array}{c}\cdots a_* \cdots\\ \diagdown\quad\diagup\\ \gamma''_*\end{array}\right]\otimes\underline{\epsilon}_r^{\#}\\
\diagdown\qquad\qquad | \qquad\qquad\diagup\\
\gamma'_{[1]}\\
|
\end{array}\right)
$$

$$
+ \sum_{\substack{\nu \\ \epsilon_* \in \{0,1,01\}}} \sum_{k=1}^{d+1} (\beta \otimes \sigma) \circ \rho \left(\begin{array}{ccc} f^{\epsilon_1} \left[\begin{array}{c} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{array} \right] \otimes \underline{\epsilon_1}^{\#} & \cdots\cdots\cdots & f^{\epsilon_r} \left[\begin{array}{c} \cdots\, a_* \,\cdots \\ \diagdown \quad \diagup \\ \gamma''_* \end{array} \right] \otimes \underline{\epsilon_r}^{\#} \\ & \diagdown \quad \diagup & \\ & \gamma'_{[k]} & \\ & | & \end{array} \right)
$$

$$
- \sum_{\nu_2} \sum_{k=2}^{d} (f^{01}_{[d+1-k]} \otimes \underline{01}^{\#}) \left(\begin{array}{ccc} & \alpha \left[\begin{array}{c} a_* \cdots\cdots a_* \\ \diagdown \quad \diagup \\ \gamma''_{[k]} \\ | \end{array} \right] & \\ a_* \cdots & & \cdots a_* \\ & \diagdown \quad \diagup & \\ & \gamma' & \\ & | & \end{array} \right) + ((f^1 - f^0) \otimes \underline{01}^{\#}) \left(\begin{array}{c} a_1 \cdots\cdots a_n \\ \diagdown \quad \diagup \\ \gamma \\ | \end{array} \right)
$$

where $\gamma'_{[1]|\mathsf{P}}$ denotes the component in P of $\gamma'_{[1]} \in \mathsf{P} \boxtimes \mathsf{E}$.

All the terms in the right hand side are already known. The left hand side can be identified with $(\partial(f^{01}_{[d]} \otimes \underline{01}^{\#}))(\gamma)$ where ∂ is the differential in $\operatorname{Hom}((\mathsf{D}(A), \partial_{\alpha_1}), (B \otimes \underline{01}^{\#}, \beta_1))$.

We have proved

3.4. **Proposition.** *If the cohomology group $H^1 \operatorname{Der}_{\tilde{\mathsf{P}}}(R_{\tilde{\mathsf{P}}}(\mathsf{D}(A), \partial_{\alpha_1}), (B \otimes \underline{01}^{\#}, \beta_1))$ is equal to* 0*, we can construct a map $f^{01}_{[d]}$ (i.e. continue our induction), and hence a map ϕ_f answering the initial problem.*

Proof. The proof follows exactly the same idea as the proof of 2.6. □

We now relate this cohomology group with one group of Γ-cohomology:

3.5. **Theorem.** *The obstructions to the existence of a homotopy of two realizations of a morphism lie in $H\Gamma^0_{\mathsf{P}}(H_*A, H_*B)$.*

Proof. The proof is almost the same as the proof of Theorem 2.7. The only difference is that working with $B \otimes \underline{01}^{\#}$ instead of B creates a shift of -1 in the degree of the codomain, and thus in the degree of the cohomology group. □

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