Lecture 6: Computation of Greeks

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Outline of The Talk

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- Let Z be a random variable on $(\Omega, \mathcal{F}, \mathbb{R}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and F a function $F : (x, Z) \longmapsto F(x, Z)$ where $x \in I \subset \mathbb{R}$. We set

$$f(x) = \mathbb{E}\left[F(x, Z)\right], \quad \text{with } F(x, Z) \in \mathcal{L}^2(\Omega), \forall x \in I.$$

- Our aim is to compute $f'(x)$.

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- Our aim is to compute $f'(x)$.
- Evaluating sensitivities is essential for portfolio hedging and gives informations on the sensitivity of option prices with respect to a given parameter.

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Forward scheme

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- Assume that we know how to simulate $F(x, Z)$ and no explicit formula is available to evaluate $f(x)$?
- In such a case a Monte Carlo method is recommended

$$f(x) \approx \frac{1}{M} \sum_{i=1}^{M} F(x, Z_i) \text{ where } (Z_i)_{1 \le i \le M} \text{ are i.i.d. copies of } Z$$

- Hence,

$$f'(x) \approx \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \overset{\varepsilon \to 0}{\sim} f'(x).$$

Then,

$$f'(x) \approx \frac{1}{M} \sum_{i=1}^{M} \frac{F(x+\varepsilon, Z_i) - F(x, Z_i)}{\varepsilon}$$

Centred Scheme

$$
\begin{aligned}
f'(x) &\overset{\varepsilon\to 0}{\sim} \frac{f(x+\varepsilon)-f(x-\varepsilon)}{2\varepsilon} \\
&\approx \frac{1}{M}\sum_{i=1}^{M}\frac{F(x+\varepsilon,Z_i)-F(x-\varepsilon,Z_i)}{2\varepsilon}
\end{aligned}
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Theorem 1

Let $x \in \mathbb{R}$, and $f \in \mathcal{C}^2$ s.t. f'' is Lipschitz.

$$\exists \varepsilon_0 > 0,\ \forall x \in]x-\varepsilon_0, x+\varepsilon_0[,\ \left\| F(x,Z) - F(x',Z) \right\|_2 \leq C_{F,Z} \left| x - x' \right|$$

$$\begin{aligned} RMSE &:= \left\| f'(x) - \frac{1}{M} \sum_{i=1}^{M} \frac{F(x+\varepsilon, Z_i) - F(x-\varepsilon, Z_i)}{2\varepsilon} \right\|_2 \\ &\leq \left[f'' \right]_{lip} \frac{\varepsilon^2}{2} + \frac{C_{F,Z}}{\sqrt{M}} \end{aligned}$$

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\end{aligned}
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- Recall that RMSE=root mean square error sasfies

$$\text{MSE} = (\text{RMSE})^2 = (\text{bias})^2 + \text{Var}$$

$$\text{bias} = f'(x) - \mathbb{E}\left[\frac{\frac{1}{M}\sum_{i=1}^{M} F(x+\varepsilon, Z_i) - F(x-\varepsilon, Z_i)}{2\varepsilon}\right]$$

$$\text{Var} = \text{Var}\left(\frac{\frac{1}{M}\sum_{i=1}^{M} F(x+\varepsilon, Z_i) - F(x-\varepsilon, Z_i)}{2\varepsilon}\right)$$

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- The error for the other schemes (Forward and Backward) is of order ε.

Optimal choice :

How to choose the sample size M ?

$$\frac{1}{\sqrt{M}} \approx \frac{\varepsilon^2}{2} \iff \frac{1}{M} = \frac{\varepsilon^4}{4} \iff M = \frac{4}{\varepsilon^4}$$

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Implementation advise: Attention :must not $\frac{1}{M\varepsilon^2} \longrightarrow \infty$

- **Particular case:** Assume that the samples

$$\left(F\left(x+\varepsilon, Z_i\right)\right)_{1\leq i\leq M} \perp \left(F\left(x-\varepsilon, Z_i\right)\right)_{1\leq i\leq M}$$

Then,

$$\begin{aligned}
\mathrm{Var}\left(\frac{\frac{1}{M}\sum_{i=1}^{M} F(x+\varepsilon, Z_i) - F(x-\varepsilon, Z_i)}{2\varepsilon}\right) \\
&= \frac{1}{4M\varepsilon^2}\left[\mathrm{Var}\left(F(x+\varepsilon, Z)\right) + \mathrm{Var}\left(F(x-\varepsilon, Z)\right)\right] \\
&\overset{\varepsilon\to 0}{\sim} \frac{1}{4M\varepsilon^2} * 2\mathrm{Var}\left(F(x, Z)\right) = \frac{\mathrm{Var}\left(F(x, Z)\right)}{M\varepsilon^2}
\end{aligned}$$

In the above Theorem 1,

$$\begin{aligned}
\left\| f'(x) - \frac{\frac{1}{M}\sum_{i=1}^{M} F(x+\varepsilon, Z_i) - F(x-\varepsilon, Z_i)}{2\varepsilon}\right\|_2^2 \\
&\leq \left([f'']_{lip}\frac{\varepsilon^2}{2}\right)^2 + \frac{\mathrm{Var}(F(x, Z))}{2M\varepsilon^2}
\end{aligned}$$

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$$\left(F\left(x+\varepsilon, Z_i\right)\right)_{1\leq i\leq M} \perp \left(F\left(x-\varepsilon, Z_i\right)\right)_{1\leq i\leq M}$$

Then,

$$\begin{aligned}\mathrm{Var}&\left(\frac{\frac{1}{M}\sum_{i=1}^{M} F(x+\varepsilon, Z_i) - F(x-\varepsilon, Z_i)}{2\varepsilon}\right)\\ &= \frac{1}{4M\varepsilon^2}\left[\mathrm{Var}\left(F(x+\varepsilon, Z)\right) + \mathrm{Var}\left(F(x-\varepsilon, Z)\right)\right]\\ &\overset{\varepsilon\to 0}{\sim} \frac{1}{4M\varepsilon^2} * 2\mathrm{Var}\left(F(x, Z)\right) = \frac{\mathrm{Var}\left(F(x, Z)\right)}{M\varepsilon^2}\end{aligned}$$

In the above Theorem 1,

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Example : The Black Scholes model

Let $Z \sim \mathcal{N}(0,1)$ and h a call or put function, then

$$F(x,Z) = \exp(-rT)\, h\left(x \exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\right)\right)$$

Theorem 1 applies and we have

$$\begin{aligned}
& \left\|F(x,Z) - F(x',Z)\right\|_2 \\
\leq \; & e^{-rT}\left(\mathbb{E}\left[\left|h\left(x\exp\left((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\right)\right)\right.\right.\right. \\
& \qquad \left.\left.\left. - \; h\left(x'\exp\left((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}Z\right)\right)\right|^2\right]\right)^{\frac{1}{2}} \\
\leq \; & e^{-rT}[h]_{lip}\left(\mathbb{E}\left[\left|x-x'\right|^2 \exp\left(2(r-\frac{\sigma^2}{2})T + 2\sigma\sqrt{T}Z\right)\right]\right)^{\frac{1}{2}} \\
\leq \; & [h]_{lip}\left|x-x'\right| e^{\frac{\sigma}{2}T}
\end{aligned}$$

```
function [deltaFT] = deltaBS(S_0, K, T, r, sigma, N_MC,h)
z = rand(N_MC,1,'norm');
S=S_0*exp((r-0.5*sigma^ 2)*T*ones(N_MC,1)+sigma*sqrt(T)*z);
payoff = max(S-K*ones(N_MC,1), 0);
BS_Price = exp(-r*T)*mean(payoff);
////////////////////////////////////////////////////////////
S_h_up=(S_0+h)*exp((r-0.5*sigma^ 2)*T*
ones(N_MC,1)+sigma*sqrt(T)*z);
payoff_h_up = max(S_h_up-K*ones(N_MC,1), 0);
BS_Price_h_up = exp(-r*T)*mean(payoff_h_up);
////////////////////////////////////////////////////////////
S_h_down=(S_0-h)*exp((r-0.5*sigma^ 2)*T*
ones(N_MC,1)+sigma*sqrt(T)*z);
payoff_h_down = max(S_h_down-K*ones(N_MC,1), 0);
BS_Price_h_down = exp(-r*T)*mean(payoff_h_down);
deltaFT=(BS_Price_h_up-BS_Price)/h;
endfunction
```

Outline

Pathwise method

Let $x \in \mathbb{R}$, Z be a r.v. on $(\Omega, \mathcal{F}, \mathbb{R})$

$$f(x) = \mathbb{E}\left[F(x, Z)\right], \; F(x, Z) \in \mathcal{L}^2(\Omega)$$
$$\frac{\partial}{\partial x} f(x) = \frac{\partial}{\partial x} \mathbb{E}\left[F(x, Z)\right]$$

Question : Under which assumption could we switch $\mathbb{E}$ et $\frac{\partial}{\partial x}$?

Theorem 2

Let $I \subset \mathbb{R}$ *an open subset* , $\psi : I \times \Omega \longrightarrow \mathbb{R}$

A. *Local version :* *Let* $x_0 \in I$, *if* ψ *satisfies:*

i. $\forall x \in I,\ \psi(x, \cdot) \in \mathcal{L}^2(\Omega)$

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iii. $\exists Y \in \mathcal{L}^1(\Omega)$, *s.t.*
$\forall x \in I\ \mathbb{P}(d\omega) p.s., \quad |\psi(x,\omega) - \psi(x_0,\omega)| \leq Y(\omega)|x - x_0|$

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Example 1 : Black-Scholes Delta

(Call under risk neutral probability $\mathbb{P}^*$)

$$F(x,Z) = e^{-rT}\left(S_T - K\right)_+$$
$$= e^{-rT}\left(x\exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z\right)-K\right)_+$$

$f(x) = \mathbb{E}\left[F(x,Z)\right]$, (Global version case).

$$f'(x) \overset{theo}{=} \mathbb{E}\left[\frac{\partial F}{\partial x}(x,Z)\right] = e^{-rT}\mathbb{E}\left[\frac{\partial}{\partial x}\left(S_T-K\right)_+\right]$$

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```
function [delta_PT] = deltaBS(S_0, K, T, r, sigma, N_MC,h)
z = rand(N_MC,1,'norm');
S_PT=(S_0)*exp((r-0.5*sigma^ 2)*T*ones(N_MC,1)+sigma*sqrt(T)*z);
payoff_PT = S_PT.*(S_PT> K)/S_0;
delta_PT = exp(-r*T)*mean(payoff_PT);
endfunction
```

Example 2 : Black-Scholes Vega

(Call under $\mathbb{P}^*$)

$$f(\sigma, Z) = \mathbb{E}\left[F\left(\sigma, Z\right)\right]$$

$$f'(\sigma) \quad = e^{-rT}\mathbb{E}\left[\frac{\partial}{\partial S_T}\left(S_T - K\right)_+ \times \frac{\partial S_T}{\partial \sigma}\right]$$

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(Call under $\mathbb{P}^*$)

$$f(\sigma, Z) = \mathbb{E}\left[F(\sigma, Z)\right]$$

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Exemple 3: Asian Delta

No explicit formula in the path dependent case for Asian options in the Black-Scholes model under $\mathbb{P}^*$. For $0 < t_1 < \ldots < t_m \leq T$

$$\begin{aligned}
\bar{S} &= \frac{1}{m}\sum_{i=1}^{m} S_{t_i} \\
F(x, \bar{S}) &= e^{-rT}\left(\bar{S} - K\right)_+ \\
S_t &= x \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)
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\end{aligned}$$

<u>Goal</u> : $f'(x) = \mathbb{E}\left[\frac{\partial}{\partial x} F\left(x, \bar{S}\right)\right]$?

$$\begin{aligned}
f'(x) &= \mathbb{E}\left[\frac{\partial}{\partial x} F\left(x, \bar{S}\right)\right] = \mathbb{E}\left[\frac{\partial}{\partial S} F\left(x, \bar{S}\right) \times \frac{\partial \bar{S}}{\partial x}\right] \\
&= e^{-rT} \mathbb{E}\left[\mathbf{1}_{\{\bar{S} \geq K\}} \frac{\partial \bar{S}}{\partial x}\right]
\end{aligned}$$

But,

$$\frac{\partial \bar{S}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{m}\sum_{i=1}^{n} S_{t_i}\right) = \frac{1}{m}\sum_{i=1}^{m}\frac{\partial}{\partial x}S_{t_i} = \frac{1}{m}\sum_{i=1}^{m}\frac{S_{t_i}}{x} = \frac{\bar{S}}{x}.$$

Therefore :

But,

$$\frac{\partial \bar{S}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{m}\sum_{i=1}^{n} S_{t_i}\right) = \frac{1}{m}\sum_{i=1}^{m}\frac{\partial}{\partial x}S_{t_i} = \frac{1}{m}\sum_{i=1}^{m}\frac{S_{t_i}}{x} = \frac{\bar{S}}{x}.$$

Therefore :

$$f'(x) = e^{-rT}\mathbb{E}\left[\mathbf{1}_{\{\bar{S}\geq K\}} \times \frac{\bar{S}}{x}\right]$$

```
function [deltaPT] = deltaAsiat(S_0, K, T, N_time, r,
sigma, N_MC)
z = rand(N_MC,N_time,'norm');
dt = T/N_time;
LogPaths= cumsum([log(S_0)*ones(N_MC,1),(r-0.5*sigma^ 2)*dt
+ sigma*sqrt(dt)*z],'c');
S = exp(LogPaths);
payoff = max(mean(S, 'c')-K, 0);
indic = mean(S, 'c') > K;
g = indic.*mean(S, 'c')/S_0;
deltaPT = exp(-r*T)*mean(g);
endfunction
```