From left modules to algebras over an operad: application to combinatorial Hopf algebras

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Motivations : Why are combinatorial Hopf algebras free or cofree ?

Definition

A Hopf algebra

$$H_* = \oplus_{n\geq 0} H_n$$

- Associative product μ
- Coassociative coproduct Δ
- Relation beetween μ and Δ :

$$\Delta(\mu(a,b)) = \mu(\Delta(a),\Delta(b)).$$

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  - $k[S_n]$, Malvenuto-Reutenauer
  - $k[P_n]$, NCQSym

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- S_n doesn't act on H_n

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- S_n acts on H_n :
  - $V^{\otimes n}$,
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  - $k[P_n]$, NCQSym
- S_n doesn't act on H_n
  - $K[Y_n]$, Loday-Ronco

Content

$\mathbb{S}$-modules and tensor product

Vector spaces, graded vector spaces, $\mathbb{S}$-modules

Definition

A $\mathbb{S}$-module is a family of vector spaces $V = (V(n))_{n \in \mathbb{N}}$ together with a right S_n action on $V(n), \forall n$.

Or equivalently

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Or equivalently

Definition

A $\mathbb{S}$-module is a contravariant functor from **Fin** to **Vect**.

Tensor products

Vector spaces	$\otimes$
Graded vector spaces	$(V \otimes^g W)_n = \oplus_{p+q=n} V_p \otimes W_q$
$\mathbb{S}$-modules	

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Graded vector spaces	$(V \otimes^g W)_n = \oplus_{p+q=n} V_p \otimes W_q$
$\mathbb{S}$-modules	$(V \otimes^{\mathbb{S}} W)(A) = \oplus_{I \sqcup J = A} V(I) \otimes W(J)$

$$(V \otimes^{\mathbb{S}} W)(n) = \otimes_{p+q=n} V(p) \otimes W(q) \otimes_{S_p \times S_q} K[S_n].$$

Notation

- $[n] = \{1, \ldots, n\}$
- $\sigma = (\sigma_1, \ldots, \sigma_n)$
- st stands for *standardization*

$$\mathrm{st}(2, 7, 9, 3) = (1, 3, 4, 2)$$

- $\sigma|_{\{a_1, \ldots, a_p\}} = \mathrm{st}(\sigma(a_1), \ldots, \sigma(a_p))$.

$$(2, 4, 5, 6, 3, 1)|_{\{1,3,6\}} = \mathrm{st}(2, 5, 1) = (2, 3, 1).$$

Shuffles $\mathrm{Sh}_{p,q}$

Definition

A (p, q)-shuffle : $(a_1, \ldots, a_p, b_1, \ldots, b_q)^{-1} = (I, J)$ with $a_1 < \ldots < a_p$ and $b_1 < \ldots < b_q$ and $I = \{a_1, \ldots, a_p\}, J = \{b_1, \ldots, b_q\}$.

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$$
\begin{aligned}
(V \otimes^{\mathbb{S}} W)(n) &= \otimes_{p+q=n} V(p) \otimes W(q) \otimes_{S_p \times S_q} K[S_n]. \\
&= \otimes_{p+q=n} V(p) \otimes W(q) \otimes \mathrm{Sh}_{p,q}.
\end{aligned}
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$$(V \otimes^{\mathbb{S}} W)(n) = \otimes_{p+q=n} V(p) \otimes W(q) \otimes_{S_p \times S_q} K[S_n].$$
$$= \otimes_{p+q=n} V(p) \otimes W(q) \otimes \mathrm{Sh}_{p,q}.$$

Notation

Any $x \in (V \otimes^{\mathbb{S}} W)(n)$ writes $v \otimes w \otimes (I, J)$. For $I \sqcup J = [n]$.
Define $\pi_{I,J} : (V \otimes^{\mathbb{S}} V)(n) \to V(|I|) \otimes V(|J|)$

The category **Smod** is monoidal symmetric

$$\begin{array}{rccc} \tau: & M \otimes^{\mathbb{S}} N & \longrightarrow & N \otimes^{\mathbb{S}} M \\ & m \otimes n \otimes (I, J) & \longmapsto & n \otimes m \otimes (J, I) \end{array}$$

Twisted associative and twisted Hopf algebras

Twisted Hopf algebras

Definition

A twisted Hopf algebra is a Hopf algebra in the category $(\mathbf{Smod}, \otimes^{\mathbb{S}})$

- Associative product $\mu : A \otimes^{\mathbb{S}} A \longrightarrow A$
- Coassociative coproduct $\Delta : A \longrightarrow A \otimes^{\mathbb{S}} A$
- Δ is a morphism of twisted associative algebras.

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$$\mu(a \otimes b \otimes ([p], p + [q])) := \mu(a \otimes b)$$

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$$\mu(a \otimes b \otimes ([p], p + [q])) := \mu(a \otimes b)$$

- Since Δ is a morphism of $\mathbb{S}$-modules it is enough to give the projection $\pi_{[p], p+[q]}$ of Δ onto $A(p) \otimes A(q)$.

Twisted commutative algebras

Definition

A twisted associative algebra is commutative if $\mu\tau = \mu$,

$$\mu(a \otimes b \otimes (I, J)) = \mu(b \otimes a \otimes (J, I))$$

which writes also

$$\mu(a \otimes b) = \mu(b \otimes a) \cdot (p+1, \ldots, p+q, 1, \ldots p),$$

for $a \in A(p), b \in B(q)$ and $a \otimes b$ represents $a \otimes b \otimes ([p], p + [q])$.

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Example

$$T(V) = \oplus_n V^{\otimes n}$$

is the free twisted commutative algebra generated by V for the concatenation product

Example 1 : the symmetric groups

$$A(n) = K[S_n]$$

- $\mu(\sigma, \tau) = \sigma \times \tau = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}$
- $\Delta(\sigma) = \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T \otimes (S, T)$

is a cocommutative twisted Hopf algebra

Example 2 : set compositions

$$P(n) = K[P_n]$$

where P_n stands for set compositions of $[n]$ or ordered set partitions.

- Symmetric group action :
  $(P_1, \dots, P_k) \cdot \sigma = (\sigma^{-1}(P_1), \dots, \sigma^{-1}(P_n))$.
- Product : quasi-shuffle.

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Example :

$$w_f((12);(2,1)) = (12,4,3) + (4,12,3) + (4,3,12) \\ + (124,3) + (4,3,12) + (4,123)$$

$$w_f((2,1);(12)) = (2,1,34) + (2,34,1) + (34,2,1) \\ + (2,134) + (234,1)$$

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$$\begin{aligned} w_f((2,1);(12)) =& (2,1,34) + (2,34,1) + (34,2,1) \\ &+ (2,134) + (234,1) = w_f((12);(21)) \cdot (3412) \end{aligned}$$

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- Coproduct :

$$\Delta(P) = \sum_{l=0}^{k} \mathrm{st}(P_1, \dots, P_l) \otimes \mathrm{st}(P_{l+1}, \dots, P_k) \otimes (\cup_{1\leq j\leq l} P_j, \cup_{l+1\leq h\leq k} P_h).$$

Theorems of Stover, Patras-Reutenauer

Theorem

A twisted Hopf algebra (H, μ, Δ) *gives rise to two Hopf algebras :*

- $\hat{H} = (H, \bar{\mu}, \hat{\Delta})$ *: the cosymmetrised one.*
- $\bar{H} = (H, \hat{\mu}, \bar{\Delta})$ *: the symmetrized one.*

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$$\begin{array}{llll} \hat{H}: & \bar{\mu}(a,b) = \mu(a \otimes b) & ; & \hat{\Delta}(a) = \sum_{S,T} \pi_{S,T}\Delta(a) \\ \bar{H}: & \hat{\mu}(a,b) = \sum_{\xi \in \mathrm{Sh}_{p,q}} \mu(a \otimes b) \cdot \xi & ; & \bar{\Delta}(a) = \sum_{p+q=n} \pi_{[p],p+[q]}\Delta(a) \end{array}$$

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Recall that for the symmetrized Hopf algebra
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$\bar{\Delta}(a) = \sum_{p+q=n} \pi_{[p], p+[q]} \Delta(a) \Rightarrow \bar{\Delta}(\sigma) = \sum_{p=0}^{n} \sigma|_{[p]} \otimes \sigma|_{p+[q]}.$

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The symmetrised Hopf algebra is the Malvenuto-Reutenauer Hopf algebra.

Example 2 : set compositions

$P(n) = K[P_n]$. w_f is the quasi-shuffle product and

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The cosymmetrized Hopf algebra $(P, \bar{w}_f, \hat{\Delta})$ is *NCQSym* (after Bergeron, Zabrocki, Novelli, Thibon).

More results

Theorem (L.)

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- $\Rightarrow$ *$(H, \bar{\mu}, \hat{\mu}, \bar{\Delta})$ is a 2-associative bialgebra.*

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Example

The Hopf algebra of Malvenuto Reutenauer is cofree.
The Hopf algebra NCQSym is free.

Left modules and algebras over an operad

Plethysm–Operad

Definition

Plethysm between two $\mathbb{S}$-modules gives a $\mathbb{S}$-module

$$(P \circ^{\mathbb{S}} Q)(n) = \oplus_{k \geq 0} P(k) \otimes_{S_k} Q^{\otimes^{\mathbb{S}} k}(n)$$

Plethysm between a $\mathbb{S}$-module P and a vector space V gives a vector space

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Definition

The category $(\mathbf{Smod}, \circ^{\mathbb{S}}, I)$ is monoidal (non symmetric). An operad is a monad in this category.

$$\mu_{\mathcal{P}} : \mathcal{P} \circ^{\mathbb{S}} \mathcal{P} \to \mathcal{P}$$

$$\eta_{\mathcal{P}} : I \to \mathcal{P}.$$

Left modules–Algebras

Definition

Let $\mathcal{P}$ be an operad.

- A left module (or twisted algebra) over $\mathcal{P}$ is a $\mathbb{S}$-module M together with
$$\mathcal{P} \circ^{\mathbb{S}} M \to M.$$
- An algebra over $\mathcal{P}$ is a vector space V together with
$$\mathcal{P} \circ V \to V.$$

- $\mathcal{P} = \mathcal{A}s$: twisted associative algebras.
- $\mathcal{P}$ is the free twisted algebra over $\mathcal{P}$ generated by I.

Hopf operads, Hopf $\mathcal{P}$-algebras

Definition

A Hopf operad is an operad in the category of co-unital co-associative algebra. For all n

$$\begin{aligned} \delta_n :& \mathcal{P}(n) \longrightarrow \mathcal{P}(n) \otimes \mathcal{P}(n) \\ \epsilon_n :& \mathcal{P}(n) \longrightarrow K \end{aligned}$$

Theorem

The tensor product of two $\mathcal{P}$-algebras is a $\mathcal{P}$-algebra, when $\mathcal{P}$ is a Hopf operad.

Connected Hopf operads

Definition

a Hopf operad $\mathcal{P}$ is connected if

$$\mathcal{P}(0) = K = K \cdot 1_0, \mathcal{P}(1) = K \cdot 1_1.$$

For $\mu \in \mathcal{P}(n), S \subset [n]$

$$\mu|_S = \mu(x_1, \dots, x_n), \text{ with, } \begin{cases} x_i = 1_1 & \text{if } i \in S \\ x_i = 1_0 & \text{if } i \notin S \end{cases}, \in \mathcal{P}(|S|).$$

Theorem

Theorem (L.,Patras)

If $\mathcal{P}$ is a connected Hopf operad then

$$\begin{array}{rccc} \Delta: & \mathcal{P} & \to & \mathcal{P} \otimes^{\mathbb{S}} \mathcal{P} \\ & \mu & \mapsto & \sum_{S \sqcup T = [n]} \mu_{(1)}|_S \otimes \mu_{(2)}|_T \otimes (S, T). \end{array}$$

is the unique morphism of twisted $\mathcal{P}$-algebras such that $\Delta(1_1) = 1_0 \otimes 1_1 + 1_1 \otimes 1_0$.

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Example

$\mathcal{P} = \mathcal{A}s$ is a connected Hopf operad : $\delta_n(\sigma) = \sigma \otimes \sigma$ and $\epsilon_n(\sigma) = 1$.
Hence

$$\Delta(\sigma) = \sum_{S \sqcup T=[n]} \sigma|_S \otimes \sigma|_T \otimes (S,T)$$

is a morphism of twisted associative algebras.

Multiplicative Hopf operads

Given data : a Hopf operad $\mathcal{P}$ and $\mathcal{A}s \to \mathcal{P}$.

- Any Hopf $\mathcal{P}$-algebra is a twisted Hopf algebra.
- In particular the $\mathbb{S}$-module $\mathcal{P}$ is a twisted Hopf algebra. So is $(\mathcal{P} \circ^{\mathbb{S}} M)$ for any connected $\mathbb{S}$-module M.

Example : Zinbiel

A Zinbiel algebra satisfies the relation

$$(ab)c = a(bc) + a(cb).$$

It is a "Hopf operad" and there is a morphism $\mathcal{C}om \to \mathcal{Z}in$.

Lemma (L.)

$\mathcal{P} = \mathcal{Z}in$ gives a twisted commutative Hopf algebra $(\oplus_n K[S_n], m, D)$. The Hopf algebra $(\oplus_n K[S_n], \bar{m}, \hat{D})$ is the Malvenuto-Reutenauer Hopf algebra. It is free associative.

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Remark

The twisted Hopf algebra on the set compositions (that gives NCQSym) comes from the operad defining commutative tridendriform algebras.

Example : $T(V)$

$T(V) = \mathcal{C}om \circ^{\mathbb{S}} V$ where V is concentrated in arity 1.

- free twisted commutative algebra generated by V.
- Product : concatenation.
- Coproduct is

$$\Delta(v) = \sum_{S \sqcup T = [k]} v|_S \otimes v|_T \otimes (S, T),$$

where if $S = \{s_1, \ldots, s_l\}$, $(v_1 \otimes \ldots \otimes v_k)|_S = v_{s_1} \otimes \ldots \otimes v_{s_l}$.

Case 1

Theorem (L., L.-Patras, Patras-Reutenauer, Loday-Ronco)

Let $\mathcal{P}$ be a multiplicative Hopf operad and M a $\mathbb{S}$-module. Let $H = \mathcal{P} \circ^{\mathbb{S}} M$.

- (H, m, δ) *is a twisted Hopf algebra.*
- $(H_*, \bar{m}, \hat{\Delta})$ *is a Hopf algebra, free.*
- $(H_*, \hat{m}, \bar{\Delta})$ *is a Hopf algebra, cofree.*
- *If δ is cocommutative, so is $\hat{\Delta}$ and in caracteristic 0, $Prim_{\hat{\Delta}}(H_*) = \mathbb{L}Prim_{\bar{\Delta}}(H_*)$.*

Case 2

Theorem (L.)

Let $\mathcal{P}$ be a multiplicative regular Hopf operad and V be a vector space. let $H_* = \mathcal{P} \circ V$.

- (H_*, m, Δ) *is a Hopf algebra*
- $(H_*, m, \bar{\Delta})$ *is a unital Hopf algebra.*
- (H_*, m, Δ) *is a free algebra*

The Hopf algebra structure on $\oplus_n K[Y_n]$ considered by Chapoton comes from the operad of tridendriform algebras defined by Loday and Ronco.