The associative operad and the weak order on the symmetric groups

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Contents

11. Non-Σ-operad

K is a field.

Notation. A *non-Σ-operad* $\mathcal{P}$ satisfies $\mathcal{P}(0) = 0$, $\mathcal{P}(1) = K$. Compositions

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$$

satisfy

$$\begin{array}{ccc} (\mu \circ_i \nu) \circ_{j+i-1} \eta & = & \mu \circ_i (\nu \circ_j \eta), \\ (\mu \circ_i \nu) \circ_{j+m-1} \eta & = & (\mu \circ_j \eta) \circ_i \nu, \\ \mu \circ_i 1 = \mu & \text{and} & 1 \circ_1 \mu = \mu \end{array}$$

Example. $\mathcal{P}(n) = S_n$. Write $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i = \sigma(i)$.

$$B_i(\sigma, \tau) := (\underbrace{a_1, \ldots, a_{i-1}}_{\sigma}, \overbrace{b_1, \ldots, b_m}^{\tau}, \underbrace{a_{i+1}, \ldots, a_n}_{\sigma})$$

where

$$a_j := \begin{cases} \sigma_j & \text{if } \sigma_j < \sigma_i \\ \sigma_j + m - 1 & \text{if } \sigma_j > \sigma_i \end{cases}$$
$$b_k := \tau_k + \sigma_i - 1 \, .$$

12. Example

$\sigma = (2, 3, 1, 4)$, $\tau = (2, 3, 1)$, and $i = 2$ then $B_2(\sigma, \tau) = (2, 4, 5, 3, 1, 6)$.

In terms of matrices

$$\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B_2(\sigma, \tau) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

13. Σ-operads

Definition. A Σ-*operad* $\mathcal{P}$ is a non-Σ-operad together with a right S_n-action on $\mathcal{P}(n)$ such that

$$(\mu \cdot \sigma) \circ_i (\nu \cdot \tau) = (\mu \circ_{\sigma(i)} \nu) \cdot B_i(\sigma, \tau)$$

There is a pair of adjoint functors

$$\mathcal{S} : \text{non-}\Sigma\text{-operad} \leftrightarrow \Sigma\text{-operads} : \mathsf{F}$$

where F is the forgetful functor and $\mathcal{SP}(n) = \mathcal{P}(n) \otimes KS_n$.

14. The asociative operad

$(F_\sigma)_{\sigma \in S_n}$ is a basis of the vector space $\mathcal{A}s(n) = KS_n$.

$\mathcal{A}s$ is the unique Σ-operad such that

$$F_{1_n} \circ_i F_{1_m} = F_{1_{n+m-1}}$$
$$F_\sigma = F_{1_n} \cdot \sigma$$

Then

$$F_\sigma \circ_i F_\tau = F_{B_i(\sigma,\tau)}$$

In the sequel $\mathcal{A}s$ is considered as a non-Σ-operad and "operads" mean non-Σ-operad.

21. The left weak Bruhat order on S_n

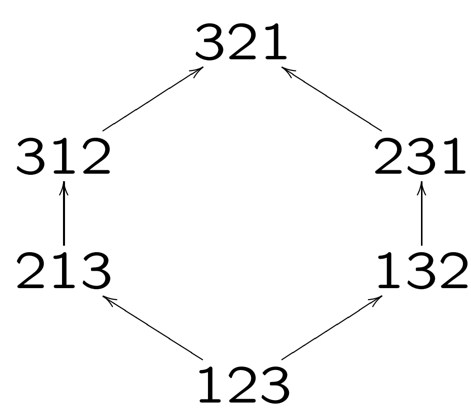
Definition. The *inversion set* of σ is

$$\mathrm{Inv}(\sigma) := \{(i,j) \in [n]\times[n] \mid i < j \text{ and } \sigma_i > \sigma_j\}.$$

The *left weak Bruhat order* on S_n is given by

$$\sigma \leq \tau \iff \mathrm{Inv}(\sigma) \subseteq \mathrm{Inv}(\tau).$$

Example.



22. The monomial basis

Definition. $(F_\sigma)_{\sigma\in S_n}$ is the *fundamental basis* of KS_n.

$$F_\sigma = \sum_{\sigma\le\tau} M_\tau$$

defines the *monomial basis* $(M_\tau)_{\tau\in S_n}$ of KS_n. By Möbius inversion

$$M_\sigma = \sum_{\sigma\le\tau} \mu(\sigma,\tau) F_\tau$$

Example.

$$\begin{aligned} M_{(1,2)} &= F_{(1,2)} - F_{(2,1)} \\ M_{(2,1)} &= F_{(2,1)} \end{aligned}$$

31. Main theorem

Compare

$$F_\sigma \circ_i F_\tau = F_{B_i(\sigma,\tau)}$$

with

Theorem.

$$M_\sigma \circ_i M_\tau = \sum_{B_i(\sigma,\tau) \leq \rho \leq T_i(\sigma,\tau)} M_\rho$$

32. Fondamental lemma

The relation

$$M_\sigma \circ_i M_\tau = \sum_{B_i(\sigma,\tau)\leq\rho\leq T_i(\sigma,\tau)} M_\rho$$

can be written

$$M_\sigma \circ_i M_\tau = \sum_{P_i(\rho)=(\sigma,\tau)} M_\rho$$

Lemma. The maps

$$S_n \times S_m \underset{\xrightarrow[B_i]{}}{\overset{\xrightarrow{T_i}}{\xleftarrow{P_i}}} S_{n+m-1}$$

satisfy the following properties :

(i) P_i and B_i are order-preserving.

(ii) $P_i \circ B_i = Id = P_i \circ T_i$.

(iii) $B_i \circ P_i \leq Id \leq T_i \circ P_i$.

The fiber of P_i is precisely the interval $[B_i, T_i]$

33. Construction of T_i and P_i

Example. $\sigma = (6, 2, 3, 4, 1, 5), \tau = (2, 1, 3)$

$$B_4(\sigma, \tau) = (8, 2, 3, 5, 4, 6, 1, 7)$$
$$T_4(\sigma, \tau) = (8, a_1, a_2, b_1, 2, 7, 1, a_3)$$
$$T_4(\sigma, \tau) = (8, 5, 6, 4, 2, 7, 1, 3)$$

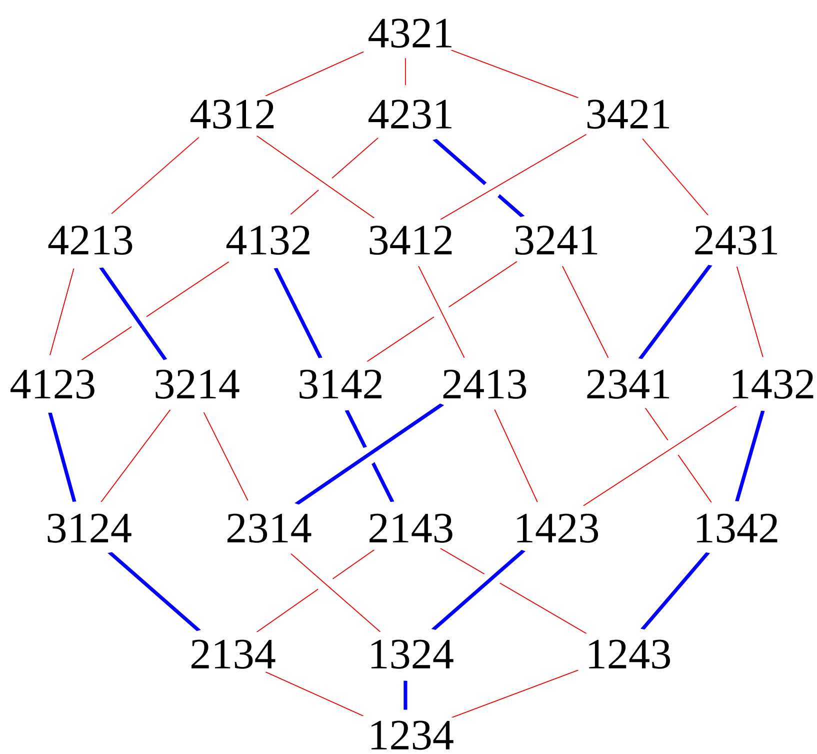
Example. Let $n = 6, m = 3, i = 4$ and

$$\rho = (8, 7, 4, \underbrace{2, 6, 3}_{m=3}, 1, 5)$$

$$P'_4(\rho) = P'_4(8, 7, 4, \underbrace{2, 6, 3}_{m=3}, 1, 5)$$
$$= (8, 7, 2, \underbrace{3, 5, 4}_{m=3}, 1, 6)$$
$$P'_4(\rho) = B_4(\mathrm{st}(8, 7, 2, 3, 1, 6), \mathrm{st}(3, 5, 4))$$
$$P_4(\rho) = ((6, 5, 2, 3, 1, 4), (1, 3, 2))$$

34. Picture

The fibers of $P_1 : S_4 \to S_3 \times S_2$



35. Proof of the theorem

Write

$$M_\sigma \tilde{\circ}_i M_\tau = \sum_{P_i(\rho)=(\sigma,\tau)} M_\rho$$

Since

$$F_\sigma = \sum_{\sigma \leq \sigma'} M_{\sigma'}$$

one gets

$$\begin{aligned} F_\sigma \tilde{\circ}_i F_\tau &= \sum_{(\sigma,\tau)\leq(\sigma',\tau')} M_{\sigma'} \tilde{\circ}_i M_{\tau'} \\ &= \sum_{(\sigma,\tau)\leq P_i(\rho)} M_\rho \\ &= \sum_{B_i(\sigma,\tau)\leq \rho} M_\rho \\ &= F_{B_i(\sigma,\tau)} = F_\sigma \circ_i F_\tau \end{aligned}$$

41. Application : the coradical filtration

$H = \oplus K S_n$ is endowed with a coproduct

$$\Delta(F_\sigma) = \sum_{i=1}^{n-1} F_{\mathrm{st}(\sigma_1,\ldots,\sigma_i)} \otimes F_{\mathrm{st}(\sigma_{i+1},\ldots,\sigma_n)}$$

Definition. $H^k = \ker(\Delta^{k+1} : H \to H^{\otimes k+2})$.
H^0 is the space of *primitive elements*.

Theorem. The sequence (H_n^k) is a filtration of the operad $\mathcal{A}s$:

$$H_n^k \circ_i H_m^h \subset H_{n+m-1}^{k+h}$$

Corollary. The space of primitives H^0 is a sub (non-Σ) operad of $\mathcal{A}s$.

42. Idea of the proof

Definition. A *global descent* p of $\sigma \in S_n$ is an integer such that

$$\forall i \leq p, \forall j > p, \quad \sigma_i > \sigma_j$$

For instance $(3, 5, 4, 1, 2)$ has for global descent set $\{3\}$.

Proposition. [A-S] H^k is generated by the M_σ's such that σ has at most k global descents.

If σ (resp. τ) has at most k (resp. h) global descents then the permutations in $P_i^{-1}(\sigma, \tau)$ have at most $k + h$ global descents.

43. Application : Dynkin's idempotents

The sub-Σ-operad $\mathcal{L}ie$ of $\mathcal{A}s$ is generated by $M_{(1,2)} = F_{(1,2)} - F_{(2,1)}$.

Fact. $\mathcal{L}ie$ is a sub (non-Σ)-operad of H^0.

Definition. $\theta_n \cdot (v_1, \ldots, v_n) = [[\ldots[v_1, v_2], \ldots], v_n]$
Dynkin's idempotent : $\frac{\theta_n}{n}$.

Theorem.

$$\theta_n = \sum_{\sigma \in S_n, \sigma(1)=1} M_\sigma$$

Introduce the twisted Lie bracket on KS_n :

$$\{F_\sigma, F_\tau\} = F_{\begin{bmatrix} \sigma & 0 \\ 0 & \tau \end{bmatrix}} - F_{\begin{bmatrix} 0 & \tau \\ \sigma & 0 \end{bmatrix}}$$

and prove $\theta_n = \{\theta_{n-1}, \theta_1\}$. A closed formula for $\{M_\sigma, M_\tau\}$ yields the result.

51. Link with hypercubes

Operad structure. The surjective map

$$\begin{array}{rccc} \mathsf{Des}: & KS_n & \rightarrow & KQ_n = K\{-1,1\}^{n-1} \\ & F_\sigma & \mapsto & F_{\mathsf{Des}(\sigma)} \end{array}$$

defined by

$$\mathsf{Des}(\sigma) = (\epsilon_1, \ldots, \epsilon_{n-1})$$
$$\epsilon_i = \begin{cases} 1 & \text{if } i \in \mathsf{Des}(\sigma) \\ -1 & \text{if not} \end{cases}$$

induces a non-Σ-operad structure on $\mathcal{Q} = (KQ_n)_n$ given by

$$F_\epsilon \circ_i F_\delta = F_{(\epsilon_1,\ldots,\epsilon_{i-1},\delta_1,\ldots,\delta_{m-1},\epsilon_i\ldots,\epsilon_{n-1})}$$

Algebra structure. Algebras over $S\mathcal{Q}$ are 2-associative algebras $(A, \cdot, *)$ such that

$$(a \cdot b) * c = a \cdot (b * c)$$
$$(a * b) \cdot c = a * (b \cdot c)$$

compare with Richter, Pirashvili, Chapoton.

52. Order structure on hypercubes

Order structure. The surjective map Des induces a weak order on Q_n given by

$$\epsilon \leq \eta \Longleftrightarrow \forall i,\ \epsilon_i \leq \eta_i$$

Monomial basis.

$$F_\epsilon = \sum_{\epsilon \leq \eta} M_\eta$$

Theorem. The map

$$\begin{array}{ccc} \mathcal{Q} & \to & \mathcal{Q} \\ F_\epsilon & \mapsto & M_\epsilon \end{array}$$

is an operad automorphism.

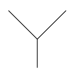
53. Link with planar binary trees

The surjective map Des : $KS_n \to KQ_n$ factors through KY_n

$$S_n \xrightarrow{\lambda} Y_n \xrightarrow{L} Q_n$$

where

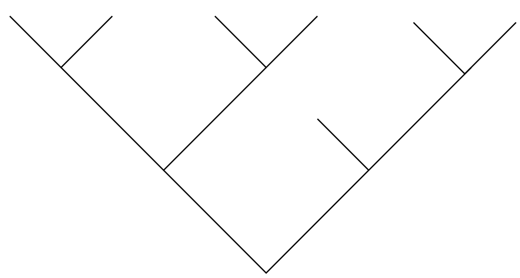
$\lambda(1_0) = |,$

$\lambda(1_1) =$  ,

$\lambda(a_1, \ldots a_{i-1}, n, a_i, \ldots, a_n) =$

$\quad \lambda(\mathrm{st}(a_1, \ldots, a_{i-1})) \vee \lambda(\mathrm{st}(a_i, \ldots, a_n))$

Example.

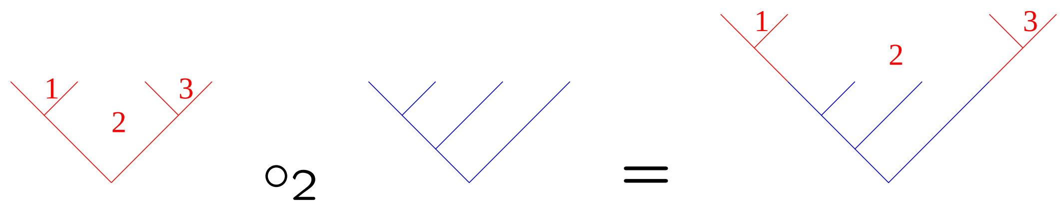
$\lambda(342651) =$ 

54. Operad structure on planar binary trees

The surjective map $\lambda : KS_n \to KY_n$ induces a non-Σ-operad structure on $\mathcal{Y} = (KY_n)_n$

Example.

$$(1,3,2) \quad \circ_2 \quad (1,2,3) \quad = \quad (1,3,4,5,2)$$



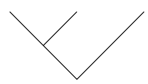
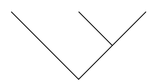
Algebra structure. Algebras over $S\mathcal{Y}$ are 2-associative algebras $(A, \cdot, *)$ such that

$$(a \cdot b) * c = a \cdot (b * c)$$

compare with Pirashvili.

55. Monomial basis and planar binary trees

Order structure. The surjective map λ induces a weak order on Y_n generated by

 $\leq$ 

Monomial basis.

$$F_t = \sum_{t \leq u} M_u$$

Theorem.

$$M_t \circ_i M_u = \sum_{B_i(t,u) \leq s \leq T_i(t,u)} M_s = \sum_{P_i(s)=(t,u)} M_s$$

61. Questions

- The algebras over $S\mathcal{Y}$ and $S\mathcal{Q}$ are well defined.
  $\Rightarrow$ What about $S\mathcal{A}s$?
- One has a morphism of Σ-operads

$$S\mathcal{A}s \to 2\,\mathcal{A}s \to S\mathcal{Y} \to S\mathcal{Q}$$

  In $S\mathcal{A}s$ there are more operations : for instance $(2, 4, 1, 3)$ is not of the form $B_i(\sigma, \tau)$.
  $\Rightarrow$ Question : count the number of permutations σ such that σ does not contain a subsequence of integers describing an interval.

62. Questions

- Work of Chapoton : there are operad structures on the faces of Stasheff polytopes and hypercubes.
  ⇒ Can we describe one on the faces of the permutohedron ? Compatible with the one of Chapoton ?
- There's an order structure (Palacios-Ronco) on the faces of the permutohedron, inducing order structure on Stasheff polytopes and hypercubes.
  ⇒ Is it compatible with the operadic structure ?