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Pre-Lie Algebras and the Rooted Trees Operad

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0 Introduction

We study here a type of algebra which deserves more attention than it has been given. People have been using these algebras, under various names, for a long time. They appeared under the name of left symmetric algebras in the work of E. Vinberg [V] on convex homogeneous cones and so were dubbed Vinberg algebras in some papers. They also appeared in the study of affine manifolds, under the name of right-symmetric algebras (see [Mat]). We propose to adopt the name of pre-Lie algebras, which has been used by M. Gerstenhaber [G]; the Lie bracket involved in the Gerstenhaber structure on the Hochschild cohomology comes from a pre-Lie algebra structure on the cochains. Besides, rooted trees have been shown to be of interest in the study of vector fields, numerical analysis (see, e.g., C. Brouder [B] and the references therein), and more recently in quantum field theory (see [CK]). We define in this paper the underlying operad of pre-Lie algebras in terms of rooted trees, which should shed light on the relationships between these different topics.

The description of the operad defining pre-Lie algebras in terms of rooted trees is the subject of Section 1. The operad arising here should not be confused with the structure on rooted trees appearing in [BV]. Section 2 is devoted to the definition of the operadic homology of pre-Lie algebras. Finally, we prove in Section 3 that the operad associated to pre-Lie algebras is a Koszul operad. To that end, we prove in fact that a free pre-Lie algebra L is a free module over the enveloping algebra of the Lie algebra underlying L. Combined with Section 1, it gives a new interpretation of the Hopf algebra appearing in the work of A. Connes and D. Kreimer [CK].

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1 A description of the operad defining pre-Lie algebras

This section is devoted to the description of the operad defining pre-Lie algebras. We prove in Theorem 1.9 that this operad is the operad of rooted trees.

We recall briefly some facts about operads (see [GiK], [GeJ]). An *operad* $\mathcal{P}$ is a sequence of vector spaces $\mathcal{P}(n)$, for $n \geq 1$, such that $\mathcal{P}(n)$ is a module over the symmetric group S_n, together with composition maps $\gamma : \mathcal{P}(n) \otimes \mathcal{P}(i_1) \otimes \cdots \otimes \mathcal{P}(i_n) \to \mathcal{P}(i_1 + \cdots + i_n)$ satisfying some relations of associativity, unitarity, and equivariance with respect to the symmetric group, called May axioms (see [May]). Note that giving an S_n-module for all n is equivalent to giving an $\mathbb{S}$-*module*, that is, a functor from the category of (finite sets, bijections) to the category of vector spaces. Hence an operad can be defined by an $\mathbb{S}$-module $\mathcal{P}$ together with composition maps $\gamma : \mathcal{P}(I) \otimes \mathcal{P}(J_1) \otimes \cdots \otimes \mathcal{P}(J_n) \to \mathcal{P}(J_1 \sqcup \cdots \sqcup J_n)$, where n is the cardinal of I. An *algebra* over an operad $\mathcal{P}$ is a vector space A together with maps $\mathcal{P}(n) \otimes A^{\otimes n} \to A$ satisfying some relations of associativity, unitarity, and equivariance.

Definition 1.1. A *pre-Lie algebra* is a vector space L together with a bilinear map $\cdot : L \times L \to L$ satisfying the relation

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (x \cdot z) \cdot y - x \cdot (z \cdot y), \quad \forall x, y, z \in L.$$

When the vector space L is graded, we define a *graded pre-Lie algebra* by the relation

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = (-1)^{|y||z|}((x \cdot z) \cdot y - x \cdot (z \cdot y)), \quad \forall x, y, z \in L.$$

Proposition 1.2. Let $(L, \cdot)$ be a pre-Lie algebra. The bracket defined by

$$[a, b] = a \cdot b - b \cdot a, \quad \forall a, b \in L,$$

endows L with a structure of Lie algebra. □

In the sequel $L_{\mathcal{L}ie}$ will denote the Lie algebra $(L, [-, -])$.

Examples 1.3. Gerstenhaber [G] introduced a structure of pre-Lie algebra on the Hochschild complex of an associative algebra A as follows. Denote by $C^m(A, A)$ the space $\mathrm{Hom}(A^{\otimes m}, A)$ in degree $m - 1$. Let $f \in C^m$ and $g \in C^n$; then the product

$$\begin{aligned}&(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1})\\ &= \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f\Big(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1}\Big)\end{aligned}$$

satisfies the graded relation defining pre-Lie algebras.

The structure of pre-Lie algebra also appears in the study of affine structures on manifolds (see [Mat]). An affine structure on an n-manifold is an atlas whose coordinate changes are in the group of affine motions of $\mathbb{R}^n$. It can also be given by a linear connection ∇ whose torsion and curvature vanish. The product $X \circ Y = -\nabla_Y X$ then defines a structure of pre-Lie algebra on the set of vector fields such that the associated Lie bracket is the usual bracket of vector fields.

1.4 The operad $\mathcal{PL}$

From Definition 1.1, it is clear that a pre-Lie algebra is an algebra over a binary quadratic operad, denoted by $\mathcal{PL}$. We recall briefly how to construct $\mathcal{PL}$ (see [GiK]). Let $\mathcal{F}$ be the free operad generated by the regular representation of S_2. A basis of $\mathcal{F}(n)$, as a vector space, is given by "parenthesized products" on n variables indexed by $\{1, \dots, n\}$. For instance, a basis of $\mathcal{F}(2)$ is given by (x_1x_2) and (x_2x_1), and a basis of $\mathcal{F}(3)$ is given by $((x_1x_2)x_3)$, $(x_1(x_2x_3))$, and all their permutations. Let R be the S_3-submodule of $\mathcal{F}(3)$ generated by the relation $r = ((x_1x_2)x_3) - (x_1(x_2x_3)) - ((x_1x_3)x_2) + (x_1(x_3x_2))$. Then $\mathcal{PL} = \mathcal{F}/(R)$, where (R) denotes the ideal of $\mathcal{F}$ generated by R. The operadic composition on $\mathcal{PL}$ is induced by the one on $\mathcal{F}$, given by

$$\gamma : \mathcal{F}(n) \otimes \mathcal{F}(i_1) \otimes \cdots \otimes \mathcal{F}(i_n) \longrightarrow \mathcal{F}(i_1 + \cdots + i_n),$$

which assigns to $(\mu, \nu_1, \dots, \nu_n)$ the word obtained by substituting ν_i for $x_i \in \mu$. Notice that the concatenation $(\rho\rho')$ is the particular case of the composition $\gamma((x_1x_2), \rho, \rho')$.

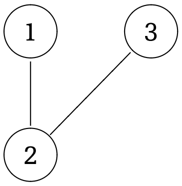
1.5 The operad of rooted trees $\mathcal{RT}$

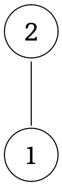
Let $n > 0$. A *rooted tree* of degree n, or n-rooted tree, is a nonempty connected graph without loops whose vertices are labeled by the set $[n] = \{1, \dots, n\}$, together with a distinguished element in this set called the root. Edges of this graph are oriented towards the root. We denote by $\mathcal{RT}(n)$ the free $\mathbb{Z}$-module generated by n-rooted trees. We can endow $\mathcal{RT} = (\mathcal{RT}(n))_{n \geq 1}$ with an operad structure, as explained below.

The action of the symmetric group is the natural one, by permutation of indices. Let T be an n-rooted tree; denote by $\mathrm{In}(T, i)$ the set of incoming edges at the vertex i of T. Let S be an m-rooted tree. In order to define the operadic composition, describing the compositions $\circ_i : \mathcal{RT}(n) \times \mathcal{RT}(m) \to \mathcal{RT}(n + m - 1)$ for $1 \leq i \leq n$ (see, e.g., [Lo]) is enough. We define the composition of T and S along the vertex i of T by

$$T \circ_i S = \sum_{f : \mathrm{In}(T,i) \to [m]} T \circ_i^f S,$$

where $T \circ_i^f S$ is the rooted tree obtained by substituting the tree S for the vertex i in T. The outgoing edge of i, if it exists, becomes the outgoing edge of the root of S, whereas incoming edges of i are grafted on the vertices of S following the map f. Then it is easy to check that these compositions endow $\mathcal{RT}$ with a structure of operad. Let us give an example. A rooted tree is drawn with its root at the bottom. Let

$T =$ 

and $S =$ 

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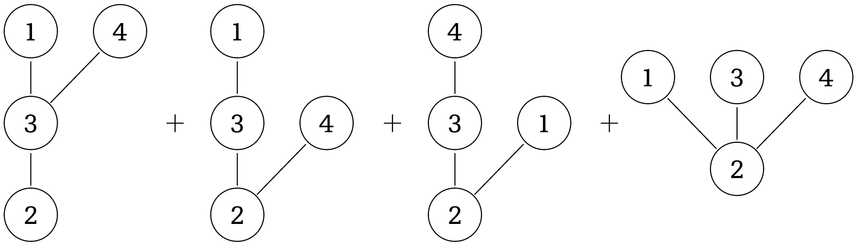
By describing maps from $\{1,3\}$ to $\{1,2\}$ and by reindexing the vertices of S and T, one gets

$T \circ_2 S =$ 

$$f : \begin{cases} 1 \longrightarrow 2 \\ 3 \longrightarrow 2 \end{cases} \quad f : \begin{cases} 1 \longrightarrow 2 \\ 3 \longrightarrow 1 \end{cases} \quad f : \begin{cases} 1 \longrightarrow 1 \\ 3 \longrightarrow 2 \end{cases} \quad f : \begin{cases} 1 \longrightarrow 1 \\ 3 \longrightarrow 1 \end{cases}.$$

1.6 The Poincaré series associated to $\mathcal{RT}$

The Poincaré series is defined by

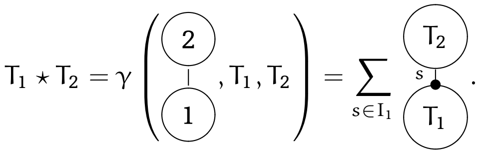
$$g_{\mathcal{RT}}(x) = \sum_{n \geq 1} \dim(\mathcal{RT}(n)) \frac{(-x)^n}{n!}.$$

Using classical results in combinatorics (see [W]), one obtains that $\dim(\mathcal{RT}(n)) = n^{n-1}$ and that $g_{\mathcal{RT}}$ is the inverse map of $x \mapsto -xe^{-x}$.

1.7 A product in the operad $\mathcal{RT}$

Recall that the operadic composition for $\mathcal{RT}$, $\gamma : \mathcal{RT}(n) \otimes \mathcal{RT}(i_1) \otimes \cdots \otimes \mathcal{RT}(i_n) \to \mathcal{RT}(i_1 + \cdots + i_n)$ is given in terms of the $\circ_i$ compositions by $\gamma(\mu, \nu_1, \ldots, \nu_n) = ((\cdots(\mu \circ_n \nu_n) \circ_{n-1} \nu_{n-1}) \cdots \circ_1 \nu_1)$.

We may then define a particular composition that corresponds to the concatenation in the $\mathcal{PL}$-case (see Section 1.4): for any rooted trees T_1 and T_2, let



$$T_1 \star T_2 = \gamma\left(\begin{array}{c}\textcircled{2}\\ | \\ \textcircled{1}\end{array}, T_1, T_2\right) = \sum_{s\in I_1} \begin{array}{c} \textcircled{T_2} \\ {}^{s}| \\ \textcircled{T_1}\end{array}.$$

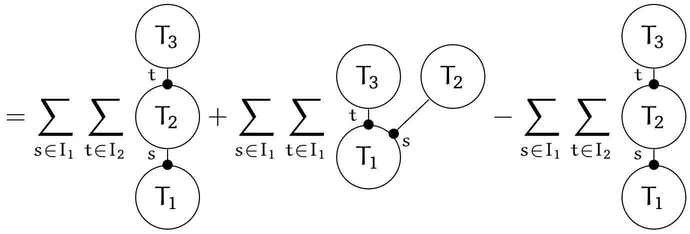
More explicitly, the operation $T_1 \star T_2$ consists of grafting the root of T_2 on every vertex of T_1.

Lemma 1.8. The following relation holds for any rooted trees T_j, $1 \le j \le 3$:

$$(T_1 \star T_2) \star T_3 - T_1 \star (T_2 \star T_3) = (T_1 \star T_3) \star T_2 - T_1 \star (T_3 \star T_2). \qquad \square$$

Proof. By computing

$$(T_1 \star T_2) \star T_3 - T_1 \star (T_2 \star T_3)$$

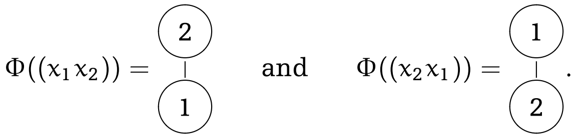


$$= \sum_{s\in I_1}\sum_{t\in I_2} \begin{array}{c} \textcircled{T_3} \\ {}^{t}| \\ \textcircled{T_2} \\ {}^{s}| \\ \textcircled{T_1}\end{array} + \sum_{s\in I_1}\sum_{t\in I_1} \begin{array}{cc} \textcircled{T_3} & \textcircled{T_2} \\ {}^{t}| & /_{s} \\ \multicolumn{2}{c}{\textcircled{T_1}}\end{array} - \sum_{s\in I_1}\sum_{t\in I_2} \begin{array}{c} \textcircled{T_3} \\ {}^{t}| \\ \textcircled{T_2} \\ {}^{s}| \\ \textcircled{T_1}\end{array}$$

and inverting the roles of T_2 and T_3, the required equality is obtained. ■

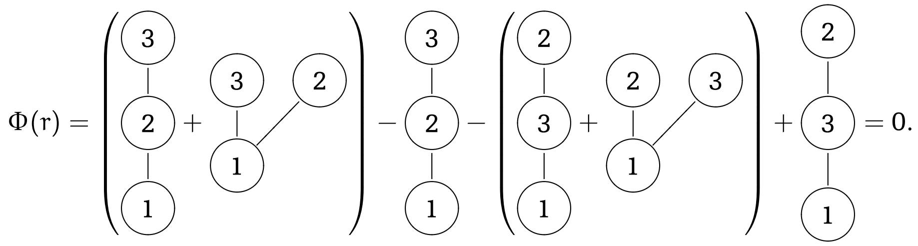
Theorem 1.9. The operad $\mathcal{PL}$ defining pre-Lie algebras is isomorphic to the operad of rooted trees $\mathcal{RT}$. □

Proof. First, we define an operadic morphism $\Phi : \mathcal{PL} \to \mathcal{RT}$. Since $\mathcal{PL} = \mathcal{F}/(R)$ (see Section 1.4), it is sufficient to define Φ on $\mathcal{PL}(2) = \mathcal{F}(2)$, then extend it on $\mathcal{F}$ by the universal property of the free operad, and check that $\Phi(r) = 0$. Set



$$\Phi((x_1x_2)) = \begin{array}{c}\textcircled{2}\\ | \\ \textcircled{1}\end{array} \quad \text{and} \quad \Phi((x_2x_1)) = \begin{array}{c}\textcircled{1}\\ | \\ \textcircled{2}\end{array}.$$

Hence



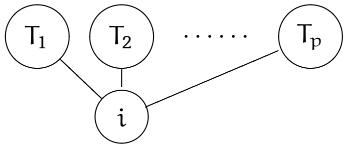
Remark that, since Φ is an operadic morphism, it sends the concatenation product (see Section 1.4) to the product $\star$ defined in Section 1.7.

The proof relies on the existence of an inverse Ψ of Φ. As we explained in the introduction of Section 1, it is more convenient for the proof to deal with $\mathbb{S}$-modules. For a finite set I, an I-*rooted tree* is a rooted tree whose vertices are labeled by I; its degree is the cardinal of I. We denote by $\Phi_I : \mathcal{PL}(I) \to \mathcal{RT}(I)$ the natural extension of Φ.

Claim. For any finite set I, there exists a map $\Psi_I : \mathcal{RT}(I) \to \mathcal{PL}(I)$ such that $\Psi_I\Phi_I = \mathrm{Id}$ and $\Phi_I\Psi_I = \mathrm{Id}$. ☐

We prove the claim by induction on $\#I$. If $I = \{i\}$, then it is trivial. Assume that the claim is true for any I such that $\#I \leq n$, and let I be a finite set of cardinal $n + 1$. Let T be an I-rooted tree, and let i be its root. Up to a permutation, we can write uniquely

$$T = B(i, T_1, \ldots, T_p) =$$



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where T_i, for $1 \leq i \leq p$, is a rooted tree of degree strictly less than $n + 1$. Let us define the map Ψ_I by induction on p.

If $p = 1$, then $T = B(i, T_1) = \textcircled{i} \star T_1$ and $\Psi_I(T) = (i\Psi_I(T_1))$ is well defined. Moreover, $\Phi_I\Psi_I(T) = T$ because Φ_I sends concatenation to the product $\star$ and because of the induction hypothesis. For $p \geq 2$, one has

$$T = B(i, T_2, \ldots, T_p) \star T_1 - \sum_{j=2}^{p} B(i, T_2, \ldots, T_j \star T_1, \ldots, T_p).$$

Consequently, we may define by induction:

$$\Psi_I(T) = (\Psi_I(B(i, T_2, \ldots, T_p))\Psi_I(T_1)) - \sum_{j=2}^{p} \Psi_I(B(i, T_2, \ldots, T_j \star T_1, \ldots, T_p)).$$

Moreover, as in the case $p = 1$, $\Phi_I\Psi_I(T) = T$.

A priori, since T is uniquely determined only up to a permutation, this definition depends on the choice of the edge we cut in the tree T. Let us prove by induction on p that it is not the case. For $p = 0, 1$, there is no choice; for $p > 1$, we prove that ungrafting the edge where T_1 lies, then ungrafting edges where T_2 lies in the trees involved in the sum, gives the same definition of $\Psi_I(T)$ as doing it by inverting T_1 and T_2. By ungrafting T_2 in the previous relation, we get

$$\begin{aligned}
T = {} & \big(B(i, T_3, \dots, T_p) \star T_2\big) \star T_1 - \sum_{k=3}^{p} B\big(i, T_3, \dots, T_k \star T_2, \dots, T_p\big) \star T_1 \\
& - B\big(i, T_3, \dots, T_p\big) \star \big(T_2 \star T_1\big) + \sum_{j=3}^{p} B\big(i, T_3, \dots, T_j \star (T_2 \star T_1), \dots, T_p\big) \\
& - \sum_{j=3}^{p} B\big(i, T_3, \dots, T_j \star T_1, \dots, T_p\big) \star T_2 + \sum_{j=3}^{p} B\big(i, T_3, \dots, (T_j \star T_1) \star T_2, \dots, T_p\big) \\
& + \sum_{j=3}^{p} \sum_{\substack{k=3 \\ k \neq j}}^{p} B\big(i, T_3, \dots, T_k \star T_2, \dots, T_j \star T_1, \dots, T_p\big).
\end{aligned}$$

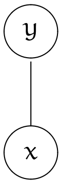
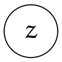
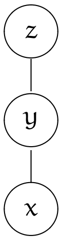
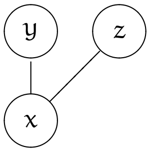
Let $A_{12} = ((\Psi_I B(i, T_3, \dots, T_p)\Psi_I(T_2))\Psi_I(T_1)) - (\Psi_I B(i, T_3, \dots, T_p)(\Psi_I(T_2)\Psi_I(T_1)))$, and let A_{21} be the same term with T_1 and T_2 inverted; since $A_{21} - A_{12} \in (R)$, these terms coincide in $\mathcal{PL}$. It is clear that the terms $B_{12}^{k} = (\Psi_I B(i, T_3, \dots, T_k \star T_2, \dots, T_p)\Psi_I(T_1)) + (\Psi_I B(i, T_3, \dots, T_k \star T_1, \dots, T_p)\Psi_I(T_2))$ and B_{21}^{k} coincide, as well as the terms $C_{12}^{jk} = \Psi_I(B(i, T_3, \dots, T_k \star T_2, \dots, T_j \star T_1, \dots, T_p))$ and C_{21}^{kj}. Finally, the terms $D_{12}^{j} = \Psi_I(B(i, T_3, \dots, (T_j \star T_1) \star T_2, \dots, T_p)) + \Psi_I(B(i, T_3, \dots, T_j \star (T_2 \star T_1), \dots, T_p))$ and D_{21}^{j} coincide, thanks to Lemma 1.8.

Furthermore, $\Psi\Phi = \mathrm{Id}$. In fact, one has $\Psi(T \star T') = (\Psi(T)\Psi(T'))$. Indeed, when $T = \textcircled{i}$, the result comes from the case $p = 1$; if $T = B(i, T_1, \dots, T_p)$, then $T \star T' = B(i, T', T_1, \dots, T_p) + \sum_{k=1}^{p} B(i, T_1, \dots, T_k \star T', \dots, T_p)$, and by choosing to ungraft T' in order to define Ψ, we get the result. As a consequence, let μ be a word in $\mathcal{PL}$; hence μ can be uniquely decomposed in a concatenation $\mu = (\rho\rho')$; since $\Phi(\mu) = \Phi(\rho) \star \Phi(\rho')$, then $\Psi\Phi(\mu) = (\Psi\Phi(\rho)\Psi\Phi(\rho'))$, and we can conclude by induction on the length of μ. This ends the proof of the theorem. ■

The description of free pre-Lie algebras is a direct consequence of the previous results.

Corollary 1.10. Let V be a vector space. The free pre-Lie algebra generated by V is the vector space generated by the rooted trees labeled by a basis of V, with the product $\star$ defined in Section 1.7. Let T_1 and T_2 be two trees labeled by a basis of V; then $T_1 \star T_2$ is the sum over the vertices v of T_1 of trees obtained by linking with an edge the root of T_2 to the vertex v of T_1. □

For instance, let

$T_1 =$  and $T_2 =$ . Then $T_1 \star T_2 =$  +  .

2 Homology of pre-Lie algebras

Since the homology of pre-Lie algebras already has been defined by A. Nijenhuis [N] and extended by A. Dzhumadildaev [D], the aim of this section is to understand how the operad theory can lead naturally to the definition of the operadic homology of a type of algebra, which hopefully coincides with some definitions given earlier.

Following V. Ginzburg and M. Kapranov [GiK], in order to define the operadic homology of a pre-Lie algebra, it is necessary to introduce the complex built on the free coalgebra on the dual operad of $\mathcal{PL}$ whose differential is the only coderivation induced by the pre-Lie product.

Notation. In the sequel, the ground field K will be of characteristic zero.

A $(k_1, \dots, k_p)$-*shuffle* is a permutation $\sigma \in S_{k_1+\dots+k_p}$ such that $\sigma(1) < \dots < \sigma(k_1)$, $\sigma(k_1+1) < \dots < \sigma(k_1+k_2)$, and so on. We denote by $Sh_{k_1,\dots,k_p}$ the set of all $(k_1, \dots, k_p)$-shuffles. A permutation $\sigma \in S_n$ acts on $V^{\otimes n}$ by $\sigma \cdot (v_1 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$ and in case V is a graded vector space we denote by $\epsilon(\sigma, \bar{v})$ the sign appearing in this action. For instance, $\epsilon((12), \bar{v}) = (-1)^{|v_1||v_2|}$.

Let V be a graded vector space; we denote by $S(V)$ the *graded symmetric algebra* generated by V. That means $S(V)$ is the quotient of the free associative algebra $T(V)$ generated by V by the ideal generated by $x \otimes y - (-1)^{|x||y|} y \otimes x, \forall x, y \in V$. If V is concentrated in degree 1, it becomes the exterior algebra on V and is usually denoted by $\bigwedge(V)$. The *suspension* of V, denoted by sV, is defined by $(sV)_n = V_{n-1}$.

Proposition 2.1. The dual operad of the pre-Lie operad is the operad $\mathcal{P}erm$ defined in [Ch]. A $\mathcal{P}erm$-algebra is a vector space A together with a bilinear product $\cdot : A \times A \to A$ satisfying the relations

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \qquad a \cdot b \cdot c = a \cdot c \cdot b, \quad \forall a, b, c \in A. \qquad \square$$

Proof. Recall the definition of the quadratic dual operad $\mathcal{P}^!$ of an operad $\mathcal{P}$ in our framework, with the notation of Section 1.4. If $\mathcal{P} = \mathcal{F}/(R)$, where $\mathcal{F}$ is the free operad generated by the regular representation of S_2, then there is a scalar product on $\mathcal{F}(3)$ defined by

$$\langle i(jk), i(jk)\rangle = \operatorname{sgn}\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}, \qquad \langle (ij)k, (ij)k\rangle = -\operatorname{sgn}\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix},$$

and $\mathcal{P}^! = \mathcal{F}/(R^\perp)$, where $R^\perp$ is the annihilator of R with respect to this scalar product [GiK].

Let R be the S_3-submodule of $\mathcal{F}(3)$ defined in Section 1.4, and let R' be the S_3-submodule of $\mathcal{F}(3)$ generated by the relations $s = ((x_1x_2)x_3) - (x_1(x_2x_3))$ and $t = ((x_1x_2)x_3) - ((x_1x_3)x_2)$. Since $\langle R, R'\rangle = 0$ and the vector spaces R and R' are, respectively, of dimension 3 and 9, we can conclude that $R' = R^\perp$. ■

Definition 2.2. A *graded $\mathcal{P}$erm-coalgebra* C is a positively graded vector space equipped with a comultiplication $\Delta : C \to C \otimes C$ of degree zero, satisfying the following identities:

$$(\mathrm{Id}\otimes\Delta)\Delta = (\Delta\otimes\mathrm{Id})\Delta, \qquad (\mathrm{Id}\otimes\Delta)\Delta = (\mathrm{Id}\otimes T)(\mathrm{Id}\otimes\Delta)\Delta,$$

where $T(a\otimes b) = (-1)^{|a||b|}b\otimes a$.

A *coderivation* of C is a linear map $d : C \to C$ satisfying $\Delta d = (\mathrm{Id}\otimes d)\Delta + (d\otimes\mathrm{Id})\Delta$. The space of all coderivations of C is denoted by $\mathrm{Coder}(C)$.

The proof of the following lemma is left to the reader.

Lemma 2.3. Let V be a reduced graded vector space; that is, $V_0 = 0$. Then the free $\mathcal{P}erm$-coalgebra generated by V, denoted by $\mathcal{P}erm^c(V)$, is the vector space $V\otimes S(V)$ equipped with the following comultiplication:

$$\Delta(v\otimes 1) = 0, \quad \forall v\in V,$$

$$\Delta(v_0\otimes v_1\cdots v_n) = \sum_{\substack{0\le k\le n-1\\ \sigma\in \mathrm{Sh}_{k,1,n-1-k}}} \epsilon(\sigma,\bar v)\, v_0\otimes v_{\sigma(1)}\cdots v_{\sigma(k)}\otimes v_{\sigma(k+1)}\otimes v_{\sigma(k+2)}\cdots v_{\sigma(n)},$$

$$\forall v_0\in V,\ v_1\cdots v_n\in S^n(V).$$

The space $\mathcal{P}erm^c(V)$ comes with a natural projection onto V, denoted by π. □

Proposition 2.4. The following isomorphism of vector spaces holds:

$$\phi : \mathrm{Coder}\left(\mathcal{P}erm^c(V)\right) \longrightarrow \mathrm{Hom}\left(\mathcal{P}erm^c(V), V\right), \qquad d \longmapsto \pi\circ d.$$

Moreover, there exists a graded pre-Lie algebra structure on $\mathrm{Hom}(\mathcal{P}erm^c(V), V)$ such that ϕ is an isomorphism of graded Lie algebras; the Lie bracket on $\mathrm{Coder}(\mathcal{P}erm^c(V))$ is defined by $[d^1, d^2] = d^1d^2 - (-1)^{|d^1||d^2|}d^2d^1$. The pre-Lie product $\circ$ is the following one.

Take $l, m \in \mathrm{Hom}(\mathcal{P}erm^c(V), V)$, and set $l = \sum_i l_i$ with $l_i : V \otimes S^i(V) \to V$. Then

$$
\begin{aligned}
&(m \circ l)_n(v_0 \otimes v_1 \cdots v_n) \\
&\quad = \sum_{\substack{0 \leq i \leq n \\ \sigma \in Sh_{i,n-i}}} \epsilon(\sigma, \bar{v}) m_{n-i}\big(l_i(v_0 \otimes v_{\sigma(1)} \cdots v_{\sigma(i)}) \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)}\big) \\
&\qquad + (-1)^{|v_0||l|} \sum_{\substack{0 \leq i \leq n-1 \\ \sigma \in Sh_{1,i,n-i-1}}} \epsilon(\sigma, \bar{v})\, m_{n-i}\big(v_0 \otimes l_i\big(v_{\sigma(1)} \\
&\qquad\qquad \otimes v_{\sigma(2)} \cdots v_{\sigma(i+1)}\big) v_{\sigma(i+2)} \cdots v_{\sigma(n)}\big).
\end{aligned}
$$

□

Proof. The proof consists essentially of computation. The first part of the proposition relies on the fact that the map $\psi : \mathrm{Hom}(\mathcal{P}erm^c(V), V) \to \mathrm{Coder}(\mathcal{P}erm^c(V))$, defined by

$$
\begin{aligned}
&\psi(l)\big(v_0 \otimes v_1 \cdots v_n\big) \\
&\quad = \sum_{i=0}^{n} \sum_{\sigma \in Sh_{i,n-i}} \epsilon(\sigma, \bar{v}) l_i\big(v_0 \otimes v_{\sigma(1)} \cdots v_{\sigma(i)}\big) \otimes v_{\sigma(i+1)} \cdots v_{\sigma(n)} \\
&\qquad + (-1)^{|v_0||l|} \sum_{i=0}^{n-1} \sum_{\sigma \in Sh_{1,n-i-1}} \epsilon(\sigma, \bar{v}) v_0 \\
&\qquad\qquad \otimes l_i\big(v_{\sigma(1)} \otimes v_{\sigma(2)} \cdots v_{\sigma(i+1)}\big) v_{\sigma(i+2)} \cdots v_{\sigma(n)},
\end{aligned}
$$

is the inverse map of ϕ; this definition implies also that ϕ is a Lie-algebra morphism. The fact that the product $\circ$ is a pre-Lie-algebra morphism is a straightforward calculation.

■

This proposition yields naturally the definition of a pre-Lie algebra up to homotopy.

Definition 2.5. A graded vector space V is a *pre-Lie algebra up to homotopy* or a $\mathcal{PL}_\infty$-*algebra* if it is equipped with a map $l \in \mathrm{Hom}(\mathcal{P}erm^c(sV), sV)$ of degree -1 such that $l \circ l = 0$ or, equivalently, $[l, l] = 0$.

Furthermore, this proposition leads to the definition of the homology of a pre-Lie algebra. Indeed, let $(L, \cdot)$ be a pre-Lie algebra, and let $\mu : (sL \otimes sL) \to (sL)$ be the map of degree -1 defined by $\mu(sx \otimes sy) = (-1)^{|sx|} s(x \cdot y)$. Hence, by Definition 1.1 (of a pre-Lie algebra), one gets $\mu \circ \mu = 0$, then $[\mu, \mu] = 0$; thus the coderivation d, induced by the isomorphism, satisfies $d^2 = 0$. The complex so obtained is the one defining the operadic homology of a pre-Lie algebra, in the sense of [GiK]. Using Koszul sign rules and the isomorphism $sL \simeq e \otimes L$, with e a formal element of degree 1, one gets the definition of the homology of a pre-Lie algebra.

2.6 Homology of pre-Lie algebras

Let L be a pre-Lie algebra. The *Pre-Lie homology* of L, denoted by $HPL(L)$, is the homology of the complex $(CPL_n(L), d)$, where $CPL_n(L) = L \otimes \bigwedge^{n-1}(L)$ and

$$\begin{aligned} d\big(v_0 \otimes v_1 \wedge \cdots \wedge v_n\big) = &\sum_{1 \le j \le n} (-1)^j v_0 \cdot v_j \otimes v_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j-1} v_0 \otimes [v_i, v_j] \wedge \cdots \wedge \widehat{v}_i \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_n, \end{aligned}$$

where the bracket is the Lie bracket in L (see Proposition 1.2).

This complex coincides with the one defined by Nijenhuis [N]; for a complete definition of the cohomology of a pre-Lie algebra with coefficients in a representation, we refer to [D].

2.7 On the link between a pre-Lie algebra and its induced Lie algebra

Let $(L, \cdot)$ be a pre-Lie algebra, and denote by $(L_{\mathcal{L}ie}, [-,-])$ its induced Lie algebra (see Proposition 1.2). Then the relation defining the pre-Lie algebra structure implies that L is a right module over $L_{\mathcal{L}ie}$ via the action

$$L \times L_{\mathcal{L}ie} \longrightarrow L, \qquad (v, g) \longmapsto v \cdot g.$$

Hence L is a right module over the enveloping algebra $\mathcal{U}(L_{\mathcal{L}ie})$ of $L_{\mathcal{L}ie}$, with the usual definition $l \cdot (a_1 \otimes \cdots \otimes a_n) = ((\cdots (l \cdot a_1) \cdot a_2) \cdots a_n)$.

As a consequence, there is a nice interpretation of the pre-Lie homology of L in terms of the Chevalley-Eilenberg homology of $L_{\mathcal{L}ie}$ with coefficients in L; one has the isomorphisms

$$HPL_{n+1}(L) \simeq H_n^{CE}(L_{\mathcal{L}ie}, L) \simeq \mathrm{Tor}_n^{\mathcal{U}(L_{\mathcal{L}ie})}(L, K).$$

3 Koszulness of the operad defining pre-Lie algebras

The aim of this section is to prove that the operad $\mathcal{PL}$ is a Koszul operad. As explained in [GiK], it is enough to prove that for any free pre-Lie algebra L, its homology $HPL(L)$ is concentrated in degree 1. In fact, the main point is that a free pre-Lie algebra L is a free right $\mathcal{U}(L_{\mathcal{L}ie})$-module (see Theorem 3.3). Then the Koszulness of the operad follows with the help of Section 2.7.

Before proving Theorem 3.3 , we would like to point out some interesting remarks on free pre-Lie algebras.

3.1 A link with the Connes-Kreimer Hopf algebra

As a Lie algebra, the free pre-Lie algebra on a single generator (see Corollary 1.10) has already appeared in the work of Connes and Kreimer [CK] on the combinatorics of renormalization. They consider a commutative Hopf algebra of polynomials in rooted trees. By the Milnor-Moore theorem, the dual Hopf algebra is the universal enveloping algebra of some Lie algebra, which they calculated in [CK]. This Lie algebra has a basis indexed by rooted trees, and one can check that the bracket is the same as the one induced by the pre-Lie structure of the free pre-Lie algebra.

Lemma 3.2. Let L be a pre-Lie algebra, let V be a vector space, and let $\sigma : V \to L$ be a morphism. The right $\mathcal{U}(L_{\mathcal{L}ie})$-module $V \otimes \mathcal{U}(L_{\mathcal{L}ie})$ can be equipped with a structure of pre-Lie algebra such that the map $\widetilde{\sigma} : V \otimes \mathcal{U}(L_{\mathcal{L}ie}) \to L$ defined by $\widetilde{\sigma}(v \otimes u) = \sigma(v) \star u$, where the action of $\mathcal{U}(L_{\mathcal{L}ie})$ on L is denoted by $\star$, becomes a morphism of pre-Lie algebras. □

Proof. An element in $V \otimes \mathcal{U}(L_{\mathcal{L}ie})$ is denoted by (v, u). The product of pre-Lie algebra on $V \otimes \mathcal{U}(L_{\mathcal{L}ie})$ is defined as

$$(v, u) * (v', u') = \big(v, u \otimes \big(\sigma(v') \star u'\big)\big), \quad \forall v, v' \in V, u, u' \in \mathcal{U}(L_{\mathcal{L}ie}).$$

Let us check the relation $R = (A * B) * C - A * (B * C) - (A * C) * B + A * (C * B) = 0$ in $V \otimes \mathcal{U}(L_{\mathcal{L}ie})$. Let $A = (v, u),\ B = (v', u'),\ C = (v'', u'')$. Then

$$\begin{aligned}(A * B) * C - A * (B * C) &= \big(v, u \otimes \sigma(v') \star u' \otimes \sigma(v'') \star u''\big)\\ &\quad - \big(v, u \otimes \sigma(v') \star \big(u' \otimes \sigma(v'') \star u''\big)\big).\end{aligned}$$

But since L is a right $\mathcal{U}(L_{\mathcal{L}ie})$-module, $\sigma(v') \star (u' \otimes \sigma(v'') \star u'') = (\sigma(v') \star u') \star (\sigma(v'') \star u'')$. Let $\alpha = \sigma(v') \star u'$, and let $\beta = \sigma(v'') \star u''$. Since α and β lie in L, the action $\alpha \star \beta$ coincides with the pre-Lie product in L; thus $\alpha \star \beta - \beta \star \alpha = [\alpha, \beta] \in L_{\mathcal{L}ie}$. As a consequence,

$$R = \big(v, u \otimes \big(\alpha \otimes \beta - \beta \otimes \alpha - [\alpha, \beta]\big)\big) = 0.$$

Then it is clear that $\widetilde{\sigma}$ is a morphism of pre-Lie algebras. ■

Theorem 3.3. Let L be a free pre-Lie algebra generated by a vector space V. Then there is an isomorphism of right $\mathcal{U}(L_{\mathcal{L}ie})$-module

$$L \simeq V \otimes \mathcal{U}(L_{\mathcal{L}ie}).$$ □

Proof. Let σ be the canonical morphism from V to L, let $U = \mathcal{U}(L_{\mathcal{L}ie})$, and let τ be the morphism from V to $V \otimes U$ such that $\tau(v) = v \otimes 1$. The product in L is denoted by $\star$ as well as the action of U on L.

By the universal property of the free right U-module $V \otimes U$, there is a unique right U-module morphism $\psi : V \otimes U \to L$ such that $\psi\tau = \sigma$. Indeed, ψ is the morphism

$$\psi : V \otimes U \longrightarrow L, \qquad v \otimes u \longmapsto \sigma(v) \star u,$$

and, by virtue of Lemma 3.2 , it is a pre-Lie algebras morphism for the product on $V \otimes U$ given by $(v, u) * (v', u') = (v, u \otimes (\sigma(v') \star u'))$. Furthermore, the universal property of the free pre-Lie algebra L implies that there is a unique pre-Lie algebras morphism $\phi : L \to V \otimes U$ such that $\phi\sigma = \tau$. As a consequence, the fact that $\psi\phi\sigma = \sigma$ and that $\psi\phi : L \to L$ is a pre-Lie algebras morphism, implies $\psi\phi = \mathrm{Id}$.

In order to conclude, it is sufficient to prove that ϕ is a right U-module morphism, because it implies that $\phi\psi = \mathrm{Id}$. The action of U on $V \otimes U$ is the concatenation denoted by $(v \otimes u) \cdot (a_1 \otimes \cdots \otimes a_n) = (v, u \otimes a_1 \otimes \cdots \otimes a_n)$. Since U is generated by L, it is sufficient to prove that $\phi(x \star y) = \phi(x) \cdot y$, $\forall x, y \in L$. Now if ϕ is a pre-Lie algebras morphism, then $\phi(x \star y) = \phi(x) * \phi(y)$. We set $\phi(x) = \sum_i (v_i, u_i) \in V \otimes U$ and $\phi(y) = \sum_j (w_j, r_j) \in V \otimes U$. Therefore

$$\begin{aligned}\phi(x) * \phi(y) &= \sum_{i,j} \big(v_i, u_i \otimes \sigma(w_j) \star r_j\big) \\ &= \sum_{i,j} \big(v_i, u_i \otimes \psi(w_j \otimes r_j)\big) \\ &= \sum_{i} \big(v_i, u_i \otimes \psi\phi(y)\big).\end{aligned}$$

But $\psi\phi = \mathrm{Id}$; hence $\phi(x) * \phi(y) = \phi(x) \cdot y$. ■

Theorem 3.4. The operad defining pre-Lie algebras is a Koszul operad. □

Proof. Let L be the free pre-Lie algebra generated by the vector space V. By virtue of Theorem 3.3 , L is the free right $\mathcal{U}(L_{\mathcal{L}ie})$-module on V; hence, by virtue of Section 2.7 ,

$$\mathrm{HPL}_n(L) = \mathrm{Tor}_{n-1}^{\mathcal{U}(L_{\mathcal{L}ie})}\big(V \otimes \mathcal{U}(L_{\mathcal{L}ie}), K\big) = \begin{cases} V & \text{if } n = 1, \\ 0 & \text{if not.} \end{cases}$$ ■

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