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On a Plus-Construction for Algebras over an Operad

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Abstract. We prove a theorem analogous to Quillen's plus-construction in the category of algebras over an operad. For that purpose we prove that this category is a closed model category and prove the existence of an obstruction theory. We apply further this plus-construction for the specific cases of Lie algebras and Leibniz algebras which are a noncommutative version of Lie algebras: let $\mathrm{sl}(A)$ be the kernel of the trace map $\mathrm{gl}(A) \to A/[A, A]$, where A is an associative algebra with unit and $\mathrm{gl}(A)$ is the Lie algebra of matrices over A. Then the homotopy of $\mathrm{sl}(A)^+$ in the category of Lie algebras is the cyclic homology of A whereas it is the Hochschild homology of A in the category of Leibniz algebras.

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0. Introduction

Quillen's plus-construction creates, from a connected CW-complex X whose π_1 is perfect, a pair (i, X^+) where X^+ is a simply connected CW-complex and $i: X \to X^+$ induces an isomorphism in homology [9, 17]. One of the most important roles played by this construction is in defining the K-theory of an associative algebra A with unit: the K-theory of A is the homotopy of $\mathrm{BGL}(A)^+$. Moreover, the K-theory of A is rationally the primitive part of the homology of $\mathrm{BGL}(A)$. One notices that the theorem of J.-L. Loday and D. Quillen [10, 12], proved independently by B. Tsygan [20], has the same form, that is the cyclic homology of A is the primitive part of $\mathrm{gl}(A)$, where $\mathrm{gl}(A)$ is the Lie algebra of matrices over A. That is to say

$$K_*(A) \otimes \mathbb{Q} \cong \mathrm{Prim}\, H_*(\mathrm{BGL}(A), \mathbb{Q}); \quad HC_*(A) \cong \mathrm{Prim}\, H_{*+1}(\mathrm{gl}(A), \mathbb{Q}).$$

Our first motivation is to prove the existence of a plus-construction in the category of Lie algebras such that cyclic homology can be interpreted as the homotopy of $\mathrm{gl}(A)^+$ (see Proposition 5.2). Moreover, we prove that there is also a plus-construction in the category of Leibniz algebras, which are a noncommutative setting of Lie algebras and we recover Hochschild homology by an analogue of Proposition 5.2 (see Proposition 5.3). Since this construction does not depend on the category of algebras, we have decided to present it in its full generality and to develop a plus-construction in the category of algebras over an operad.

The first section provides material needed about operads for the next sections. In the second section, we consider the category of algebras over an operad as a model category in the sense of Quillen, then define the homotopy and the Quillen homology of an algebra over an operad. In that context, we prove an analogue of the Hurewicz theorem. The third section is devoted to an obstruction theory which is useful to prove the functoriality up to homotopy of the plus-construction. The plus-construction is developed in the fourth section. The last section is concerned with the proof of Propositions 5.2 and 5.3.

NOTATION. From now on K denotes a field of characteristic 0 and all vector spaces will be considered over K. The category of K-vector spaces is denoted by Vect_K. The group of permutations on k elements is denoted by S_k.

A graded vector space V is a non-negative lower graded vector space. It is said to be *n-reduced* if $V_i = 0,\ \forall i < n$ and *reduced* if it is 1-reduced. Its *suspension* is the graded vector space sV defined by $(sV)_n = V_{n-1}$.

A graded vector space V is a *differential graded vector space* or a *complex* if it is equipped with a morphism $d : V \to V$ of degree -1 such that $d^2 = 0$. In that context, the space of cycles is $Z_r(V) = \mathrm{Ker}(d : V_r \to V_{r-1})$ and the space of boundaries is $B_r(V) = \mathrm{Im}(d : V_{r+1} \to V_r)$; the homology of the complex V is $H_*(V) = Z_*(V)/B_*(V)$. A differential graded vector space is *connected* if $H_0(V) = 0$. A *quasi-isomorphism* of complexes is a morphism inducing an isomorphism in homology.

The free commutative algebra (resp. co-commutative coalgebra) generated by a graded vector space V will be denoted either by $S(V)$ or $\Lambda(V)$. The reason for the second notation comes from the Chevalley-Eilenberg complex associated to a Lie algebra $\mathcal{G}$: since it is the free commutative algebra generated by $s\mathcal{G}$ it is the exterior algebra generated by $\mathcal{G}$.

1. Algebras over an Operad

Topological operads were introduced by P. May [13] in the 70's in the context of homotopy theory. Since algebraic operads were reintroduced by V. Ginzburg and M. Kapranov in 1994 [6], they have become objects of great interest. In this section, we give definitions and results useful for the next sections.

An operad is an algebraic object which determines a type of algebras. For instance, the operads $\mathcal{L}$ie, $\mathcal{A}$s and $\mathcal{C}$om, encode respectively Lie algebras, associative algebras, and commutative algebras. In this section, we give two equivalent definitions of operads and define differential graded algebras over an operad. Following E. Getzler and J.D.S. Jones [5] we explain the construction of an almost free resolution of a differential graded algebra over an operad. We refer to [5, 6, 11] for more details.

DEFINITION 1.1. A *S-module* $\mathcal{M}$ is a sequence $\{\mathcal{M}(n)\}$ such that $\mathcal{M}(n)$ is a S_n-module for each n. We associate to any S-module $\mathcal{M}$ an endofunctor $T(\mathcal{M}, -)$ of Vect_K defined by

$$T(\mathcal{M}, V) = \bigoplus_{k\geqslant 0} (\mathcal{M}(k) \otimes_{S_k} V^{\otimes k}),\ \forall\, V \in \mathrm{Vect}_K,$$

where S_k acts on $V^{\otimes k}$ by permutation. The category of S-modules is denoted by S-mod.

The composite of these endofuntors is an endofunctor of the same form [8]. More precisely, there exists an associative bifunctor, $\circ\colon$ S-mod $\times$ S-mod $\to$ S-mod, such that $T(\mathcal{M} \circ \mathcal{N}, -) \cong T(\mathcal{M}, T(\mathcal{N}, -))$.

Let I be the S-module defined by $I(n) = K$ if $n = 1$ and $I(n) = 0$ if not; then I is the unit for this product.

DEFINITION 1.2. An *operad* is an associative algebra in the category S-mod. Consequently, an operad is a S-module $\mathcal{P}$ with an associative product $\mu\colon\ \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and a unit $\eta\colon\ I \to \mathcal{P}$ commuting with μ. In fact, the operad structure may be described in terms of a collection of equivariant morphisms, called *products*

$$\mathcal{P}(n) \otimes \mathcal{P}(j_1) \otimes \cdots \otimes \mathcal{P}(j_n) \to \mathcal{P}(j_1 + \cdots + j_n)$$

and a unit $1 \in \mathcal{P}(1)$ satisfying the axioms of P. May [13].

A *unital operad* is an operad $\mathcal{P}$ such that $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = K$.

The existence of the bifunctor $\circ$ asserts that giving an operad $\mathcal{P}$ is equivalent to giving a monad of the form $T(\mathcal{P}, -)$ in the category $\mathrm{End}(\mathrm{Vect}_K)$. An algebra over this monad is a *$\mathcal{P}$-algebra*.

PROPOSITION 1.3. *Let $\mathcal{P}$ be an operad. A $\mathcal{P}$-algebra A is a vector space equipped with S_n-equivariant morphisms, called products, $\mathcal{P}(n) \otimes A^{\otimes n} \to A$ which are associative with respect to the product in $\mathcal{P}$ and which make the operad unit act as the identity.*

We denote by $\mu(a_1, \ldots, a_n)$ the image of $\mu \otimes a_1 \otimes \cdots \otimes a_n$ under the product. As in the case of monads, $T(\mathcal{P}, V)$ is the free $\mathcal{P}$-algebra generated by the vector space V. Further $\mathcal{P}(n)$ can be identified with the n-multilinear part of $T(\mathcal{P}, \langle x_1, \ldots, x_n\rangle)$, as a S_n-module.

EXAMPLES 1.4. Consider the free (nonunitary) associative algebra $\mathcal{A}\mathrm{s}(x_1, \ldots, x_n)$ generated by $x_1, \ldots, x_n$ and let $\mathcal{A}\mathrm{s}(n)$ be the sub-vector space of $\mathcal{A}\mathrm{s}(x_1, \ldots, x_n)$ generated by the monomials containing each x_i only once; then $\mathcal{A}\mathrm{s}(n)$ is the vector space generated by $x_{\sigma(1)} \cdots x_{\sigma(n)}$ for $\sigma \in S_n$ and it is naturally endowed with a structure of S_n-module which corresponds to the regular representation. The collection $\mathcal{A}s = \{\mathcal{A}s(n), n \geqslant 0\}$ forms an operad, namely *the associative operad*: products are morphisms

$$\mathcal{A}s(l) \otimes \mathcal{A}s(m_1) \otimes \cdots \otimes \mathcal{A}s(m_l) \to \mathcal{A}s(m_1 + \cdots + m_l)$$

which associate to a monomial $\mu, \nu_1, \ldots, \nu_l$ the one obtained by substituting monomials ν_i for elements x_i in μ. An $\mathcal{A}s$-algebra is nothing but an associative algebra and the product $\mathcal{A}s(n) \otimes A^{\otimes n} \to A$ is given by the polynomial evaluation. Moreover, the free algebra generated by a vector space V is

$$T(\mathcal{A}s, V) = \bar{T}(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}.$$

Similarly, we can define an operad $\mathcal{C}om = \{\mathcal{C}om(n), n \geqslant 0\}$, where $\mathcal{C}om(n)$ is the vector space generated by $x_1 \cdots x_n$ seen as the trivial representation of S_n. A $\mathcal{C}om$-algebra is nothing but a (nonunitary) commutative algebra, and the free commutative algebra generated by a vector space V is

$$T(\mathcal{C}om, V) = \bar{S}(V) = \bigoplus_{n=1}^{\infty} (V^{\otimes n})_{S_n}.$$

We can define as well the operad $\mathcal{L}ie$ where algebras over this operad are Lie algebras.

DEFINITION 1.5. (Differential Graded $\mathcal{P}$-algebras). We consider the category of differential graded vector spaces equipped with the graded tensor product; the symmetric operator is the following one:

$$\begin{aligned} &U \otimes V \to V \otimes U \\ &u \otimes v \mapsto (-1)^{|u||v|} v \otimes u. \end{aligned}$$

The differential on $U \otimes V$ is given by $\mathrm{d}(u \otimes v) = \mathrm{d}u \otimes v + (-1)^{|u|} u \otimes \mathrm{d}v$.

Let $\mathcal{P}$ be an operad. A graded $\mathcal{P}$-algebra is a graded vector space A with morphisms of degree 0, $\mathcal{P}(n) \otimes A^{\otimes n} \to A$, satisfying the same relations as in the nongraded case. A *derivation* of A is a morphism $d : A \to A$ of degree -1 such that

$$\mathrm{d}(\mu(a_1, \ldots, a_n)) = \sum_{i=1}^{n} \pm \mu(a_1, \ldots, \mathrm{d}a_i, \ldots, a_n).$$

A is said to be a *differential graded $\mathcal{P}$-algebra* if it is endowed with a derivation d such that $d^2 = 0$.

DEFINITION 1.6. The *homotopy* of a differential graded $\mathcal{P}$-algebra A, denoted by $\pi_*(A)$, is the homology of the underlying complex.

A non-graded $\mathcal{P}$-algebra can be considered as a differential graded $\mathcal{P}$-algebra concentrated in degree 0 with the differential 0.

DEFINITION 1.7. (Differential Graded $\mathcal{P}$-coalgebras). A *coalgebra* over an operad $\mathcal{P}$ is a vector space X equipped with S_n-equivariant coproducts $\mathcal{P}(n) \otimes X \to X^{\otimes n}$, which are associative with respect to the product in $\mathcal{P}$ and which make the operad unit act as the identity. The image of $p \otimes v$ under this coproduct is written, following Sweedler's notation:

$$p(v) = \sum v_{(1)} \otimes \cdots \otimes v_{(n)}.$$

If $\mathcal{P}(n)$ is finite-dimensional for all n, the linear dual of the product in $\mathcal{P}$

$$\mathcal{P}(n) \otimes \mathcal{P}(l_1) \otimes \cdots \otimes \mathcal{P}(l_n) \to \mathcal{P}(l_1 + \cdots + l_n)$$

induces a coproduct of $\mathcal{P}$-coalgebras on $C(\mathcal{P}, V) = \oplus_{n\geqslant 0}(\mathcal{P}(n)^* \otimes V^{\otimes n})^{S_n}$. When V is reduced, we obtain the reduced (co)free $\mathcal{P}$-coalgebra (co)generated by V. Then a $\mathcal{P}$-coalgebra structure on X is specified by a coproduct $X \to C(\mathcal{P}, X)$. Let X be a graded $\mathcal{P}$-coalgebra. A *coderivation* of X is a morphism $d : X \to X$ of degree -1 satisfying

$$\mathrm{d}p(v) = \sum \pm v_{(1)} \otimes \cdots \otimes \mathrm{d}v_{(i)} \otimes \cdots \otimes v_{(n)}.$$

We define a differential graded $\mathcal{P}$-coalgebra as for differential graded $\mathcal{P}$-algebras.

DEFINITION 1.8. (Indecomposable Elements). Let A be a differential graded $\mathcal{P}$-algebra. An *ideal* of A is a sub-vector space I such that

$$\mu(a_1, \ldots, a_{n-1}, s) \in I, \quad \forall \mu \in \mathcal{P}(n), \quad \forall a_i \in A,\ \forall s \in I.$$

Let $\mathcal{P}$ be a unital operad and A^n be the image of the product $\bigoplus_{k\geqslant n} (\mathcal{P}(k) \otimes_{S_k} A^{\otimes k}) \to A$. Then A^n is an ideal of A called the *nth power of the augmentation ideal*. For instance A^2 is the vector space generated by the elements in A which can be written as a product of elements in A; the vector space A/A^2 denoted by QA is called the K-module of *indecomposable elements*. If A is the free graded $\mathcal{P}$-algebra generated by V, one has

$$A^n = \bigoplus_{k\geqslant n} (\mathcal{P}(k) \otimes_{S_k} V^{\otimes k}) \quad \text{and} \quad QA \cong V.$$

When A is a differential graded $\mathcal{P}$-algebra, one sees that $H_0 QA \cong Q\pi_0 A$.

DEFINITION 1.9. (Almost Free Algebras). The category of differential graded $\mathcal{P}$-algebras is equipped with a coproduct [4, 5], denoted $\vee$, and one has the following isomorphism:

$$T(\mathcal{P}, V) \vee T(\mathcal{P}, W) \cong T(\mathcal{P}, V \oplus W).$$

A differential graded $\mathcal{P}$-algebra is *almost free* if it is free as a graded $\mathcal{P}$-algebra. Specifically, it is of the form $(T(\mathcal{P}, V), \mathrm{d})$ where V is a graded vector space and d is not necessarily induced by a differential on V. An *almost free morphism* is a map of the form $A \to A \vee T(\mathcal{P}, V)$ where $T(\mathcal{P}, V)$ is an almost free algebra. Notice that A is almost free if and only if the map $0 \to A$ is almost free.

THEOREM 1.10 [5, 6]. *Any differential graded algebra over an operad admits an almost free resolution, that is for any differential graded $\mathcal{P}$-algebra A, there exists a quasi-isomorphism $F \to A$ with F almost free.*

Remark 1.11. We have seen that an operad is an associative algebra in the category of S-modules. The principle of this resolution is the same as the bar resolution for associative algebras, though it is rather more complicated. Note also that V. Ginzburg and M. Kapranov construct a bar resolution of an operad $\mathcal{P}$, which is a free cooperad denoted by $\mathbb{B}(\mathcal{P})$ and the almost free resolution is $T(\mathcal{P}, C(\mathbb{B}(\mathcal{P}), A)) \to A$, where $C(\mathbb{B}(\mathcal{P}), A)$ is the free $\mathbb{B}(\mathcal{P})$-coalgebra generated by A. What we need for the next sections is that if A is n-reduced, so is the almost free resolution.

V. Ginzburg and M. Kapranov have built a duality for binary quadratic operads [6], called Koszul duality. For instance, the dual of the operad $\mathcal{C}$om is the operad $\mathcal{L}$ie and the operad $\mathcal{A}$s is self-dual. The dual of an operad $\mathcal{P}$ is denoted $\mathcal{P}^!$. If the operad $\mathcal{P}$ happens to be a unital Koszul operad [6], with $\mathcal{P}(n)$ finite dimensional for all n, E. Getzler and J.D.S. Jones [5] have defined a pair of adjoint functors from the category of differential graded $\mathcal{P}$-algebras (dg$\mathcal{P}$ − alg) and the category of reduced differential graded $\mathcal{P}^!$-coalgebras (dg$\mathcal{P}^!$ − Cog_0)

$$C_{\mathcal{P}}\colon\ \mathrm{dg}\mathcal{P} - \mathrm{alg} \leftrightarrows \mathrm{dg}\mathcal{P}^! - \mathrm{Cog}_0\colon\ T_{\mathcal{P}}.$$

These functors induce a smaller resolution, called the *Koszul resolution*:

THEOREM 1.12 [5, 6]. *If $\mathcal{P}$ is a Koszul operad, the unit and the co-unit of the adjunction are quasi-isomorphisms. In particular, for every differential graded $\mathcal{P}$-algebras A, there exists a quasi-isomorphism $T_{\mathcal{P}}(C_{\mathcal{P}}(A)) \to A$.*

In fact the functors $C_{\mathcal{P}}$ and $T_{\mathcal{P}}$ are built as below:

(1.13) Construction of the Functor $C_{\mathcal{P}}$. Let (A, ∂) be a differential graded $\mathcal{P}$-algebra. We define $C_{\mathcal{P}}(A)$ as the reduced free graded $\mathcal{P}^!$-coalgebra generated by sA and equipped with a differential d which is the sum of a linear differential d_1 induced by ∂ and a quadratic differential d_2 induced by the structure of $\mathcal{P}$-algebra on A. Indeed we can set $C_{p,q} = (\mathcal{P}^!(p+1) \otimes_{\mathrm{S}_{p+1}} (sA)^{\otimes p+1})_{p+q}$, then

$$\mathrm{d}_1\colon\ C_{p,q} \to C_{p,q-1}, \qquad \mathrm{d}_2\colon\ C_{p,q} \to C_{p-1,q},$$

and $C_{\mathcal{P}}(A) = \oplus_{p,q} C_{p,q}$ is the total complex of a first quadrant bicomplex; hence the functor $C_{\mathcal{P}}$ preserves quasi-isomorphisms. Let A be a differential graded $\mathcal{P}$-algebra. The *operadic homology* of A is the homology of the total complex $C_{\mathcal{P}}(A)$.

Note that the operads $\mathcal{L}$ie, $\mathcal{A}$s and $\mathcal{C}$om are Koszul operads and the operadic homology obtained is nothing but respectively Chevalley–Eilenberg homology and Hochschild homology and Harrison homology [6]. The functor $C_{\mathcal{P}}$ is the functor $\mathcal{C}$ of Quillen for Lie algebras [18], and the bar construction for associative algebras.

(1.14) Construction of the Functor $T_{\mathcal{P}}$. It is a generalization of the cobar construction for associative coalgebras in the case of $\mathcal{P}^!$-coalgebras. Let (C, d) be a reduced differential graded $\mathcal{P}^!$-coalgebra. We define $T_{\mathcal{P}}(C)$ as the free graded $\mathcal{P}$-algebra generated by $s^{-1}C$ and equipped with a differential ∂ which is the sum of a linear differential ∂_1 induced by d and a quadratic differential ∂_2 induced by the structure of $\mathcal{P}^!$-coalgebra on C. Indeed $T_{\mathcal{P}}(C)$ is a second quadrant bicomplex and we have the following lemma.

LEMMA 1.15. *The functor $T_{\mathcal{P}}$ preserves quasi-isomorphisms between 2-reduced differential graded $\mathcal{P}^!$-coalgebras.*

Proof. Let $\psi : (B, d) \to (B', d')$ be a quasi-isomorphism between 2-reduced differential graded $\mathcal{P}^!$-coalgebras. We define a filtration on $T_{\mathcal{P}}$ by $F^p = \oplus_{k \geqslant p} \mathcal{P}(k) \otimes_{S_k} (s^{-1}B)^{\otimes k}$. The differential ∂_1 respects the filtration and since ∂_2 is quadratic, we get $\partial_2(F^p) \subset F^{p+1}$. Let us consider the diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & F^{p+1} & \to & F^p & \to & F^p/F^{p+1} & \to & 0 \\ & & {\scriptstyle T_{\mathcal{P}}(\psi)}\downarrow & & & & \downarrow{\scriptstyle \overline{T_{\mathcal{P}}(\psi)}} & & \\ 0 & \to & F'^{p+1} & \to & F'^p & \to & F'^p/F'^{p+1} & \to & 0. \end{array}$$

However, the differential on F^p/F^{p+1} is induced by ∂_1 because ∂_2 is quadratic, so by hypothesis $\overline{T_{\mathcal{P}}(\psi)}$ is a quasi-isomorphism. Consequently, if the map $H(T_{\mathcal{P}}(\psi))\colon H(F^{p+1}) \to H(F'^{p+1})$ is an isomorphism, so is $H(T_{\mathcal{P}}(\psi))\colon H(F^p) \to H(F'^p)$. Since B and B' are 2-reduced, we get $H_k(F^{p+1}) = H_k(F'^{p+1}) = 0$, for $k \leqslant p$. The result now follows. □

2. Quillen Homology and Hurewicz Theorem

In this section we first recall that the category of differential graded algebras over an operad is a model category in the sense of D. Quillen [3, 16]. The proof is due to E. Getzler and J.D.S. Jones based on the almost free resolution (see 1.10) [5]. Further, we will define the Quillen homology of an algebra over an operad and prove a Hurewicz's theorem in this framework.

Throughout the section, let $\mathcal{P}$ be a unital operad and let dg$\mathcal{P}$-alg be the category of differential graded $\mathcal{P}$-algebras: henceforth called 'algebras'. Recall that the homotopy of an algebra A is the homology of the underlying complex.

DEFINITION 2.1. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is said to be a *model category* if it has three distinguished classes of morphisms called weak equivalences, fibrations and cofibrations, and if the following axioms hold:

(CM1) The category $\mathcal{C}$ has all finite limits and colimits.

(CM2) If f and g are morphisms such that gf is defined, then if any two of f, g and gf are weak equivalences, then so is the third.
(CM3) If f is a retract of g, and if g is a weak equivalence (resp. a fibration, resp. a cofibration), then so is f.
(CM4) (i) Cofibrations have the left lifting property with respect to every trivial fibration;
(ii) fibrations have the right lifting property with respect to every trivial cofibration.
(CM5) (i) Every morphism f can be factored $f = pi$ where p is a trivial fibration and i a cofibration;
(ii) every morphism f can be factored $f = pi$ where p is a fibration and i a trivial cofibration.

In these axioms, a *trivial fibration* is a morphism which is both a fibration and a weak equivalence; similarly for trivial cofibrations. A morphism $f\colon A \to B$ has the *left lifting property* with respect to a morphism $g\colon X \to Y$ if given any diagram

$$\begin{array}{ccc} A & \to & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \to & Y \end{array}$$

there is a morphism $B \to X$ making both triangles commute. Similarly for right lifting property. A morphism $f : A \to B$ is a *retract* of a morphism $g\colon X \to Y$ if there exists a diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{r} & A \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ B & \xrightarrow{i'} & Y & \xrightarrow{r'} & B \end{array}$$

such that $ri = A$ and $r'i' = B$.

THEOREM 2.2 [5]. *The category of differential graded $\mathcal{P}$-algebras, equipped with the following three classes of morphisms*:

(i) *weak equivalences which are morphisms inducing isomorphisms in homotopy,*
(ii) *fibrations which are surjections in degrees* > 0,
(iii) *cofibrations which are morphisms having the left lifting property with respect to every trivial fibration,*

is a model category.

The proof is based on the following theorem.

THEOREM 2.3 [5]. (i) *Any map $A \to B$ admits a factorization $A \to X \to B$ in which $A \to X$ is an almost free morphism and $X \to B$ is a trivial fibration.*
(ii) *Any almost free morphism is a cofibration.*

Recall that a *fibrant* object in a model category is an object A such that $A \to 0$ is a fibration whereas a *cofibrant* object is an object B such that $0 \to B$ is a cofibration. By the definition of a fibration, it follows that every algebra is fibrant. Furthermore, every cofibrant algebra is a retract of an almost free algebra. We give now a canonical path space that will enable us to treat right homotopies.

DEFINITION 2.4. (A Canonical Path Space). First recall some definitions. Let X be an object of a category $\mathcal{C}$. A *path space* for X is an object X' of $\mathcal{C}$ together with a weak equivalence $\sigma\colon X \to X'$ and a morphism $\rho\colon X' \to X \times X$ satisfying $\rho\sigma = (X, X)$. It is called a *good path space* if ρ is a fibration.

Let I be the free differential graded commutative algebra generated by two elements: t in degree 0 and $\mathrm{d}t$ in degree -1, with the differential d given by $\mathrm{d}(t) = \mathrm{d}t$ and $\mathrm{d}(\mathrm{d}t) = 0$. Consider the field K as a commutative algebra and let $s_0 : K \to I$ be the canonical inclusion and $p_0, p_1\colon I \to K$ be defined by

$$p_0(\phi(t, \mathrm{d}t)) := \phi(0, 0), \qquad p_1(\phi(t, \mathrm{d}t)) := \phi(1, 0), \quad \forall \phi(t, \mathrm{d}t) \in I.$$

From these definitions, it follows that s_0 is a quasi-isomorphism and that: $H_*(I) = K$ and $p_0 s_0 = p_1 s_0 = K$.

For any differential graded $\mathcal{P}$-algebra X, there exists a structure of differential graded $\mathcal{P}$-algebra on $X \otimes I$ described as follows:

(i) Set $\mu \in \mathcal{P}(n)$ and $x_i \otimes \alpha_i \in X \otimes I$ for $1 \leqslant i \leqslant n$; we define the product $\mu(x_1 \otimes \alpha_1, \ldots, x_n \otimes \alpha_n)$ by $\pm\, \mu(x_1, \ldots, x_n) \otimes \alpha_1 \cdots \alpha_n$;
(ii) The differential is the classical differential on a tensor product of differential graded vector spaces.

Hence, we get a factorization

$$X \xrightarrow{X\otimes s_0} X \otimes I \underset{X\otimes p_1}{\overset{X\otimes p_0}{\rightrightarrows}} X.$$

We define then the *canonical path space* X^I as

$$(X^I)_i = \begin{cases} 0, & \text{if } i < 0,\\ \mathrm{Ker}\big(\mathrm{d}\colon (X \otimes I)_0 \to (X \otimes I)_{-1}\big), & \text{if } i = 0,\\ (X \otimes I)_i, & \text{if } i > 0. \end{cases}$$

The induced morphisms yield the diagram

$$X \xrightarrow{s_0^X} X^I \underset{p_1^X}{\overset{p_0^X}{\rightrightarrows}} X,$$

and it is not hard to check that X^I is a good path space. Therefore p_0^X and p_1^X are trivial fibrations.

LEMMA 2.5. *Let $f\colon A \to B$ be a weak equivalence. Any morphism $g\colon X \to B$ with X cofibrant factorizes through A up to homotopy. More precisely, there exists a morphism $\phi\colon X \to A$ such that $f\phi$ is homotopic to g.*

Proof. The morphism f can be factored through the fiber product $A \times_B B^I$ of f and p_0^B

$$A \xrightarrow{j=(A,s_0^B f)} A \times_B B^I \xrightarrow{q=p_1^B \mathrm{pr}_2} B.$$

With the definition of the canonical path space, it is not hard to check that q is a fibration and j is a weak equivalence. Since f is a weak equivalence, so is q. Hence the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & A \times_B B^I \\ \Big\downarrow & \overset{s}{\nearrow} & \Big\downarrow{\scriptstyle q} \\ X & \xrightarrow[g]{} & B \end{array}$$

has a lift $s = (\phi, H)$. However, the image of s lies in $A \times_B B^I$, so $p_0^B H = f\phi$. Furthermore, $p_1^B H = p_1^B \mathrm{pr}_2 s = qs = g$. □

PROPOSITION 2.6. *Let $F_A \to A$ (resp. $F_B \to B$) be a weak equivalence with F_A (resp. F_B) cofibrant. For any morphism $f\colon A \to B$ there exists a morphism $\tilde{f}\colon F_A \to F_B$ such that the diagram*

$$\begin{array}{ccc} F_A & \overset{\tilde{f}}{\dashrightarrow} & F_B \\ \Big\downarrow & & \Big\downarrow \\ A & \xrightarrow{f} & B \end{array}$$

commutes up to homotopy.

PROPOSITION AND DEFINITION 2.7. *Let $i\colon F \to A$ be a weak equivalence with F cofibrant. The homology of the module of indecomposables QF of F does not depend on the choice of F. It is called the Quillen homology of A and is denoted by $H_*^{\mathrm{Q}}(A)$.*

The proof involves the two following lemmas which can be found in [3, 16].

LEMMA 2.8. *Let $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ be adjoint functors between model categories. The following are equivalent:*

(i) *F preserves cofibrations and G preserves fibrations,*
(ii) *F preserves cofibrations and trivial cofibrations,*
(iii) *G preserves fibrations and trivial fibrations.*

LEMMA 2.9. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between model categories. If F preserves trivial cofibrations between cofibrant objects, then F preserves weak equivalences between cofibrant objects.*

Proof of Proposition 2.7. Let $F_1 \to A$ and $F_2 \to A$ be weak equivalences with F_1 and F_2 cofibrant. By Lemma 2.5, there exists a morphism h such that the diagram

$$\begin{array}{ccc} & & F_2 \\ & \overset{h}{\nearrow} & \downarrow \\ F_1 & \longrightarrow & A \end{array}$$

commutes up to homotopy. Hence h is a weak equivalence between cofibrant objects.

It suffices to notice that the indecomposable elements functor Q is left adjoint to the functor $(-)_+$ from the category of differential graded vector spaces to dg$\mathcal{P}$ − alg which assigns to a vector space V the trivial algebra V_+. Since $(-)_+$ preserves fibrations and trivial fibrations then Q preserves cofibrations and trivial cofibrations (Lemma 2.8), then it preserves weak equivalences between cofibrant objects (Lemma 2.9). Consequently h induces an isomorphism $H_*(QF_1) \cong H_*(QF_2)$. □

COROLLARY 2.10. *If a morphism of differential graded $\mathcal{P}$-algebras is a weak equivalence then it induces an isomorphism in homology.*

Remark 2.11. When the operad $\mathcal{P}$ is a Koszul operad, there is a canonical free resolution (Theorem 1.12) $T(\mathcal{P}, s^{-1}C_{\mathcal{P}}(A)) \to A$; hence the Quillen homology of A (see 2.7) is $H_*(s^{-1}C_{\mathcal{P}}(A))$, so it is isomorphic to the operadic homology $H_*^{\mathcal{P}}$. More precisely $H_*^{\mathcal{P}}(A) = H_{*-1}^{\mathrm{Q}}(A)$.

DEFINITION 2.12. (Hurewicz Morphism). Let A be a differential graded $\mathcal{P}$-algebra and $F \to A$ be a weak equivalence with F cofibrant. The map $F \to QF$ induces a morphism $\pi_*(A) = \pi_*(F) \longrightarrow H_*(QF) = H_*^{\mathrm{Q}}(A)$ called the *Hurewicz morphism* and denoted by ϕ.

THEOREM 2.13. *Let A be a differential graded $\mathcal{P}$-algebra and n a nonnegative integer.*

(a) *If $\pi_k(A) = 0$ for $0 \leqslant k \leqslant n$, then the Hurewicz morphism is an isomorphism for $k \leqslant 2n+1$ and an epimorphism for $k = 2n+2$.*
(b) *If $\pi_0(A) = 0$ and $H_k^{\mathrm{Q}}(A) = 0$ for $0 \leqslant k \leqslant n$, then the Hurewicz morphism is an isomorphism for $k \leqslant 2n+1$ and an epimorphism for $k = 2n+2$.*

Proof. (a) Assume that $\pi_k(A) = 0$ for $0 \leqslant k \leqslant n$. We consider the graded $\mathcal{P}$-algebra B defined by

$B_i = A_i$ for $i \geqslant n+2$,

$B_i = 0$ for $i \leqslant n$ and

$B_{n+1} = \mathrm{Ker}(d : A_{n+1} \to A_n)$.

Then (B, d) is a differential graded $\mathcal{P}$-algebra. Furthermore the map $B \to A$ is a weak equivalence. Since B is n-reduced then so is its canonical resolution F (see remark 1.11). We obtain then a weak equivalence $i : F \to A$ with F almost free. Since F is n reduced, the second ideal of augmentation F^2 is $(2n+1)$-reduced.

The short exact sequence $0 \to F^2 \to F \to QF \to 0$ yields a long exact sequence in homology

$$\cdots \to H_k(F^2) \to H_k(F) \to H_k(QF) \to H_{k-1}(F^2) \to \cdots .$$

It suffices to replace $H_k(F)$ by $\pi_k(A)$ and $H_k(QF)$ by $H_k^{\mathrm{Q}}(A)$.

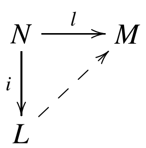
(b) We prove by induction on k that $\pi_k(A) = 0$ for $0 \leqslant k \leqslant n$. It is trivial for $k = 0$. Assume that $\pi_i(A) = 0$ for $0 \leqslant i \leqslant k$ where $k < n$. The result follows by (a): the Hurewicz morphism ϕ_i is an isomorphism for $0 \leqslant i \leqslant 2k+1$, in particular, $\pi_{k+1}(A) = 0$ since $H_{k+1}^{\mathrm{Q}}(A) = 0$. Hence $\pi_k(A) = 0$ for $0 \leqslant k \leqslant n$ and we apply (a) to conclude.

3. Obstruction Theory

Obstruction to extending maps has been studied by M. Rothenberg and G. Triantafillou [19] in the category of differential graded Lie algebras. In this section, we develop an obstruction theory (to extending maps and homotopies) in the category of differential graded $\mathcal{P}$-algebras for any unital operad $\mathcal{P}$. Proofs are similar to the classical case, so we do not give any detail.

Let N be an almost free algebra and $i\colon N \to L$ an almost free morphism. Obstruction theory is based on two extension problems.

The first one is the following: given a morphism $l\colon N \to M$ with M connected and a diagram



what is the obstruction to the existence of a lift in that diagram?

The second one is concerned with homotopy extensions. That is, given some morphisms $f, g : L \to M$, with M connected and such that f is homotopic to g on N, what is the obstruction to the existence of a homotopy from f to g? By homotopy we mean a right homotopy (in the sense of Quillen) and since all objects considered are fibrant and cofibrant it is equivalent to a homotopy via the canonical path space. Thus, f is *homotopic* to g if there exists a morphism of differential graded $\mathcal{P}$-algebras $H\colon L \to M^I$ such that $p_0^M H = f$ and $p_1^M H = g$ (see definition 2.4).

NOTATION. Let $\mathcal{P}$ be a unital operad (i.e. $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = K$). We write

$$N = T(\mathcal{P}, V) = \bigoplus_{n \geqslant 1} \mathcal{P}(n) \otimes_{S_n} V^{\otimes n} \quad \text{and} \quad L = N \vee T(\mathcal{P}, W) = T(\mathcal{P}, V \oplus W).$$

Recall that the module of indecomposables $QL = L/L^2$ is isomorphic to $V \oplus W$. We define the skeletal filtration of L over N by

$$\mathrm{sk}_n L := N \vee T\big(\mathcal{P}, \bigoplus_{i=0}^{n} W_i\big) \quad \text{and} \quad \mathrm{sk}_{-1} L := N.$$

Theorem 3.7 gives an answer to the first problem and Theorem 3.8 answers the second one. First of all, we compute the obstruction to extending a map $f\colon\ \mathrm{sk}_n L \to M$ to a map $f'\colon\ \mathrm{sk}_{n+1} L \to M$.

LEMMA 3.1. *Let $f\colon\ \mathrm{sk}_n L \to M$ be a morphism of differential graded $\mathcal{P}$-algebras with M connected. Then f determines an element $\mathcal{O}(f) \in Z^{n+1}\,\mathrm{Hom}(QL/QN, \pi_n(M))$ which is the obstruction to extending f to $\mathrm{sk}_{n+1} L$. Moreover, $\mathcal{O}(f)$ depends only on the homotopy class of f.*

This lemma is the first step towards an obstruction to extending maps. If $\mathcal{O}(f) \neq 0$ but if its class vanishes in $H^{n+1}\,\mathrm{Hom}(QL/QN, \pi_n(M))$, we prove that we can extend f to a map $\tilde{f}\colon\ \mathrm{sk}_{n+1} L \to M$ which coincides with f only on $\mathrm{sk}_{n-1} L$ (see Theorem 3.7). From now on, we need to treat homotopies and for that purpose we define an integration operator analogous to the one defined by P. Deligne, P. Griffiths, J. Morgan and D. Sullivan [2, 7] for the needs of homotopy theory in the category of commutative algebras.

DEFINITION 3.2 (Integration of a Homotopy). Let B be a differential graded $\mathcal{P}$-algebra and $\int_0^1 : B^I \to B$ and $\int_0^t : B^I \to B^I$ be linear operators of degree 1 defined by:

$$\int_0^1 b \otimes t^i = 0 \quad \text{and} \quad \int_0^1 b \otimes t^i \mathrm{d}t = (-1)^{|b|} \frac{b}{i+1};$$

$$\int_0^t b \otimes t^i = 0 \quad \text{and} \quad \int_0^t b \otimes t^i \mathrm{d}t = (-1)^{|b|} b \otimes \frac{t^{i+1}}{i+1}.$$

The following proposition is a straightforward calculation.

PROPOSITION 3.3. *Suppose $\beta \in B^I$ and $H\colon\ A \to B^I$ is a homotopy from f to g; then we have*

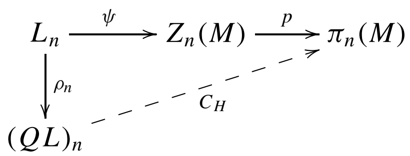
(a) $\mathrm{d}\int_0^t \beta + \int_0^t \mathrm{d}\beta = \beta - p_0^B \beta \otimes 1,$

(b) $\mathrm{d}\int_0^1 H(a) + \int_0^1 \mathrm{d}H(a) = g(a) - f(a).$

The relation (b) implies that $\int_0^1 H$ is a chain homotopy from f to g.

LEMMA 3.4. *Let $f, g\colon\ \mathrm{sk}_n L \to M$ be morphisms of differential graded $\mathcal{P}$-algebras. Assume the existence of a homotopy $H : \mathrm{sk}_{n-1} L \to M^I$ from f to g. These hypotheses determine an element $\Delta(f, g, H) \in \mathrm{Hom}(QL/QN, \pi_n(M))^n$ which is the obstruction to extending H to a homotopy $H'\colon\ \mathrm{sk}_n L \to M^I$ from f to g. Furthermore $\Delta(f, g, H)$ depends only on the homotopy classes of f and g.*

Proof. Because of 3.3 (b), the map $\psi = g - f - \int_0^1 H\mathrm{d}\colon\ L_n \longrightarrow M$ has its values in $Z_n(M)$. It is easy to check that $p\psi$ vanishes on $(L^2)_n$. Hence, we can factor $p\psi$:



Now define $\Delta(f, g, H) \in \mathrm{Hom}(QL/QN, \pi_n(M))^n$ by $\Delta(f, g, H)([u]) = C_H(u)$ and the lemma easily follows. □

LEMMA 3.5. *Let $f, g\colon\ \mathrm{sk}_n L \to M$ be morphisms of differential graded $\mathcal{P}$-algebras, and $H\colon\ \mathrm{sk}_{n-1} L \to M^I$ be a homotopy from f to g. Then $\mathcal{O}(g) - \mathcal{O}(f) = \mathrm{d}\Delta(f, g, H)$.*

If two maps $f, g\colon\ \mathrm{sk}_n L \to M$ coincide on $\mathrm{sk}_{n-1} L$, we can define the homotopy $s_0^M f\colon\ \mathrm{sk}_{n-1} L \to M^I$ from f to g. Then we denote $\Delta(f, g, s_0^M f)$ by $\Delta(f, g)$. Since $\int_0^1 s_0^M f\mathrm{d} = 0$, the map ψ defined in the proof of Lemma 3.4 is just equal to $g - f$. The proof of the next lemma is left to the reader.

LEMMA 3.6. *Let $f\colon\ \mathrm{sk}_n L \to M$ be a morphism of differential graded $\mathcal{P}$-algebras with M connected and $\alpha \in \mathrm{Hom}(QL/QN, \pi_n(M))^n$. Then there exists a morphism $g\colon\ \mathrm{sk}_n L \to M$ such that $g|_{\mathrm{sk}_{n-1} L} = f|_{\mathrm{sk}_{n-1} L}$ and $\Delta(f, g) = \alpha$.*

The main theorem of obstruction to extending maps follows by Lemmas 3.5 and 3.6:

THEOREM 3.7. *Let $f\colon\ \mathrm{sk}_n L \to M$ be a morphism of differential graded $\mathcal{P}$-algebras with M connected. Then the class of $\mathcal{O}(f)$ in $H^{n+1}\,\mathrm{Hom}(QL/QN, \pi_n(M))$ defined in Lemma 3.1 vanishes if and only if there exists $g\colon\ \mathrm{sk}_{n+1} L \to M$ such that $g|_{\mathrm{sk}_{n-1} L} = f|_{\mathrm{sk}_{n-1} L}$.*

By virtue of 3.4, obstruction to extending a homotopy lies in $\mathrm{Hom}(QL/QN, \pi_n M)^n$. It is not hard to prove, following the plan of the proof of the theorem 3.7, the next theorem concerning the obstruction to extending homotopies.

THEOREM 3.8. *Assume $n \geqslant 1$. Let $f, g\colon \mathrm{sk}_{n+1} L \to M$ be morphisms of differential graded $\mathcal{P}$-algebras with M connected and let $H\colon \mathrm{sk}_{n-1} L \to M^I$ be a homotopy from f to g on $\mathrm{sk}_{n-1} L$. Then the class of $\Delta(f, g, H)$ vanishes in $H^n(QL/QN, \pi_n M)$ if and only if there exists a homotopy $\tilde{H}\colon \mathrm{sk}_n L \to M^I$ from f to g on $\mathrm{sk}_n L$ which coincides with H on $\mathrm{sk}_{n-2} L$.*

4. Plus-Construction

Recall first Quillen's plus-construction: let X be a connected CW-complex with $\pi_1(X)$ perfect; then there exists a map $i\colon X \to X^+$ inducing an isomorphism in homology with X^+ simply connected [9, 17]. T. Pirashvili [15] has adapted this construction to simplicial Lie algebras. We prove that this construction exists in the framework of differential graded algebras over an operad. More precisely, let $\mathcal{P}$ be a unital operad and A be a differential graded $\mathcal{P}$-algebra with $\pi_0(A)$ *perfect*, that means $Q\pi_0(A) = 0$. We take an almost free resolution F of A and obtain a plus-construction on F.

THEOREM 4.1. *Let F be an almost free differential graded $\mathcal{P}$-algebra with $\pi_0(F)$ perfect. Then there exists an almost free morphism $i\colon F \to F^+$ such that $\pi_0(F^+) = 0$ and $H_*^{\mathrm{Q}}(i)$ is an isomorphism. Moreover, i is universal up to homotopy among morphisms $F \to G$ with G connected. In particular the pair (i, F^+) is unique up to homotopy.*

Proof. In order to prove the existence of such a construction, we use the same techniques as in the topological case. The first step consists of adding some 1-cells so that $\pi_0(F)$ is 'killed'. In the second step we add some 2-cells to annihilate the homology appeared in the first step. We set $F = T(\mathcal{P}, V)$.

First step. Let M be a subset of F_0 generating the $\mathcal{P}$-algebra $\pi_0(F)$. Set

$$F' = T\Big(\mathcal{P}, V \bigoplus_{m\in M} Ke_m\Big)$$

with $|e_m| = 1$ and $\mathrm{d}(e_m) = m$. The morphism $j\colon F \to F'$ is almost free and it is clear that $\pi_0(F') = 0$. Furthermore the diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & F & \xrightarrow{j} & F' & \to & \mathrm{Coker}(j) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & QF & \xrightarrow{Qj} & QF' & \to & \mathrm{Coker}(Qj) & \to & 0 \end{array}$$

yields a homology diagram:

$$H_1(F') \xrightarrow{\phi_1} H_1(QF') \xrightarrow{p} H_1\,\mathrm{Coker}(Qj) \to H_0(QF).$$

By definition, one has: $H_1(F') = \pi_1(F')$ and $H_1(QF') = H_1^{\mathrm{Q}}(F')$; in fact ϕ_1 is nothing but the Hurewicz morphism. Since $\pi_0(F') = 0$, then ϕ_1 is an isomorphism by Theorem 2.13. Now $p\phi_1$ is surjective because $H_0(QF) = H_0^{\mathrm{Q}}(F) = 0$. The isomorphism

$$\mathrm{Coker}(Qj) \cong \bigoplus_{m\in M} Ke_m$$

provides for all $m \in M$ an element $\alpha_m \in F_1'$ such that $\mathrm{d}\alpha_m = 0$ and such that its projection onto $\oplus_{m\in M} Ke_m$ is e_m.

Second step. Set $F^+ = T(\mathcal{P}, V \oplus_{m\in M} Ke_m \oplus_{m\in M} K\sigma_m)$ with $|\sigma_m| = 2$ and $\mathrm{d}\sigma_m = \alpha_m$. Clearly the map $i\colon\ F \to F^+$ is an almost free morphism and F^+ is connected. We deduce from the isomorphism

$$\mathrm{Coker}(Qi) \cong \bigoplus_{m\in M} Ke_m \bigoplus_{m\in M} K\sigma_m,$$

that $\mathrm{Coker}(Qi)$ is acyclic. Whence i realizes an isomorphism in homology as required. □

The next general lemma insures the universality as well as the uniqueness up to homotopy of the construction. The proof relies on obstruction theory.

LEMMA 4.2. *Let $i\colon\ F \to F'$ be an almost free morphism between almost free differential graded $\mathcal{P}$-algebras. If $H_*^{\mathrm{Q}}(i)$ is an isomorphism, then for every morphism $f\colon\ F \to G$ with G connected, there exists a map $g\colon\ F' \to G$ unique up to homotopy such that $gi = f$.*

COROLLARY 4.3. *The plus-construction is functorial up to homotopy in the category of almost free differential graded $\mathcal{P}$-algebras.*

The next proposition shows that the plus-construction does not depend on the choice of an almost free resolution. Hence, we can define a plus-construction for every differential graded $\mathcal{P}$-algebras.

PROPOSITION AND DEFINITION 4.4. *Let A be a differential graded $\mathcal{P}$-algebra whose π_0 is perfect, F an almost free resolution of A and F^+ its plus-construction. Then the homotopy and the Quillen homology of F^+ does not depend on the choice of F. Then we define the homology and homotopy of A^+ by the relations*

$$\pi_*(A^+) := \pi_*(F^+) \quad \text{and} \quad H_*^{\mathrm{Q}}(A^+) := H_*^{\mathrm{Q}}(F^+).$$

Proof. By Lemma 2.5, if $\alpha_F\colon\ F \to A$ and $\alpha_G\colon\ G \to B$ are weak equivalences with F and G cofibrant, then there exists $u\colon\ F \to G$ such that $\alpha_G u$ is homotopic to α_F. As a consequence u is an isomorphism in homotopy between objects which are both cofibrant and fibrant, so it is a homotopy equivalence [3, 16]. Thus, there exists $v\colon G \to F$ such that uv and vu are homotopic to the identity. Applying the plus-construction, there exist morphisms $u^+\colon F^+ \to G^+$ and $v^+\colon G^+ \to F^+$ such that u^+v^+ and v^+u^+ are homotopic to the identity. As a conclusion $\pi_*(u^+)$ is an isomorphism. □

We conclude this section by computing homotopy lower degree terms.

PROPOSITION 4.5. *Let A be a differential graded $\mathcal{P}$-algebra whose π_0 is perfect. The following relations hold:*

$$\pi_0(A^+) = 0 \quad \text{and} \quad \pi_1(A^+) \cong H_1^{\mathrm{Q}}(A).$$

Proof. The first relation is contained in the definition of the plus-construction. The second one follows by the Hurewicz theorem (Theorem 2.13). □

5. Homotopy of sl$(A)^+$

Let A be an associative algebra with unit. Recall that $\mathrm{gl}_r(A)$ denotes the Lie algebra of $r\times r$-matrices over A equipped with the bracket $[\alpha, \beta] = \alpha\beta - \beta\alpha$. We denote by $\mathrm{gl}(A)$ the inductive limit of the canonical inclusions $\mathrm{gl}_r(A) \to \mathrm{gl}_{r+1}(A)$. Let $\mathrm{sl}(A)$ be the kernel of the trace map $\mathrm{Tr}\colon\ \mathrm{gl}(A) \to A/[A, A]$. Then $\mathrm{sl}(A)$ is a perfect Lie algebra. Recall that the Chevalley–Eilenberg homology of $\mathrm{sl}(A)$ is the homology of the following complex:

$$\cdots \to \Lambda^n \mathrm{sl}(A) \xrightarrow{\mathrm{d}} \Lambda^{n-1}\mathrm{sl}(A) \to \cdots \to \mathrm{sl}(A) \to 0,$$

with

$$\mathrm{d}(g_1 \wedge \cdots \wedge g_n) = \sum_{1 \leqslant i<j \leqslant n} (-1)^{i+j-1}[g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge \hat{g_j} \wedge \cdots \wedge g_n.$$

It is also the operadic homology of $\mathrm{sl}(A)$, denoted by $H_*^{\mathcal{L}\mathrm{ie}}(A)$ and also $H_{*-1}^{\mathrm{Q}}(A)$ (see 2.11).

A *Leibniz algebra* is a vector space L endowed with a bilinear map $[-,-]$: $L \times L \to L$ which satisfies:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \forall x, y, z \in L.$$

If the bracket is antisymmetric, then this relation is equivalent to the Jacobi relation. Consequently a Lie algebra is a Leibniz algebra. The Leibniz homology of a Leibniz algebra L is the homology of the complex

$$\cdots \to L^{\otimes n} \xrightarrow{\mathrm{d}} L^{\otimes n-1} \to \cdots \to L \to 0,$$

with [10]

$$\mathrm{d}(l_1 \otimes \cdots \otimes l_n) = \sum_{1 \leqslant i < j \leqslant n} (-1)^j l_1 \otimes \cdots \otimes l_{i-1} \otimes [l_i, l_j] \otimes \cdots \otimes \hat{l_j} \otimes \cdots \otimes l_n.$$

It is also the operadic homology of L, $H_*^{\mathcal{L}\mathrm{eib}}(L)$, or the Quillen homology of L, $H_{*-1}^{\mathrm{Q}}(L)$.

As a conclusion, $\mathrm{sl}(A)$ can be considered as a Leibniz algebra. We will compute the homotopy of $\mathrm{sl}(A)^+$ in the category of Lie algebras as well as in the category of Leibniz algebras (Propositions 5.2 and 5.3).

Let $\mathcal{P}$ be a finite dimensional unital Koszul operad. The symbol $\mathcal{P}^!$ denotes the dual operad associated to $\mathcal{P}$. Recall that a reduced free graded $\mathcal{P}^!$-coalgebra has the form

$$C(\mathcal{P}^!, V) = \bigoplus_{n \geqslant 1} (\mathcal{P}^!(n)^* \otimes V^{\otimes n})^{\mathrm{S}_n}, \text{ (see 1.7).}$$

The operadic homology of a differential graded $\mathcal{P}$-algebra X is the homology of the total bicomplex $C_{\mathcal{P}}(X)$ which is the reduced free graded $\mathcal{P}^!$-coalgebra generated by sX equipped with a specific differential (see 1.13).

PROPOSITION 5.1. *Let X be a differential graded $\mathcal{P}$-algebra whose homology is a reduced free graded $\mathcal{P}^!$-coalgebra: $H_*^{\mathcal{P}}(X) = C(\mathcal{P}^!, V)$. Then*

(a) *there exists a quasi-isomorphism of $\mathcal{P}^!$-coalgebras $C_{\mathcal{P}}(X) \to H_*^{\mathcal{P}}(X)$;*
(b) *if X is connected, then $\pi_*(X)$ is isomorphic to $s^{-1}V$.*

Proof. (a) There is the following isomorphism of vector spaces:

$$C_{\mathcal{P}}(X) \cong H_*^{\mathcal{P}}(X) \oplus B_*(X) \oplus Q,$$

where $B_*(X)$ denotes the sub-vector space of boundaries of X. We then get a morphism of differential graded vector spaces $\phi\colon C_{\mathcal{P}}(X) \to H_*^{\mathcal{P}}(X)$, and thus, a morphism of differential graded vector spaces $p\phi\colon C_{\mathcal{P}}(X) \to V$, where p is the projection of $C(\mathcal{P}^!, V)$ onto V. On the one hand, the universal property of a reduced free coalgebra yields a unique morphism of differential graded $\mathcal{P}^!$-coalgebras $\psi\colon C_{\mathcal{P}}(X) \to H_*^{\mathcal{P}}(X)$ such that $p\psi = p\phi$. On the other hand, because of the universal property, $\pi_*(\psi)$ is the unique morphism of graded $\mathcal{P}^!$-coalgebras such that $p\pi_*(\psi) = p\pi_*(\phi)$. It follows that ψ is a weak equivalence.

(b) Assume that X is connected. By replacing X by its truncation

$$Z_n = X_n \quad \text{for } n > 1, \quad Z_1 = \mathrm{Ker}(\mathrm{d}\colon X_1 \to X_0) \quad \text{and} \quad Z_0 = 0,$$

we can assume $X_0 = 0$. These conditions imply $H_i^{\mathcal{P}}(X) = 0$ for $i = 0, 1$; thus $V_0 = V_1 = 0$. Therefore, ψ is a weak equivalence between 2-reduced differential graded $\mathcal{P}^!$-coalgebras. The functor $T_{\mathcal{P}}$ then preserves this weak equivalence (see

1.15). Notice that $C(\mathcal{P}^!, V) = C_{\mathcal{P}}(s^{-1}V)$, where $s^{-1}V$ is regarded as a trivial $\mathcal{P}$-algebra with zero differential. As a conclusion, every morphism in the diagram

$$\begin{array}{ccc} T_{\mathcal{P}}C_{\mathcal{P}}(X) & \to & T_{\mathcal{P}}(H_*^{\mathcal{P}}(X)) \\ \downarrow & & \downarrow \\ X & & s^{-1}V \end{array}$$

is a weak equivalence; hence we get

$$\pi_*(X) \cong \pi_*(s^{-1}V) = s^{-1}V. \qquad \square$$

PROPOSITION 5.2. *Let A be a unital associative algebra. Consider the plus-construction of* $\mathrm{sl}(A)$ *in the category of differential graded Lie algebras. Then*

$$\pi_*(\mathrm{sl}(A)^+) \cong \overline{HC}_*(A),$$

where $\overline{HC}_*(A)$ is the reduced cyclic homology of A, i.e. $\overline{HC}_0(A) = 0$ and $\overline{HC}_i(A) = HC_i(A)$, for $i > 0$.

Proof. The proof is based on well-known results. By the theorem of J.-L. Loday and D. Quillen [10, 12], proved independently by B. Tsygan [20], there exists an isomorphism of Hopf algebras

$$H_*^{\mathcal{L}\mathrm{ie}}(\mathrm{gl}(A)) \cong \Lambda_*(HC(A)[1]).$$

The proof of this theorem goes in two steps. The first one consists of computing the primitive part of the coalgebra $H_*^{\mathcal{L}\mathrm{ie}}(\mathrm{gl}(A))$. In the second step, the direct sum of matrices endows $H_*^{\mathcal{L}\mathrm{ie}}(\mathrm{gl}(A))$ with a structure of commutative and co-commutative Hopf algebra. The result follows by applying the theorem of Cartan–Milnor–Moore. With the same method, it is not hard to prove that there is an isomorphism of co-commutative coalgebras:

$$H_*^{\mathcal{L}\mathrm{ie}}(\mathrm{sl}(A)) \cong \Lambda_*(\overline{HC}(A)[1]).$$

However, $H_*^{\mathcal{L}\mathrm{ie}}(\mathrm{sl}(A)^+) = H_*^{\mathcal{L}\mathrm{ie}}(\mathrm{sl}(A))$ and $\mathrm{sl}(A)^+$ is connected, hence by virtue of Proposition 5.1, we get the isomorphism:

$$\pi_*(\mathrm{sl}(A)^+) \cong \overline{HC}_*(A). \qquad \square$$

PROPOSITION 5.3. *Let A be a unital associative algebra. Consider the plus-construction of* $\mathrm{sl}(A)$ *in the category of differential graded Leibniz algebras. Then*

$$\pi_*(\mathrm{sl}(A)^+) \cong \overline{HH}_*(A),$$

where $\overline{HH}_*(A)$ is the reduced Hochschild homology of A, i.e. $\overline{HH}_0(A) = 0$ and $\overline{HH}_i(A) = HH_i(A)$, for $i > 0$.

Proof. The theorem of C. Cuvier and J.-L. Loday [1, 10] implies the following isomorphism of graded vector spaces:

$$HL(\mathrm{gl}(A)) \cong \overline{T}(HH(A)[1]).$$

J.-M. Oudom improved this theorem by proving that this isomorphism is an isomorphism of graded Leibniz-dual coalgebras [14], and is an isomorphism of abelian groups in the category of graded Leibniz-dual coalgebras: the group structure is induced by the direct sum of matrices. With the same method of proving the theorem of C. Cuvier, J.-L. Loday and J.-M. Oudom, we get the following isomorphism of graded Leibniz-dual coalgebras:

$$HL(\mathrm{sl}(A)) \cong \overline{T}(\overline{HH}(A)[1])\,.$$

The result now follows by Proposition 5.1. □

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