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Lie theory for Hopf operads

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Abstract

The present article takes advantage of the properties of algebras in the category of $\mathbb{S}$-modules (twisted algebras) to investigate further the fine algebraic structure of Hopf operads. We prove that any Hopf operad $\mathcal{P}$ carries naturally the structure of a comonoid in the category of twisted $\mathcal{P}$-algebras. Many properties of Hopf algebraic structures are then shown to be encapsulated in this remarquable structure. In particular, various classical theorems of the theory of free Lie algebras are lifted to arbitrary Hopf operads.

Keywords: Hopf operad; Twisted Hopf algebra; Cartier–Milnor–Moore theorem; Poisson operad

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Introduction

Let $\mathcal{P}$ be an arbitrary algebraic operad, that is, a monoid in the category of $\mathbb{S}$-modules or, equivalently, the analytic functor associated to a given (suitable) class of algebras. The usual

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definition of algebras over $\mathcal{P}$ in the category of vector spaces carries over to algebras over $\mathcal{P}$ in the category of $\mathbb{S}$-modules. These algebras are classically referred to as *twisted $\mathcal{P}$-algebras*. The meaning and importance of twisted algebraic structures was pointed out in 1978 by Barratt, whose work was motivated by the study of homotopy invariants [Bar78]. He introduced the notion of *twisted* Lie algebras (Lie algebras in the category of $\mathbb{S}$-modules). His constructions were later extended to various other algebraic structures by Joyal [Joy86].

The general definition of an algebra over an operad $\mathcal{P}$ in the category of $\mathbb{S}$-modules is more recent and appears often in the recent literature [MSS02,Fre04] under the categorical name of "left $\mathcal{P}$-module." The terminology twisted $\mathcal{P}$-algebra is used here in order to emphasize the fact that we embed the category of $\mathcal{P}$-algebras into that of twisted $\mathcal{P}$-algebras.

Similarly to what happens with groups or usual algebras, a natural way to study operads is to study their representations (in the category of $\mathbb{S}$-modules). Accordingly, the purpose of the present article is to take advantage of the properties of *twisted* algebraic structures to investigate the internal algebraic structure of operads. We are mainly interested in Hopf operads, that is, the particular class of operads $\mathcal{P}$ for which the tensor product of two $\mathcal{P}$-algebras can be provided with the structure of a $\mathcal{P}$-algebra. Hopf operads are widespread objects in algebraic topology: this is because the diagonal map (in the category of topological spaces) often induces a connected Hopf operad structure on the homology of topological operads. Indeed, the notion of Hopf operad first appeared in the work of Getzler–Jones on iterated loop spaces and two-dimensional topological field theory. Hopf operads also appear naturally in algebra: the associative, commutative, magmatic and Poisson operads are Hopf operads—see for example [Fre98], where these structures were used to investigate various Hopf algebra structures, including commutative Hopf–Poisson algebras [Dri86]. Recently, various new Hopf operads appeared, often linked with combinatorial structures (trees, combinatorial Hopf algebras...). Examples include the Ramanujan operad [Cha03], the dendriform, pre-dendriform operads, and other operads satisfying "splitting of associativity" conditions (see [Lod04]). A striking application of these ideas to the solution of the Duchamp–Hivert–Thibon conjecture (the Lie algebra of primitive elements in the Malvenuto–Reutenauer Hopf algebra is a free Lie algebra) is contained in [Foi05].

The present article focuses mostly on connected Hopf operads $\mathcal{P}$ such that the category of twisted $\mathcal{P}$-algebras is a tensor category—with, in particular, a unit for the tensor product. This class of Hopf operads includes the associative, magmatic, commutative, Poisson operads and variants thereof.

The article is organized as follows. We first show that a connected Hopf operad $\mathcal{P}$ is naturally a twisted Hopf $\mathcal{P}$-algebra, that is, a comonoid in the category of twisted $\mathcal{P}$-algebras. We then prove that the set of primitive elements for this comonoidal structure is a sub-operad of $\mathcal{P}$. When this result is applied to the associative operad $\mathcal{A}s$, one obtains a twisted Hopf algebra structure on the direct sum S_* of the symmetric group algebras. As could be expected, the operad of its primitive elements is the Lie operad, which makes more precise the results obtained in [PR04] on the fine twisted Hopf algebra structure of S_*. Furthermore, there is a primitive functor from twisted Hopf $\mathcal{P}$-algebras to twisted $\mathcal{Q}$-algebras where $\mathcal{Q}$ is the operad of primitive elements of $\mathcal{P}$. In Section 3, we prove that this functor admits a left adjoint: the enveloping algebra functor. We also show that many classical properties in the theory of free associative algebras and free Lie algebras go over to arbitrary connective Hopf operads and their primitive sub-operads. Our general results are illustrated in Section 4 on the most classical examples of algebra structures in the Lie-theoretical setting, namely the associative and Poisson algebras and the corresponding Hopf operads.

1. Twisted algebras over an operad

This section gives a brief account of the notion of twisted algebras. For further details on this notion, we refer to Barratt [Bar78], Joyal [Joy86], Markl, Schnider, and Stasheff [MSS02], Fresse [Fre04] and Patras and Reutenauer [PR04].

1.1. $\mathbb{S}$-*modules*

Let $\mathbf{k}$ be the ground field.

1.1.1. Definition. An $\mathbb{S}$-*module* $M = \{M(n)\}_{n\geqslant 0}$ is a collection of right S_n-modules over the field $\mathbf{k}$. Permutations in S_n are written as sequences $\sigma = (\sigma(1), \dots, \sigma(n))$.

Following Joyal, an $\mathbb{S}$-module is equivalent to a *vector species*, that is, a contravariant functor from the category of finite sets **Bij** (and set isomorphisms) to the category **Vect** of $\mathbf{k}$-vector spaces. This equivalence goes as follows: from an $\mathbb{S}$-module M one defines the vector species

$$S \mapsto M(S) := \bigoplus_{i_*\colon S\to\{1,\dots,r\}} M(r)/\equiv$$

where r is the number of elements of S and i_* is a bijection. The equivalence relation is given by $(m\cdot\sigma, i_*) \equiv (m, \sigma\circ i_*)$.

Conversely, the skeleton of a vector species M is an $\mathbb{S}$-module. The action of S_n on $M(n) := M(\{1,\dots,n\})$ is given by $m\cdot\sigma = M(\sigma)(m)$.

1.1.2. Operations on $\mathbb{S}$-modules. The category of $\mathbb{S}$-modules is provided with three monoidal structures: $(\cdot, \mathbf{k})$, $(\otimes, \mathcal{C}\mathrm{om})$ and $(\circ, I)$, the first two being symmetric. The notation here follows the one of [BLL98].

(i) First, the category of vector species is a linear symmetric monoidal category for the *tensor product* $\cdot$:

$$(M\cdot N)(U) = \bigoplus_{I\sqcup J=U} M(I)\otimes N(J)$$

where $I\sqcup J$ runs over the partitions of U. Translated to $\mathbb{S}$-modules, the definition of $\cdot$ reads:

$$\begin{aligned}(M\cdot N)(n) &= \bigoplus_{p+q=n} \bigl(M(p)\otimes N(q)\bigr)\otimes_{S_p\times S_q}\mathbf{k}[S_n]\\ &= \bigoplus_{p+q=n} \bigl(M(p)\otimes M(q)\bigr)\otimes\mathbf{k}[\mathrm{Sh}_{p,q}]\end{aligned}$$

where $\mathrm{Sh}_{p,q}$ is the set of (p,q)-shuffles, that is, permutations of S_n that can be written $(\tau_1,\dots,\tau_p,\rho_1,\dots,\rho_q)^{-1}$, where $\tau_1<\dots<\tau_p$ and $\rho_1<\dots<\rho_q$. The second equality follows from the existence, for any permutation $\sigma\in S_n$, of a unique decomposition $\sigma = (\sigma_1\times\sigma_2)\cdot\alpha$ where α is a (p,q)-shuffle and $S_p\times S_q$ stands for the usual embedding of $S_p\times S_q$ into S_{p+q}.

Hence an element in $M \cdot N$ can be written uniquely as $m \otimes n \otimes \sigma$ where $m \in M(p)$, $n \in N(q)$ and $\sigma \in \mathrm{Sh}_{p,q}$. If $\sigma = \mathrm{id}_{S_{p+q}}$ we write $m \otimes n$ instead of $m \otimes n \otimes \sigma$.

The unit for this tensor product is the following $\mathbb{S}$-module, still denoted $\mathbf{k}$ by a slight abuse of notation:

$$\mathbf{k}(n) = \begin{cases} \mathbf{k}, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Replacing $\mathbf{k}$ by an arbitrary vector space in this definition allows us to consider more generally an arbitrary $\mathbf{k}$-vector space as an $\mathbb{S}$-module concentrated in degree 0. In other terms, the category of vector spaces embeds as a full subcategory in the category of $\mathbb{S}$-modules. Furthermore, the tensor product of two vector spaces in the category of $\mathbb{S}$-modules coincides with the usual tensor product of vector spaces.

In the species framework, the symmetry isomorphism $\tau_{M,N} : M \cdot N \to N \cdot M$ is given by

$$\tau_{M,N} : M(I) \otimes N(J) \to N(J) \otimes M(I)$$

that is, in the $\mathbb{S}$-modules framework

$$\tau_{M,N}(m \otimes n) = n \otimes m \otimes \zeta_{p,q} \tag{1.1}$$

where $m \in M(p)$, $n \in N(q)$ and $\zeta_{p,q} = (q+1, \ldots, q+p, 1, \ldots, q)$.

For any $\sigma \in S_n$, the symmetry isomorphism induces an isomorphism τ_σ of $\mathbb{S}$-modules from $M_1 \cdot \cdots \cdot M_n$ to $M_{\sigma^{-1}(1)} \cdot \cdots \cdot M_{\sigma^{-1}(n)}$ given by

$$\tau_\sigma(m_1 \otimes \cdots \otimes m_n) = m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(n)} \otimes \sigma(l_1, \ldots, l_n) \tag{1.2}$$

where $m_i \in M_i(l_i)$ and $\sigma(l_1, \ldots, l_k)$ is the permutation of $S_{l_1+\cdots+l_k}$ obtained by replacing $\sigma(i)$ by the block Id_{l_i}. More precisely

$$\sigma(l_1, \ldots, l_k) = (B_1, \ldots, B_k)$$

where B_i is the block $\sum_{j=1}^{\sigma(i)-1} l_{\sigma^{-1}(j)} + [l_i]$. For instance

$$(2,3,1)(a,b,c) = (c+1, \ldots, c+a, c+a+1, \ldots, c+a+b, 1, \ldots, c).$$

(ii) The notation $M \otimes N$ is devoted to the $\mathbb{S}$-module given by the collection $M(n) \otimes N(n)$ together with the diagonal action of the symmetric groups. It is the vector species $(M \otimes N)(U) = M(U) \otimes N(U)$. The unit for this second linear symmetric monoidal structure is the $\mathbb{S}$-module $\mathcal{C}\mathrm{om}$ where $\mathcal{C}\mathrm{om}(n) = \mathbf{k}$ with the trivial action.

(iii) Finally, the category of $\mathbb{S}$-modules is endowed with still another monoidal structure (which is not symmetric): the *plethysm* $\circ$ defined by

$$(M \circ N)(n) := \bigoplus_{k \geqslant 0} M(k) \otimes_{S_k} \left(N^{\cdot k}\right)(n),$$

where S_k acts on the left of $(N^{\cdot k})$ by formula (1.2).

An alternative definition in the framework of vector species is given by

$$(M \circ N)(U) := \bigoplus_{k \geqslant 1} M(k) \otimes_{S_k} \Biggl(\bigoplus_{I_1 \sqcup \cdots \sqcup I_k = U} N(I_1) \otimes \cdots \otimes N(I_k) \Biggr),$$

when $U \neq \emptyset$ and where the action of S_k is given by

$$\sigma \cdot \bigl(N(I_1) \otimes \cdots \otimes N(I_k)\bigr) = N(I_{\sigma^{-1}(1)}) \otimes \cdots \otimes N(I_{\sigma^{-1}(k)}).$$

The unit for the plethysm is the $\mathbb{S}$-module I given by

$$I(n) = \begin{cases} \mathbf{k}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

1.1.3. Natural transformations relating the operations on $\mathbb{S}$-modules. Two natural transformations relate the monoidal structures:

$$T_1 : (A \otimes B) \circ (C \otimes D) \to (A \circ C) \otimes (B \circ D),$$
$$T_2 : (A \otimes B) \circ (C \cdot D) \to (A \circ C) \cdot (B \circ D).$$

They are obtained by interchanging terms in the corresponding direct sums. The symmetry isomorphism τ has to be taken into account in order to define T_2.

In terms of $\mathbb{S}$-modules, one has the following description: let $a \in A(k)$ and $b \in B(k)$, $c_i \in C(l_i)$, $d_i \in D(l_i)$, $e_i \in C(r_i)$, $f_i \in D(s_i)$:

$$T_1\bigl((a \otimes b) \otimes (c_1, d_1, c_2, d_2, \ldots, c_k, d_k)\bigr) = (a \otimes c_1, \ldots, c_k) \otimes (b \otimes d_1, \ldots, d_k),$$
$$T_2\bigl((a \otimes b) \otimes (e_1, f_1, e_2, f_2, \ldots, e_k, f_k)\bigr)$$
$$= (a \otimes e_1, \ldots, e_k) \otimes (b \otimes f_1, \ldots, f_k) \cdot \sigma(r_1, s_1, r_2, s_2, \ldots, r_k, s_k)$$

where $\sigma = (1, k+1, 2, k+2, 3, \ldots, 2k-1, 2k)$.

The relations are even more natural when written in terms of vector species. Let us consider, for example, the relation defining T_2. We have, for $e_i \in C(S_i)$ and $f_i \in D(T_i)$ with $\coprod_{i=1}^{k}(S_i \amalg T_i) = U$:

$$T_2 : \bigl(A(k) \otimes B(k)\bigr) \otimes_{S_k} \bigl(\bigl(C(S_1) \otimes D(T_1)\bigr) \otimes \cdots \otimes \bigl(C(S_k) \otimes D(T_k)\bigr)\bigr)$$
$$\longrightarrow \bigl(A(k) \otimes_{S_k} \bigl(C(S_1) \otimes \cdots \otimes C(S_k)\bigr)\bigr) \otimes \bigl(B(k) \otimes_{S_k} \bigl(D(T_1) \otimes \cdots \otimes D(T_k)\bigr)\bigr),$$

$$T_2\bigl(a \otimes b \otimes (e_1, f_1, e_2, f_2, \ldots, e_k, f_k)\bigr) = (a \otimes e_1, \ldots, e_k) \otimes (b \otimes f_1, \ldots, f_k).$$

As a consequence of the definitions, the following relations hold

$$(A \cdot B) \circ C = (A \circ C) \cdot (B \circ C), \tag{1.3}$$

$$T_1(T_1 \circ \mathrm{Id}) = T_1(\mathrm{Id} \circ T_1), \tag{1.4}$$

$$T_2(T_1 \circ \mathrm{Id}) = T_2(\mathrm{Id} \circ T_2). \tag{1.5}$$

The last identity expresses in two different ways the natural transformation

$$(A\otimes B)\circ(C\otimes D)\circ(E\cdot F)\to(A\circ C\circ E)\cdot(B\circ D\circ F).$$

1.2. Operads

1.2.1. Definition. An *operad* is a monoid in the category of $\mathbb{S}$-modules with respect to the plethysm. That is, an operad is an $\mathbb{S}$-module $\mathcal{P}$ together with a product $\mu_{\mathcal{P}}:\mathcal{P}\circ\mathcal{P}\to\mathcal{P}$ and a unit $u_{\mathcal{P}}:I\to\mathcal{P}$ satisfying

$$\mu_{\mathcal{P}}(\mathcal{P}\circ\mu_{\mathcal{P}})=\mu_{\mathcal{P}}(\mu_{\mathcal{P}}\circ\mathcal{P}),$$
$$\mu_{\mathcal{P}}(\mathcal{P}\circ u_{\mathcal{P}})=\mu_{\mathcal{P}}(u_{\mathcal{P}}\circ\mathcal{P})=\mathcal{P}.$$

In other terms an operad $\mathcal{P}$ is a collection of S_n-modules $(\mathcal{P}(n))_{n\geqslant 0}$ together with an element $1_1\in\mathcal{P}(1)$ and compositions

$$\gamma:\mathcal{P}(k)\otimes\mathcal{P}(l_1)\otimes\cdots\otimes\mathcal{P}(l_k)\to\mathcal{P}(l_1+\cdots+l_k)$$

satisfying associativity, unitary conditions and equivariance conditions reflecting the action of $S_k\times S_n$ on $\mathcal{P}^{\cdot k}(n)$ given by:

$$\gamma(p\cdot\sigma,p_1,\ldots,p_k)=\gamma(p,p_{\sigma^{-1}(1)},\ldots,p_{\sigma^{-1}(k)})\cdot\sigma(l_1,\ldots,l_k),$$
$$\gamma(p,p_1\cdot\tau_1,\ldots,p_k\cdot\tau_k)=\gamma(p,p_1,\ldots,p_k)\cdot(\tau_1\oplus\cdots\oplus\tau_k).$$

Usually, $\gamma(p,p_1,\ldots,p_k)$ is written $p(p_1,\ldots,p_k)$.

When $p_j=1_1$ for all j except i the latter composition is written $p\circ_i p_i$. The associativity, unitarity and equivariance read as follows. For $p\in\mathcal{P}(n)$, $q\in\mathcal{P}(m)$, $r\in\mathcal{P}(l)$, $\sigma\in S_n$, $\tau\in S_m$, we have:

$$(p\circ_i q)\circ_{j+i-1}r=p\circ_i(q\circ_j r), \tag{1.6}$$
$$(p\circ_i q)\circ_{j+m-1}r=(p\circ_j r)\circ_i q,\quad i<j, \tag{1.7}$$
$$p\circ_i 1_1=p=1_1\circ_1 p, \tag{1.8}$$
$$(p\cdot\sigma)\circ_i(q\cdot\tau)=(p\circ_{\sigma(i)}q)\cdot(\sigma\circ_i\tau), \tag{1.9}$$

where $\sigma\circ_i\tau$ is the permutation of S_{n+m-1} obtained by replacing τ for $\sigma(i)$. For instance

$$(3,4,2,5,1)\circ_2(a,b,c)=(3,a+3,b+3,c+3,2,7,1).$$

Notice that the definition of operads generalizes to arbitrary symmetric monoidal categories.

1.2.2. Connected operads. An operad $\mathcal{P}$ is *connected* if $\mathcal{P}(0)$ is isomorphic to $\mathbf{k}$. In the sequel, to avoid confusion, we will denote by 1_0 the unit in $\mathcal{P}(0)$ and 1_1 the one in $\mathcal{P}(1)$. If $\mathcal{P}$ is a connected operad, then it is endowed with degeneracy maps, for all $S\subset[n]$ of cardinality l,

$$\begin{array}{rcl} |_S:\mathcal{P}(n) & \to & \mathcal{P}(l),\\ p & \mapsto & p|_S=p(x_1,\ldots,x_n)\end{array}$$

where $x_i = 1_1$, if $i \in S$ and $x_i = 1_0$ if not. As a consequence $p|_\emptyset \in \mathcal{P}(0) \cong \mathbf{k}$ since the operad $\mathcal{P}$ is connected. A *morphism of connected operads* $\phi : \mathcal{P} \to \mathcal{Q}$ is a morphism of operads such that $\phi(1_0) = 1_0$. As a consequence $\phi(p|_S) = \phi(p)|_S$.

1.2.3. Examples. On the $\mathbb{S}$-module $\mathcal{C}\mathrm{om}$ (the unit for $\otimes$ defined in Section 1.1.2), one can define an operad structure, where the composition is given by the product in $\mathbf{k}$: for any $\lambda \in \mathcal{C}\mathrm{om}(n)$, $\lambda_i \in \mathcal{C}\mathrm{om}(l_i)$,

$$\lambda(\lambda_1, \dots, \lambda_n) = \lambda\lambda_1 \cdots \lambda_n.$$

It is a connected operad whose degeneracy maps are the identity on $\mathbf{k}$.

Another fundamental example is the operad $\mathcal{A}\mathrm{s}$ defined by $\mathcal{A}\mathrm{s}(n) = \mathbf{k}[S_n]$ for all $n \geqslant 0$. The composition $\sigma \circ_i \tau$ is the one given in Definition 1.2.1. This operad is connected. The degeneracy maps are the following: for $S = \{s_1 < \cdots < s_l\}$ the permutation $\sigma|_S$ is the standardization of $(\sigma(s_1), \dots, \sigma(s_l))$. Recall that, in general, the standardization of a sequence of length l of distinct non-negative integers is the process by which the elements of the sequence are replaced by the integers $1, \dots, l$ in such a way that the relative order of the elements in the sequence is preserved. For instance, for $\sigma = (3, 2, 6, 1, 8, 7, 5, 4) \in S_8$, we have

$$\sigma|_{\{1,3,6,7\}} = \mathrm{st}(3, 6, 7, 5) = (1, 3, 4, 2) \in S_4.$$

Furthermore, for any $\sigma \in S_n$

$$\sigma|_\emptyset = 1.$$

Let $\mathcal{P}$ be an operad. Given n sets $S_i \subset [l_i]$ for $1 \leqslant i \leqslant n$ the set $S_1 \star \cdots \star S_n$ is the subset of $[l_1 + \cdots + l_n]$ of all $l_1 + \cdots + l_{i-1} + \alpha$, $i \leqslant n$, $\alpha \in S_i$. Using the associativity of $\mu_{\mathcal{P}}$ one gets, for all $p \in \mathcal{P}(n)$, $q_i \in \mathcal{P}(l_i)$,

$$p(q_1, \dots, q_n)|_{S_1 \star \cdots \star S_n} = p(q_1|_{S_1}, \dots, q_n|_{S_n}).$$

This relation yields the following lemma.

1.2.4. Lemma. *Let $\mathcal{P}$ be a connected operad. Let $p \in \mathcal{P}(n)$, $q_i \in \mathcal{P}(l_i)$ and $S_i \subset [l_i]$. For any set $J = \{j_1 < \cdots < j_l\} \subset [n]$ such that S_i is empty for all $i \notin J$, one has*

$$p(q_1, \dots, q_n)|_{S_1 \star \cdots \star S_n} = \left(\prod_{i \notin J} q_i|_\emptyset\right) p|_J(q_{j_1}|_{S_{j_1}}, \dots, q_{j_l}|_{S_{j_l}}).$$

1.3. Twisted algebras over an operad

1.3.1. Definition. Let $\mathcal{P}$ be an operad. An $\mathbb{S}$-module M is a *twisted $\mathcal{P}$-algebra* if M is endowed with a product $\mu_M : \mathcal{P} \circ M \to M$ such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{P} \circ \mathcal{P} \circ M \arrow[r, "\mathcal{P} \circ \mu_M"] \arrow[d, "\mu_{\mathcal{P}} \circ M"'] & \mathcal{P} \circ M \arrow[d, "\mu_M"] \\
\mathcal{P} \circ M \arrow[r, "\mu_M"] & M,
\end{tikzcd}
\qquad
\begin{tikzcd}
I \circ M \arrow[r, "="] \arrow[d, "u_{\mathcal{P}} \circ M"'] & M. \\
\mathcal{P} \circ M \arrow[ur, "\mu_M"'] &
\end{tikzcd}$$

Recall that the product $\circ$ involves the tensor product $\cdot$ in the category of $\mathbb{S}$-modules. This is the ground for the Barratt–Joyal terminology of twisted algebras, that refers to the existence of symmetric group actions "twisting" the usual definition of algebras over operads [Bar78,Joy86].

Recall also that the category of vector spaces **Vect** embeds as a full linear symmetric monoidal subcategory of the category of $\mathbb{S}$-modules. As a corollary, twisted $\mathcal{P}$-algebras concentrated in degree 0 are $\mathcal{P}$-algebras, in the usual terminology (see e.g. [MSS02]).

1.3.2. Example: Twisted Lie algebras. Let $\mathcal{L}$ie be the Lie operad. As an operad it is generated by $\mu \in \mathcal{L}\mathrm{ie}(2)$ satisfying the following relations:

$$\mu \cdot (2,1) = -\mu,$$
$$\mu \circ_2 \mu \cdot \big((1,2,3) + (2,3,1) + (3,1,2)\big) = 0.$$

A *twisted Lie algebra* is a twisted algebra over the operad $\mathcal{L}$ie, that is, an $\mathbb{S}$-module M endowed with a multiplication, the bracket, defined by $[a,b] = \mu(a,b)$ and satisfying the following relations: let $a \in M(p)$, $b \in M(q)$, $c \in M(r)$,

$$[b,a] \cdot \zeta_{p,q} = -[a,b],$$
$$\big[a,[b,c]\big] + \big[c,[a,b]\big] \cdot \zeta_{p+q,r} + \big[b,[c,a]\big] \cdot \zeta_{p,q+r} = 0,$$

where $\zeta_{p,q}$ was defined in relation (1.1). This is exactly Definition 4 in [Bar78]. These relations are a direct consequence of the computation of $(\mu \cdot (2,1))(a,b)$ and $((\mu \circ_2 \mu) \cdot (1,2,3) + (2,3,1) + (3,1,2))(a,b,c)$ using the symmetry isomorphism τ. For instance,

$$\big((\mu \circ_2 \mu) \cdot (2,3,1)\big)(a,b,c) = \big[c,[a,b]\big] \cdot (2,3,1)(p,q,r) = \big[c,[a,b]\big] \cdot \zeta_{p+q,r}.$$

1.3.3. Connected twisted $\mathcal{P}$-algebras. Assume $\mathcal{P}$ is a connected operad. The $\mathbb{S}$-module $\mathbf{k}$ (the unit for $\cdot$) is a twisted $\mathcal{P}$-algebra for the product

$$p(\lambda_1, \ldots, \lambda_n) = p|_{\emptyset}\lambda_1 \cdots \lambda_n, \quad \forall p \in \mathcal{P}(n),\ \lambda_i \in \mathbf{k}.$$

As a consequence, for any twisted $\mathcal{P}$-algebra M, the restriction to $\mathcal{P}(0)$ of the morphism $\mu_M : (\mathcal{P} \circ M)(0) \to M(0)$ induces a morphism of twisted $\mathcal{P}$-algebras

$$\eta_M : \mathbf{k} \to M$$

satisfying

$$p(a_1, \ldots, a_n) = p|_{\{i_1,\ldots,i_r\}}(a_{i_1}, \ldots, a_{i_r}) \tag{1.10}$$

for all $p \in \mathcal{P}(n)$, $a_i \in M$ as soon as $a_j = \eta_M(1)$ for all $j \notin \{i_1, \ldots, i_r\}$. Thus, connected operads allow us to deal with (twisted) $\mathcal{P}$-algebras M with unit.

A *connected twisted $\mathcal{P}$-algebra M* is a twisted $\mathcal{P}$-algebra such that the morphism η_M realizes an isomorphism onto $M(0)$.

1.3.4. Free twisted $\mathcal{P}$-algebras. Since I is the unit for the plethysm, the $\mathbb{S}$-module $\mathcal{P} = \mathcal{P} \circ I$ is the free twisted $\mathcal{P}$-algebra generated by I. More generally, $\mathcal{P} \circ M$ is the free twisted $\mathcal{P}$-algebra generated by an $\mathbb{S}$-module M.

If $\mathcal{P}$ is connected and $M(0) = 0$, then $\mathcal{P} \circ M$ is a connected twisted $\mathcal{P}$-algebra.

2. Twisted Hopf algebras over a Hopf operad

This section gives the first definitions and results on Hopf operads. A Hopf operad $\mathcal{P}$ has the important property that the tensor product of twisted $\mathcal{P}$-algebras remains a twisted $\mathcal{P}$-algebra (see Theorem 2.2.1). If in addition the Hopf operad $\mathcal{P}$ is connected, then the $\mathbb{S}$-module $\mathbf{k}$ is a twisted $\mathcal{P}$-algebra. In that case, the category of twisted $\mathcal{P}$-algebras is a monoidal category with unit $\mathbf{k}$. It is symmetric if $\mathcal{P}$ is cocommutative.

If the operad $\mathcal{P}$ is connected, then there exists a morphism of twisted Hopf algebras $\Delta : \mathcal{P} \to \mathcal{P} \cdot \mathcal{P}$ making $\mathcal{P}$ into a twisted Hopf $\mathcal{P}$-algebra (see Theorem 2.3.3). The latter structure is defined in Section 2.3. The main theorem of this section states that the primitive elements with respect to Δ form a sub-operad of $\mathcal{P}$ (see Theorem 2.4.2).

2.1. Hopf operads

Let **Coalg** be the category of coassociative counital coalgebras, that is, vector spaces V endowed with a coassociative coproduct $\delta : V \to V \otimes V$ and a linear map, the counit, $\epsilon : V \to \mathbf{k}$ satisfying

$$(\epsilon \otimes V)\delta = (V \otimes \epsilon)\delta = V.$$

2.1.1. Definition. (See [GJ94,Moe02].) Since the category **Coalg** is a symmetric monoidal category for the tensor product one can define an operad in this category: a *Hopf operad* $\mathcal{P}$ is an operad in the category of coalgebras: $\mu_{\mathcal{P}}$ and $u_{\mathcal{P}}$ are morphisms of coalgebras. Equivalently, there exist morphisms of operads $\delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ and $\epsilon : \mathcal{P} \to \mathcal{C}om$, such that δ is coassociative and $(\epsilon \otimes \mathcal{P})\delta = (\mathcal{P} \otimes \epsilon)\delta = \mathcal{P}$. This makes sense since $\mathcal{C}om$ is the unit for the tensor product $\otimes$ and $\mathcal{P} \otimes \mathcal{P}$ is equipped with the structure of an operad by the natural transformation T_1 defined in 1.1.3.

A *connected Hopf operad* is a Hopf operad which is connected and such that δ and ϵ are morphisms of connected operads. As a consequence,

$$\epsilon(p) = \epsilon(p)|_{\emptyset} = \epsilon(p|_{\emptyset}) = p|_{\emptyset}.$$

A Hopf operad is *cocommutative* whenever δ is cocommutative.

2.1.2. Remark. One may alternatively consider operads in the category of coassociative coalgebras (without a counit). This gives rise to a weaker notion of Hopf operad, obtained by removing the conditions relative to ϵ. We call these operads *non-counital Hopf operads*. Although most of our results will be stated for Hopf operads, in many cases of interest it is possible to extend them to the non-counital case.

2.1.3. Example 1. The operad $\mathcal{C}om$ is a connected cocommutative Hopf operad for the coproduct $\delta(\lambda) = \lambda 1 \otimes 1$ for $\lambda \in \mathcal{C}om(n) = \mathbf{k}$ and the counit the identity map.

This construction may be extended to any operad provided with a map of operads $\eta : \mathcal{C}\mathrm{om} \to \mathcal{P}$. Indeed, the map $\mathcal{P} \otimes \eta$:

$$\mathcal{P} = \mathcal{P} \otimes \mathcal{C}\mathrm{om} \to \mathcal{P} \otimes \mathcal{P}$$

induces a non-counital Hopf algebra structure on $\mathcal{P}$. The observation applies to the Poisson operad $\mathcal{P}\mathrm{ois}$ and, more generally, to any operad obtained from $\mathcal{C}\mathrm{om}$, another operad $\mathcal{Q}$, and a distributive law between $\mathcal{C}\mathrm{om}$ and $\mathcal{Q}$ in the sense of Beck–Markl, see [Mar96]. However, the non-counital Hopf algebra structure on $\mathcal{P}\mathrm{ois}$ that can be defined by this process is not the usual one. The latter will be described in the last section of the article and provides $\mathcal{P}\mathrm{ois}$ with a connected cocommutative Hopf operad structure.

The same construction can be adapted to the case where $\mathcal{P}(0) = 0$. Let $\mathcal{C}\mathrm{omn}$ be the operad of non-unital commutative algebras (so that $\mathcal{C}\mathrm{omn}(0) = 0$ and $\mathcal{C}\mathrm{omn}(i) = \mathbf{k}$, $i > 0$). For any map $\eta : \mathcal{C}\mathrm{omn} \to \mathcal{P}$, the map $\mathcal{P} \otimes \eta$ provides $\mathcal{P}$ with the structure of a non-counital Hopf operad. This construction applies for example to the operad of commutative p-nilpotent algebras, that is, non-unital commutative algebras such that $x_1 \cdots x_n = 0$ for $n > p$.

2.1.4. Example 2. The operad $\mathcal{A}\mathrm{s}$ is a connected cocommutative Hopf operad for the coproduct $\delta(\sigma) = \sigma \otimes \sigma$ and $\epsilon(\sigma) = 1$ for all $\sigma \in S_n$.

This construction generalizes to arbitrary set operads. Recall that an operad $\mathcal{P}$ is a set operad, if the underlying vector space $\mathcal{P}(n)$ admit a basis stable under $\mu_{\mathcal{P}}$ and the action of the symmetric group. Let (e_α) be such a basis. The set operad $\mathcal{P}$ is then a cocommutative Hopf operad for the coproduct $\delta(e_\alpha) = e_\alpha \otimes e_\alpha$ and $\epsilon(e_\alpha) = 1$.

Examples of such operads include the Tree operad [MSS02, Sect. 1.5] and the operads freely generated by a finite set of multilinear operations (with no relations between the operations) such as Holtkamp's magmatic operads (the description of which is postponed to Section 2.4.3).

2.2. *Tensor products of $\mathcal{P}$-algebras*

2.2.1. Theorem. *Let $\mathcal{P}$ be a non-counital Hopf operad and V, W be twisted $\mathcal{P}$-algebras. The $\mathbb{S}$-module $V \cdot W$ is a twisted $\mathcal{P}$-algebra for the following product: $\mu_{V \cdot W}$ is the composite of*

$$\mathcal{P} \circ (V \cdot W) \xrightarrow{\delta \circ (V \cdot W)} (\mathcal{P} \otimes \mathcal{P}) \circ (V \cdot W) \xrightarrow{T_2} (\mathcal{P} \circ V) \cdot (\mathcal{P} \circ W) \xrightarrow{\mu_V \cdot \mu_W} V \cdot W.$$

If the coalgebra structure on $\mathcal{P}$ is cocommutative, then the symmetry isomorphism τ is a morphism of twisted $\mathcal{P}$-algebras. If $\mathcal{P}$ is a connected Hopf operad, the category of $\mathcal{P}$-algebras is a tensor category, symmetric if $\mathcal{P}$ is cocommutative—with the $\mathcal{P}$-algebra $\mathbf{k}$ as a unit.

Proof. Let $\delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ be the Hopf coproduct of $\mathcal{P}$ and $M := V \cdot W$. One has to prove that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{P} \circ \mathcal{P} \circ M \arrow[r, "\mathcal{P} \circ \mu_M"] \arrow[d, "\mu_{\mathcal{P}} \circ M"'] & \mathcal{P} \circ M \arrow[d, "\mu_M"] \\
\mathcal{P} \circ M \arrow[r, "\mu_M"] & M,
\end{tikzcd}
\qquad
\begin{tikzcd}
I \circ M \arrow[r, "="] \arrow[d, "\eta_{\mathcal{P}} \circ M"'] & M. \\
\mathcal{P} \circ M \arrow[ur, "\mu_M"'] &
\end{tikzcd}$$

For the first diagram, one has

$$\begin{aligned}\mu_M(\mu_{\mathcal{P}}\circ M) &= (\mu_V\cdot\mu_W)T_2\big(\delta\circ(V\cdot W)\big)\big(\mu_{\mathcal{P}}\circ(V\cdot W)\big)\\ &= (\mu_V\cdot\mu_W)T_2\big((\delta\mu_{\mathcal{P}})\circ(V\cdot W)\big).\end{aligned}$$

The following relations hold

$$\begin{aligned}(\delta\mu_{\mathcal{P}})\circ(V\cdot W) &= (\mu_{\mathcal{P}}\otimes\mu_{\mathcal{P}})T_1(\delta\circ\delta)\circ(V\cdot W),\\ T_2\big((\mu_{\mathcal{P}}\otimes\mu_{\mathcal{P}})\circ V\cdot W\big) &= \big((\mu_{\mathcal{P}}\circ V)\cdot(\mu_{\mathcal{P}}\circ W)\big)T_2,\\ (\mu_V\cdot\mu_W)\big((\mu_{\mathcal{P}}\circ V)\cdot(\mu_{\mathcal{P}}\circ W)\big) &= (\mu_V\cdot\mu_W)\big((\mathcal{P}\circ\mu_V)\cdot(\mathcal{P}\circ\mu_W)\big).\end{aligned}$$

The first equality follows from Definition 2.1.1. The second one comes from the naturality of T_2. The last one comes from the definition of twisted $\mathcal{P}$-algebras applied to V and W (see 1.3.1). Combining these equalities, we get

$$\mu_M(\mu_{\mathcal{P}}\circ M) = (\mu_V\cdot\mu_W)\big((\mathcal{P}\circ\mu_V)\cdot(\mathcal{P}\circ\mu_W)\big)T_2(T_1\circ M)\big((\delta\circ\delta)\circ M\big).$$

Relation (1.5) implies

$$T_2(T_1\circ M)\big((\delta\circ\delta)\circ M\big) = T_2\big((\mathcal{P}\otimes\mathcal{P})\circ T_2\big)\big((\delta\circ\delta)\circ M\big).$$

The naturality of T_2 implies

$$\big((\mathcal{P}\circ\mu_V)\cdot(\mathcal{P}\circ\mu_W)\big)T_2 = T_2\big((\mathcal{P}\otimes\mathcal{P})\circ(\mu_V\cdot\mu_W)\big).$$

As a consequence we have

$$\begin{aligned}\mu_M(\mu_{\mathcal{P}}\circ M) &= (\mu_V\cdot\mu_W)T_2\big((\mathcal{P}\otimes\mathcal{P})\circ(\mu_V\cdot\mu_W)\big)\big((\mathcal{P}\otimes\mathcal{P})\circ T_2\big)\big((\delta\circ\delta)\circ M\big)\\ &= (\mu_V\cdot\mu_W)T_2\big(\delta\circ(\mu_V\cdot\mu_W)\big)(\mathcal{P}\circ T_2)\big((\mathcal{P}\circ\delta)\circ M\big)\\ &= (\mu_V\cdot\mu_W)T_2(\delta\circ M)(\mathcal{P}\circ\mu_M) = \mu_M(\mathcal{P}\circ\mu_M).\end{aligned}$$

We leave to the reader the task to check that μ_M makes the second diagram commute and that, when δ is cocommutative, $\tau: V\cdot W\to W\cdot V$ is a morphism of twisted $\mathcal{P}$-algebras.

When $\mathcal{P}$ is a connected Hopf operad, the $\mathbb{S}$-module **k** is naturally provided by the map ϵ with a twisted $\mathcal{P}$-algebra structure that coincides with the one given in 1.3.3. The last part of the theorem follows, verifications are left to the reader. □

2.2.2. Remark. Moerdijk's theorem [Moe02, Proposition 1.4] states that the category of S-algebras is a tensor category when S is a Hopf monad. The last statement of our theorem is a consequence of Moerdijk's theorem. For the proof above implies that the monad

$$\begin{aligned}\mathbb{P}:\mathbb{S}\text{-}mod &\to \mathbb{S}\text{-}mod,\\ M &\mapsto \mathcal{P}\circ M\end{aligned}$$

is a *Hopf monad* when $\mathcal{P}$ is a connected Hopf operad.

2.3. Twisted Hopf algebras over a connected Hopf operad

2.3.1. Definition. Let $\mathcal{P}$ be a connected Hopf operad so that the unit $\mathbf{k}$ for $\cdot$ is a twisted $\mathcal{P}$-algebra. A twisted $\mathcal{P}$-algebra M is a *twisted Hopf $\mathcal{P}$-algebra* if M is a comonoid in the category of twisted $\mathcal{P}$-algebras. More precisely M is endowed with a coassociative counital coproduct

$$\Delta : M \to M \cdot M, \qquad \epsilon_M : M \to \mathbf{k}$$

where Δ and ϵ_M are morphisms of twisted $\mathcal{P}$-algebras.

2.3.2. Remark. In case $\mathcal{P} = \mathcal{A}s$ one recovers twisted bialgebras as defined in [Sto93,PR04]. In that particular case, there is a straightforward way to extend the use of the classical Hopf algebraic terminology: for example the notion of antipode [MM65] makes sense, and leads to a distinction between the two notions of twisted bialgebras and twisted Hopf algebras. However, under connectedness assumptions, there is an equivalence of categories between connected twisted bialgebras and connected twisted Hopf algebras (see [Sto93]).

The notion of antipode does not hold usually for comonoids in the category of algebras or twisted algebras over a given operad $\mathcal{P}$. Besides, when the operad is a Hopf operad, the suitable notion of coproduct (or comonoidal structure, see the definition above) is no longer obtained through a duality between the two notions of algebras and coalgebras, as it is the case for Hopf (and twisted Hopf algebras) over $\mathcal{A}s$—the ground motivation for the Bourbaki-type terminology of "bialgebras." For these various reasons, we prefer to use the terminology "twisted Hopf $\mathcal{P}$-algebras" rather than "twisted $\mathcal{P}$-bialgebras."

2.3.3. Theorem. *If $\mathcal{P}$ is a connected Hopf operad then $\mathcal{P}$ is a twisted Hopf $\mathcal{P}$-algebra for the coproduct*

$$\Delta(p) = \sum_{\substack{(1),(2)\\ S \sqcup T = [n]}} p_{(1)}|_S \otimes p_{(2)}|_T \otimes \sigma(S,T)^{-1} \tag{2.1}$$

where the Hopf structure on $\mathcal{P}$ is given by $\delta(p) = \sum_{(1),(2)} p_{(1)} \otimes p_{(2)}$ and for $S = \{s_1 < \cdots < s_k\}$ and $T = \{t_1 < \cdots < t_{n-k}\}$ the permutation $\sigma(S,T)$ is $(s_1, \ldots, s_k, t_1, \ldots, t_{n-k})$. The coproduct Δ is cocommutative if $\mathcal{P}$ is a cocommutative Hopf operad.

Proof. Consider the map of $\mathbb{S}$-modules induced by the embeddings $I \oplus I \subset \mathcal{P}(1) \oplus \mathcal{P}(1) = \mathcal{P}(1) \otimes \mathcal{P}(0) \oplus \mathcal{P}(0) \otimes \mathcal{P}(1) \subset \mathcal{P} \cdot \mathcal{P}$:

$$\begin{array}{rcl} \phi : I & \to & \mathcal{P} \cdot \mathcal{P}, \\ 1_1 & \mapsto & (1_1 \otimes 1_0) + (1_0 \otimes 1_1). \end{array}$$

Since $\mathcal{P}$ is the free twisted $\mathcal{P}$-algebra on I and since $\mathcal{P} \cdot \mathcal{P}$ is endowed with a twisted $\mathcal{P}$-algebra structure thanks to Theorem 2.2.1, there is a unique morphism of twisted $\mathcal{P}$-algebras

$$\Delta : \mathcal{P} \circ I = \mathcal{P} \to \mathcal{P} \cdot \mathcal{P}$$

extending ϕ. Indeed

$$\Delta = \mu_{\mathcal{P}.\mathcal{P}}(\mathcal{P} \circ \phi) = (\mu_{\mathcal{P}} \cdot \mu_{\mathcal{P}}) T_2(\delta \circ \phi).$$

As a consequence

$$\begin{aligned}
\Delta(p) &= (\mu_{\mathcal{P}} \cdot \mu_{\mathcal{P}}) T_2 \Bigl(\sum_{(1),(2)} p_{(1)} \otimes p_{(2)} \otimes (1_0 \otimes 1_1 + 1_1 \otimes 1_0, \ldots, 1_0 \otimes 1_1 + 1_1 \otimes 1_0) \Bigr) \\
&= (\mu_{\mathcal{P}} \cdot \mu_{\mathcal{P}}) \Bigl(\sum_{\substack{(1),(2) \\ S \sqcup T = [n]}} p_{(1)} \otimes (x_1 \otimes \cdots \otimes x_n) \otimes p_{(2)} \otimes (y_1 \otimes \cdots \otimes y_n) \otimes \sigma(S,T)^{-1} \Bigr) \\
&= \sum_{\substack{(1),(2) \\ S \sqcup T = [n]}} p_{(1)}|_S \otimes p_{(2)}|_T \otimes \sigma(S,T)^{-1}
\end{aligned}$$

where

$$\begin{cases} x_i = 1_1 \text{ and } y_i = 1_0, & \text{if } i \in S, \\ x_i = 1_0 \text{ and } y_i = 1_1, & \text{if } i \in T, \end{cases}$$

and $\sigma_{S,T}^{-1}$ is the shuffle coming from T_2. The counit $\epsilon_{\mathcal{P}} : \mathcal{P} \to \mathbf{k}$ is the isomorphism between $\mathbf{k}$ and $\mathcal{P}(0)$. Indeed

$$(\epsilon_{\mathcal{P}} \otimes \mathrm{Id}) \Delta(p) = \sum_{(1),(2)} (p_{(1)}|_{\emptyset}) p_{(2)} = p$$

because $p_{(1)}|_{\emptyset} = \epsilon(p_{(1)})$ and ϵ is the counit for δ.

The map Δ is clearly coassociative on the generator 1_1, then the uniqueness of the construction implies that Δ is coassociative. □

2.3.4. Example. In case $\mathcal{P} = \mathcal{A}s$ one has

$$\Delta(\sigma) = \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T \otimes \sigma(S,T)^{-1}.$$

This is the twisted bialgebra structure on the direct sum of the symmetric group algebras described and studied in [PR04].

2.3.5. Corollary. *Let $\mathcal{P}$ be a connected Hopf operad. Any free twisted $\mathcal{P}$-algebra has a canonical twisted Hopf $\mathcal{P}$-algebra structure, cocommutative whenever $\mathcal{P}$ is.*

Proof. It follows from Theorem 2.3.3. The map $\mathcal{P} \circ M \to (\mathcal{P} \circ M) \cdot (\mathcal{P} \circ M)$ is given by the composite

$$\mathcal{P} \circ M \to (\mathcal{P} \cdot \mathcal{P}) \circ M = (\mathcal{P} \circ M) \cdot (\mathcal{P} \circ M)$$

thanks to relation (1.3). It maps $p \otimes \overline{m}$ to $1_0 \otimes (p \otimes \overline{m}) + (p \otimes \overline{m}) \otimes 1_0 + \sum_{S,T \neq \emptyset} (p_{(1)}|_S \otimes \overline{m}_S) \otimes (p_{(2)}|_T \otimes \overline{m}_T) \otimes \sigma(S,T)^{-1}$.

The counit $\epsilon_{\mathcal{P} \circ M}$ is induced by the one on $\mathcal{P}$, since $\mathbf{k} \circ M = \mathbf{k}$. One has $\epsilon_{\mathcal{P} \circ M} = \epsilon_{\mathcal{P}} \circ M : \mathcal{P} \circ M \to \mathbf{k} \circ M = \mathbf{k}$. □

2.4. *Primitive elements*

2.4.1. Definition. Let $\mathcal{P}$ be a connected Hopf operad and M be a twisted Hopf $\mathcal{P}$-algebra. From 1.3.3 we get that the unit $\eta_M : \mathbf{k} \to M$ and the counit $\epsilon_M : M \to \mathbf{k}$ satisfy $\epsilon_M \eta_M = \mathrm{Id}_{\mathbf{k}}$. Let $\overline{M} := \mathrm{Ker}\, \epsilon_M$. We have

$$M = \mathbf{k} \oplus \overline{M}$$

and for all $x \in \overline{M}$ we have

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \underbrace{\sum_{S,T} x_{(S)} \otimes x_{(T)} \otimes \sigma(S,T)^{-1}}_{\in \overline{M} \cdot \overline{M}}.$$

The space $\mathrm{Prim}(M)$ of *primitive elements* of M is defined by

$$\mathrm{Prim}(M) = \bigl\{ x \in \overline{M} \bigm| \Delta(x) = 1 \otimes x + x \otimes 1 \bigr\}.$$

Since Δ is a morphism of $\mathbb{S}$-modules, $\mathrm{Prim}(M)$ is a sub-$\mathbb{S}$-module of M. In the sequel, $\overline{\Delta}$ denotes the projection of Δ onto $\overline{M} \cdot \overline{M}$. The space of primitive elements is then $\mathrm{Prim}(M) = \ker(\overline{\Delta} : \overline{M} \to \overline{M} \cdot \overline{M})$. Note that if V is an $\mathbb{S}$-module then $\mathcal{P} \circ V$ is a twisted Hopf $\mathcal{P}$-algebra and

$$\overline{\mathcal{P} \circ V} = \overline{\mathcal{P}} \circ V. \tag{2.2}$$

Since $\overline{\Delta}$ is coassociative, one can define consistently $\overline{\Delta}^{[n]}$ as a map from $\overline{M}$ to $\overline{M}^{\cdot n}$ (e.g. $\overline{\Delta}^{[3]} := (\overline{\Delta} \cdot \overline{M})\overline{\Delta} = (\overline{M} \cdot \overline{\Delta})\overline{\Delta}$).

The twisted Hopf $\mathcal{P}$-algebra M is a *connected twisted Hopf $\mathcal{P}$-algebra* if for any $x \in \overline{M}$ there exists n such that

$$\overline{\Delta}^{[n]}(x) = 0.$$

For instance, $\mathcal{P}$ is a connected twisted Hopf $\mathcal{P}$-algebra.

2.4.2. Theorem. *Let $\mathcal{P}$ be a connected Hopf operad. The space of primitive elements of the twisted Hopf $\mathcal{P}$-algebra $\mathcal{P}$ is a sub-operad of $\mathcal{P}$.*

Proof. Notice that $p \in \mathcal{P}(n)$ is primitive if and only if

$$\sum_{S \sqcup T = [n],\, S,T \neq \emptyset} p_{(1)}|_S \otimes p_{(2)}|_T \otimes \sigma(S,T)^{-1} = 0,$$

that is, if and only if $\Delta_{S,T}(p) := p_{(1)}|_S \otimes p_{(2)}|_T = 0$.

As pointed out in Definition 2.4.1, the space $Q = \mathrm{Prim}(\mathcal{P})$ is a sub-$\mathbb{S}$-module of $\mathcal{P}$; moreover $1_1 \in Q(1)$.

Assume $p \in Q(n)$, $q_i \in Q(l_i)$, $l_i > 0$ for all $1 \leqslant i \leqslant n$ and $S \sqcup T = [l_1 + \cdots + l_n]$, $S, T \neq \emptyset$. Let us write $S = S_1 \star \cdots \star S_n$ and $T = T_1 \star \cdots \star T_n$ with $S_i \sqcup T_i = [l_i]$. Since $\Delta : \mathcal{P} \to \mathcal{P} \cdot \mathcal{P}$ is a morphism of twisted $\mathcal{P}$-algebras one has

$$\begin{aligned}&\Delta_{S,T}\bigl(p(q_1,\dots,q_n)\bigr)\\&\quad=\sum_{(a),(b),(a_1,\dots,a_n),(b_1,\dots,b_n)} p_{(a)}(q_{1(a_1)}|_{S_1},\dots,q_{n(a_n)}|_{S_n})\otimes p_{(b)}(q_{1(b_1)}|_{T_1},\dots,q_{n(b_n)}|_{T_n}),\end{aligned}$$

where $q_{i(a_i)}\otimes q_{i(b_i)}$ stands for $\delta(q_i)$. Since q_i is a primitive element, the displayed quantity is equal to 0 if there exists $i\leqslant n$ with $S_i\neq\emptyset$ and $T_i\neq\emptyset$. So, let us assume that, for any i, $S_i=\emptyset$ or $T_i=\emptyset$, and let us set: $\{i_1,\dots,i_k\}=\{i,\ S_i\neq\emptyset\}$ and $\{j_1,\dots,j_{n-k}\}=[n]-\{i_1,\dots,i_k\}=\{j,\ T_j\neq\emptyset\}$. By Lemma 1.2.4 we get

$$\Delta_{S,T}\bigl(p(q_1,\dots,q_n)\bigr)=\sum_{(1),(2)} p_{(1)}|_{\{i_1,\dots,i_k\}}(q_{i_1},\dots,q_{i_k})\otimes p_{(2)}|_{\{j_1,\dots,j_{n-k}\}}(q_{j_1},\dots,q_{j_{n-k}})$$

which is equal to zero because p is primitive, and $S,T\neq\emptyset$ implies that $k,n-k\neq 0$. □

2.4.3. Magmatic operads. As an illustration of these phenomena, and as another example of connected Hopf operads, let us consider Holtkamp's generalizations of the magmatic operad. A magma is traditionally a set provided with a binary map. The notion gives rise to an operad (the operad $\mathcal{M}\mathrm{ag}_2$ freely generated by a bilinear map), and to higher order generalizations, Holtkamp's $\mathcal{M}\mathrm{ag}_N$ (respectively $\mathcal{M}\mathrm{ag}_\omega$) operads [Hol05]. These are the free operads generated by an operation in k variables $\vee^k$ for each $2\leqslant k\leqslant N$ (respectively, for each $2\leqslant k$). They become connected operads if we define $\vee^k\circ_i 1_0=\vee^{k-1}$. More generally $\vee^k|_S=\vee^{|S|}$, where $\vee^1=1_1$ and $\vee^0=1_0$. These operads are set operads (in freely generated operads, one can parametrize the compositions of the generators by decorated trees, see e.g. [MSS02]) and therefore connected Hopf operads with $\delta(\vee^k)=\vee^k\otimes\vee^k$ and $\epsilon(\vee^k)=1_{\mathbf{k}}$. Holtkamp's theorem stating that $\mathrm{Prim}(\mathcal{M}\mathrm{ag}_N)$ is a sub-operad of $\mathcal{M}\mathrm{ag}_N$ and $\mathrm{Prim}(\mathcal{M}\mathrm{ag}_\omega)$ is a sub-operad $\mathcal{M}\mathrm{ag}_\omega$, is then a consequence of our Theorem 2.4.2.

2.4.4. Remark. When $\mathcal{P}=\mathcal{A}s$, it follows from the last theorem and from [PR04, Proposition 17] that the operad of primitive elements of $\mathcal{A}s$ is the Lie operad $\mathcal{L}ie$. This is not a surprising result in view of the classical Lie theory and structure theorems for Hopf algebras such as the Cartier–Milnor–Moore theorem [MM65,Pat94]. However this observation shows that the theory of Hopf operads and twisted algebras is a good framework to understand and extend the classical Lie theory—a phenomenon that we would like to emphasize. The next sections are devoted to a finer study of these phenomena.

3. Fine structure of twisted Hopf $\mathcal{P}$-algebras

The present section investigates the relations between a connected Hopf operad $\mathcal{P}$ and the structure of (twisted) Hopf $\mathcal{P}$-algebras. Notions of Lie theory such as the enveloping algebra functor (from Lie algebras to cocommutative Hopf algebras) are lifted to the twisted Hopf operadic setting.

3.1. The primitive part

Here, and in the next subsection, $\mathcal{P}$ is a connected Hopf operad, and $\mathcal{Q}$ is the operad of primitive elements of $\mathcal{P}$.

3.1.1. Theorem. *Let H be a twisted Hopf $\mathcal{P}$-algebra. The $\mathbb{S}$-module* Prim(H) *is a twisted $\mathcal{Q}$-algebra. As a consequence,* Prim *is a functor from the category of twisted Hopf $\mathcal{P}$-algebras to the category of twisted $\mathcal{Q}$-algebras.*

Proof. Since any element in Prim(H) satisfies $\Delta_H(h) = 1 \otimes h + h \otimes 1$ the same proof as in Theorem 2.4.2 holds using relation (1.10) instead of Lemma 1.2.4. □

In particular, since the category of $\mathcal{P}$-algebras embeds as a full subcategory in the category of twisted $\mathcal{P}$-algebras (as a direct consequence of the existence of a natural embedding of **Vect** in the category of $\mathbb{S}$-modules), the set of primitive elements of any Hopf $\mathcal{P}$-algebra is a Prim$(\mathcal{P})$-algebra.

3.1.2. Left adjoint functor to Prim. Let ι be the embedding of $\mathcal{Q}$ in $\mathcal{P}$. The forgetful functor from the category of twisted $\mathcal{P}$-algebras to the category of twisted $\mathcal{Q}$-algebras is denoted by ι^*. This functor has a left adjoint denoted by $\mathcal{U}$ (see e.g. [GJ94]). We recall its construction. Let M be a twisted $\mathcal{Q}$-algebra. The twisted $\mathcal{P}$-algebra $\mathcal{U}(M)$ is the coequalizer of the maps $\mathcal{P} \circ \mu_M$, $(\mu_{\mathcal{P}} \circ M)(\mathcal{P} \circ \iota \circ M) : \mathcal{P} \circ \mathcal{Q} \circ M \to \mathcal{P} \circ M$. In other terms:

$$\mathcal{U}(M) = (\mathcal{P} \circ M)/J(M)$$

where $J(M)$ is the ideal generated by elements of the form $\iota(q) \otimes m_1 \otimes \cdots \otimes m_n - 1_1 \otimes q(m_1, \ldots, m_n)$, for $q \in \mathcal{Q}(n)$, $m_i \in M$.

Let us write $\iota^*_{\mathcal{H}}$ for the restriction of ι^* to the functor from twisted Hopf $\mathcal{P}$-algebras to twisted $\mathcal{Q}$-algebras. The functor Prim from the category of twisted Hopf $\mathcal{P}$-algebras to the category of twisted $\mathcal{Q}$-algebras is a subfunctor of $\iota^*_{\mathcal{H}}$ since for any twisted Hopf $\mathcal{P}$-algebra H the space Prim(H) is a sub-twisted $\mathcal{Q}$-algebra of $\iota^*_{\mathcal{H}}(H)$. We denote by j the natural transformation from Prim to $\iota^*_{\mathcal{H}}$.

3.1.3. Theorem. *The functor $\mathcal{U}$ from the category of twisted $\mathcal{Q}$-algebras to the category of twisted Hopf $\mathcal{P}$-algebras is left adjoint to* Prim. *When $\mathcal{P}$ is cocommutative, the same construction gives rise to an adjunction between the category of twisted $\mathcal{Q}$-algebras and the category of cocommutative twisted Hopf $\mathcal{P}$-algebras.*

Proof. Let us show first that, for any twisted $\mathcal{Q}$-algebra M, $\mathcal{U}(M)$ carries naturally a Hopf $\mathcal{P}$-algebra structure. Denote by π the morphism of twisted $\mathcal{P}$-algebras from $\mathcal{P} \circ M$ to $\mathcal{U}(M)$. By Corollary 2.3.5, $\mathcal{P} \circ M$ is a twisted Hopf $\mathcal{P}$-algebra (cocommutative if $\mathcal{P}$ is), with coproduct Δ and counit ϵ. Let us prove that $(\pi \cdot \pi)\Delta$ and ϵ factor through $\mathcal{U}(M)$. Let $q \in \mathcal{Q}(n)$, $m_i \in M$, $i = 1 \ldots n$. Since $\iota(q)$ is primitive, we have

$$(\pi \cdot \pi)\Delta\big(\iota(q) \otimes m_1 \otimes \cdots \otimes m_n - 1_1 \otimes q(m_1, \ldots, m_n)\big) = 0.$$

Besides, the map ϵ vanishes on $J(M)$ since $\mathcal{Q}(0) = 0$ and ϵ is 0 excepted on $\mathcal{P}(0)$. The existence of a Hopf $\mathcal{P}$-algebra structure on $\mathcal{U}(M)$ follows.

Let us consider now the morphisms:

$$\psi : \mathrm{Hom}_{\mathcal{P}\text{-alg}}\big(\mathcal{U}(M), H\big) \to \mathrm{Hom}_{\mathcal{Q}\text{-alg}}\big(M, \iota^*(H)\big) \quad \text{and}$$
$$\phi : \mathrm{Hom}_{\mathrm{Hopf}\,\mathcal{P}\text{-alg}}\big(\mathcal{U}(M), H\big) \to \mathrm{Hom}_{\mathcal{Q}\text{-alg}}\big(M, \mathrm{Prim}(H)\big)$$

where ψ is the natural isomorphism expressing that $\mathcal{U}$ is left-adjoint to ι^*. For a morphism of twisted $\mathcal{P}$-algebras $f : \mathcal{U}(M) \to H$, the morphism of twisted $\mathcal{Q}$-algebras $\psi(f)$ is given by the composition of $\iota^*(f)$ with the unit of the adjunction $M \to \iota^*(\mathcal{U}(M))$. The map ϕ is induced by the Prim functor and sends a morphism of twisted Hopf $\mathcal{P}$-algebras $f : \mathcal{U}(M) \to H$ to the composition of the map $M \to \mathrm{Prim}(\mathcal{U}(M))$ and $\mathrm{Prim}(f)$, since $M \subset \mathrm{Prim}(\mathcal{U}(M))$.

We have then the following commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[dr] & \mathrm{Prim}(\mathcal{U}(M)) \arrow[r, "\mathrm{Prim}(f)"] \arrow[d, "j_{\mathcal{U}(M)}"] & \mathrm{Prim}(H) \arrow[d, "j_H"] \\
 & \iota^*_{\mathcal{H}}(\mathcal{U}(M)) \arrow[r, "\iota^*_{\mathcal{H}}(f)"] & \iota^*_{\mathcal{H}}(H).
\end{tikzcd}$$

The composition of the upper arrows gives $\phi(f)$, and the composition of the lower ones gives $\psi(f)$. Hence $\psi(f) = j_H\phi(f)$. As a consequence ϕ is injective. Let us prove that ϕ is surjective. Let $l : M \to \mathrm{Prim}(H)$ be a morphism of twisted $\mathcal{Q}$-algebras. Then there exists a morphism of twisted $\mathcal{P}$-algebras $f : \mathcal{U}(M) \to H$ such that $j_H l = \psi(f)$. To prove that f is a morphism of twisted Hopf $\mathcal{P}$-algebras it is enough to prove that $\Delta_H f = (f \cdot f)\Delta_{\mathcal{U}(M)}$ on M. Indeed both morphisms are morphisms of twisted $\mathcal{P}$-algebras from $\mathcal{U}(M)$ to $H \cdot H$ and we can use again the adjunction between $\mathcal{U}$ and ι^*. But M is primitive in $\mathcal{U}(M)$ as well as $f(M)$ in H, and the identity follows. As a consequence f is a morphism of twisted Hopf $\mathcal{P}$-algebras such that $j_H\phi(f) = j_H l$. The injectivity of j_H implies that ϕ is surjective. The last statement follows for the same reasons. □

When $\mathcal{P}$ is the associative operad, its primitive operad is the Lie operad and $\mathcal{U}$ is the twisted enveloping algebra functor defined by Stover. The theorem above is a generalization of Proposition 7.10 in [Sto93].

3.2. Free algebras

In the present section, we assume that **k** is a field of characteristic zero.

3.2.1. Theorem. *The twisted $\mathcal{Q}$-algebra of primitive elements of the free twisted Hopf $\mathcal{P}$-algebra $\mathcal{P} \circ V$ over an $\mathbb{S}$-module V is canonically isomorphic to the free twisted $\mathcal{Q}$-algebra $\mathcal{Q} \circ V$.*

As a consequence of the existence of a natural embedding of **Vect** in the category of $\mathbb{S}$-modules, the theorem holds in particular for free $\mathcal{P}$-algebras over a vector space V: the $\mathcal{Q}$-algebra of primitive elements of the free $\mathcal{P}$-algebra over a vector space V identifies with the free $\mathcal{Q}$-algebra over V. This result generalizes to arbitrary Hopf operads the fundamental property of associative algebras: the primitive part of a tensor algebra $T(V)$ over a vector space V (i.e. of a free associative algebra, naturally provided with a cocommutative Hopf algebra structure) is a free Lie algebra, see e.g. [Wig89] for a very short self-contained account. The result goes back to the introduction of the shuffle product in the study of Lie polynomials by Ree [Ree58]: see [Reu93] and the references therein for further insights on these classical topics.

Proof. By definition of Q, the following sequence of $\mathbb{S}$-modules is left exact:

$$Q \xrightarrow{i} \overline{\mathcal{P}} \xrightarrow{\overline{\Delta}} \overline{\mathcal{P}} \cdot \overline{\mathcal{P}}$$

(see Section 2.4).

In particular, for any n, the sequence

$$Q(n) \xrightarrow{i} \overline{\mathcal{P}}(n) \xrightarrow{\overline{\Delta}} \overline{\mathcal{P}} \cdot \overline{\mathcal{P}}(n)$$

is a left exact sequence of right S_n-modules. Besides, recall that, for any finite group G and any field $\mathbf{k}$ of characteristic 0, every $\mathbf{k}[G]$-module is projective. In particular, for any left S_n-module M, the tensor product $- \otimes_{S_n} M$ is an exact functor (see e.g. [Bro94, Sect. I.8]) and we have finally, for any $\mathbb{S}$-module V, a left exact sequence:

$$Q \circ V \to \overline{\mathcal{P}} \circ V \xrightarrow{\overline{\Delta} \circ V} (\overline{\mathcal{P}} \cdot \overline{\mathcal{P}}) \circ V.$$

From (2.2) we have $\overline{\mathcal{P}} \circ V = \overline{\mathcal{P} \circ V}$ and from (1.3) we have $(\overline{\mathcal{P}} \cdot \overline{\mathcal{P}}) \circ V = \overline{\mathcal{P} \circ V} \cdot \overline{\mathcal{P} \circ V}$. As a consequence, $Q \circ V = \operatorname{Prim}(\mathcal{P} \circ V)$. □

3.2.2. Remark. There is still an open question related to the enveloping algebra functor: when does it induce an equivalence of categories? In the case of connected twisted cocommutative Hopf/connected twisted Lie, this question has been answered (in any characteristic) by Stover in [Sto93, Theorem 8.4]. In general, we have only a partial answer. Analyzing carefully the previous proof, we get that in any characteristic, $\operatorname{Prim}(\mathcal{P} \circ M) = \operatorname{Prim}(\mathcal{P}) \circ M$, as soon as M is a free $\mathbb{S}$-module. Consequently, applying $\mathcal{U}$ on both sides we get

$$\mathcal{U} \operatorname{Prim}(\mathcal{P} \circ M) = \mathcal{P} \circ M,$$

since $\mathcal{U}$ is left adjoint to the forgetful functor. Hence, if M is a free $\mathbb{S}$-module, $\mathcal{P} \circ M$ is the enveloping twisted Hopf $\mathcal{P}$-algebra of its primitive elements, in any characteristic. Taking $M = I$ gives that any connected Hopf operad $\mathcal{P}$ is the enveloping twisted Hopf $\mathcal{P}$-algebra of its primitive elements. Note that in characteristic 0 the result is valid for any $\mathbb{S}$-module M.

4. Multiplicative Hopf operads

Previously we have proved that any connected Hopf operad $\mathcal{P}$ is the enveloping twisted Hopf $\mathcal{P}$-algebra of its primitive elements. In this section we focus on multiplicative Hopf operads. This structure reflects, in some sense, the fact that the operad is naturally provided with a twisted Hopf algebra structure. Then, by Stover's result, the operad is the twisting enveloping algebra of its twisted Lie algebra of primitive elements. The second part of this section is devoted to the particular case of the Poisson operad.

4.1. Multiplicative Hopf operads

Let $\mathcal{P}$ be a connected Hopf operad. Then, $\mathcal{P}$ has naturally the structure of a twisted Hopf $\mathcal{P}$-algebra. That is, there is a coproduct map from $\mathcal{P}$ to $\mathcal{P} \cdot \mathcal{P}$ which is a morphism of twisted $\mathcal{P}$-algebras.

Assume that $\phi : U \to \mathcal{P}$ is a morphism of connected Hopf operads. In view of Theorem 2.2.1, this requirement amounts to the following condition. The morphism ϕ, as any morphism of operads from U to $\mathcal{P}$, provides an arbitrary twisted $\mathcal{P}$-algebra with the structure of a twisted U-algebra. In particular, the tensor product $A \cdot B$ of two twisted $\mathcal{P}$-algebras, which is a twisted $\mathcal{P}$-algebra (since $\mathcal{P}$ is a Hopf operad) carries naturally the structure of a twisted U-algebra.

On the other hand, ϕ induces on A and B a structure of twisted U-algebra and, since U is a connected Hopf operad, the tensor product $A \cdot B$ carries the structure of a twisted U-algebra. The hypothesis that ϕ is a morphism of connected Hopf operads ensures that the two structures of twisted U-algebras on $A \cdot B$ are identical.

It follows in particular that the coproduct map from $\mathcal{P}$ to $\mathcal{P} \cdot \mathcal{P}$ is also a morphism of twisted U-algebras, and $\mathcal{P}$ inherits from this construction the structure of a twisted Hopf U-algebra. More generally, we have:

4.1.1. Proposition. *Let $\phi : U \to \mathcal{P}$ be a morphism of connected Hopf operads. Then, $\mathcal{P}$ and, more generally, any twisted Hopf $\mathcal{P}$-algebra, is naturally provided with the structure of a twisted Hopf U-algebra.*

4.1.2. Definition. A *multiplicative Hopf operad* $\mathcal{P}$ is a connected Hopf operad together with a morphism of connected Hopf operads $\phi : \mathcal{A}s \to \mathcal{P}$.

As a consequence of Proposition 4.1.1 any multiplicative Hopf operad is a connected twisted Hopf algebra. In that case the following extension of the Cartier–Milnor–Moore theorem holds (recall that, when $\mathcal{P} = \mathcal{A}s$, our functor $\mathcal{U}$ coincides with the twisted enveloping algebra functor of Joyal and Stover [Joy86,Sto93]).

4.1.3. Theorem. *(See [Sto93, Theorem 8.4].) Any multiplicative cocommutative Hopf operad is—as a twisted Hopf algebra—the twisted enveloping algebra of its primitive elements.*

Theorem 4.1.3 allows, for example, to use the Poincaré–Birkhoff–Witt theorem of Stover (for twisted enveloping algebras) to construct basis of multiplicative cocommutative Hopf operads.

Let us illustrate Theorem 4.1.3 with the Poisson operad—set operads containing $\mathcal{C}om$ as a sub-operad would provide other examples.

4.2. *Example: the Poisson operad*

In this section we assume that $\mathbf{k}$ is of characteristic 0.

Recall that a Poisson algebra A is a commutative algebra with a unit 1 provided with a Lie bracket $[\,,]$ which is a biderivation. That is, besides the antisymmetry and Jacobi identities for $[\,,]$, we have the Poisson distributivity formula:

$$[f, gh] = [f, g]h + g[f, h]. \tag{4.1}$$

In particular, we have $[f, 1] = 0$.

The simplest way to describe the Poisson operad $\mathcal{P}ois$ is through the corresponding functor:

$$\mathcal{P}ois(V) = \mathcal{C}om \circ \mathcal{L}ie(V). \tag{4.2}$$

Explicitly, an element of the free Poisson algebra over V is a commutative polynomial in the Lie polynomials (the elements of $\mathcal{L}\mathrm{ie}(V)$), and the bracket of two such commutative polynomials is computed using (iteratively) the Poisson distributivity formula and the Lie bracket in $\mathcal{L}\mathrm{ie}(V)$.

Due to the Poincaré–Birkhoff–Witt theorem, which states that $\mathcal{A}\mathrm{s}(V)$ is isomorphic to $\mathcal{C}\mathrm{om} \circ \mathcal{L}\mathrm{ie}(V)$ as an analytic functor, $\mathcal{P}\mathrm{ois}(V)$ and $\mathcal{A}s(V)$ are isomorphic as analytic functors. Therefore $\mathcal{A}\mathrm{s}$ and $\mathcal{P}\mathrm{ois}$ are also isomorphic as $\mathbb{S}$-modules, as a consequence of the correspondence between polynomial functors and symmetric group representations. We refer to [Mac95, Appendix A] for further details on analytic functors, polynomial functors and symmetric group representations. In particular, $\mathcal{P}\mathrm{ois}(n)$ is isomorphic, as a right S_n-module to the regular representation of S_n.

Recall however that the Poincaré–Birkhoff–Witt theorem also holds in the category of $\mathbb{S}$-modules: the twisted enveloping algebra of a connected twisted Lie algebra L is isomorphic, as an $\mathbb{S}$-module, to the free twisted commutative algebra over L [Joy86, Theorem 2].

We are going to show that the (classical) Poincaré–Birkhoff–Witt isomorphism between $\mathcal{P}\mathrm{ois}(V)$ and $\mathcal{A}\mathrm{s}(V)$ can be lifted to the Hopf operadic setting, and understood directly by means of Joyal's Poincaré–Birkhoff–Witt theorem for the twisted enveloping algebras of twisted Lie algebras.

4.2.1. Primitive elements of the Poisson operad. Recall that the Poisson operad is a connected cocommutative Hopf operad. Let $[\,,]$ and μ be the two generators in $\mathcal{P}\mathrm{ois}(2)$ representing the Lie structure and commutative structure. Then we have

$$\begin{cases} \mu|_\emptyset = 1_0, \\ \mu|_S = 1_1, \quad \text{for } |S| = 1 \end{cases} \quad \text{and} \quad [\,,]|_T = 0, \quad \text{for } |T| < 2$$

and

$$\delta(\mu) = \mu \otimes \mu, \qquad \delta\big([\,,]\big) = [\,,] \otimes \mu + \mu \otimes [\,,].$$

The last two equations mean in terms of Poisson algebras, that if A and B are two Poisson algebras, the tensor product $A \otimes B$ is provided with a Poisson structure as follows. As a commutative algebra, $A \otimes B$ is provided with the usual commutative product, $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$. The bracket on $A \otimes B$ is defined by: $[a_1 \otimes b_1, a_2 \otimes b_2] = a_1a_2 \otimes [b_1, b_2] + [a_1, a_2] \otimes b_1b_2$.

It follows in particular from the definition of δ that the natural inclusion gives a morphism of connected Hopf operad $\mathcal{C}\mathrm{om} \to \mathcal{P}\mathrm{ois}$, so that Proposition 4.1.1 applies: any twisted Hopf $\mathcal{P}\mathrm{ois}$-algebra is naturally provided with the structure of a commutative twisted Hopf algebra.

4.2.2. Lemma. *The Lie operad is a sub-operad of the operad of primitive elements in* $\mathcal{P}\mathrm{ois}$.

Proof. Since the Lie operad is a sub-operad of $\mathcal{P}\mathrm{ois}$ and is generated by the Lie bracket $[\,,]$, it is enough to check that $[\,,] \in \mathcal{P}\mathrm{ois}(2)$ is a primitive element. Thanks to Theorem 2.3.3 one has to compute

$$\Delta\big([\,,]\big) = \sum_{(1),(2)} \sum_{S \sqcup T = [2]} \big([\,,]_{(1)}\big)\big|_S \otimes \big([\,,]_{(2)}\big)\big|_T.$$

Since $[\,,\,]|_S = 0$ for $|S| < 2$ the latter equality writes

$$\Delta\big([\,,\,]\big) = 1_0 \otimes [\,,\,] + [\,,\,] \otimes 1_0,$$

and $[\,,\,]$ is primitive. □

4.2.3. Theorem. *The Poisson operad is naturally provided with the structure of a commutative and cocommutative twisted Hopf algebra. The sub-operad of primitive elements of* $\mathcal{P}\mathrm{ois}$ *is the Lie operad. Moreover, as a twisted Hopf algebra,* $\mathcal{P}\mathrm{ois}$ *is isomorphic to the free commutative twisted algebra over the* $\mathbb{S}$*-module* $\mathcal{L}\mathrm{ie}$*, with trivial twisted Lie structure.*

Proof. As pointed out before, the inclusion $\mathcal{C}\mathrm{om} \subset \mathcal{P}\mathrm{ois}$ induces a morphism of connected Hopf operads $\mathcal{C}\mathrm{om} \to \mathcal{P}\mathrm{ois}$, thus, by composition with $\mathcal{A}\mathrm{s} \to \mathcal{C}\mathrm{om}$, a morphism of connected Hopf operads $\mathcal{A}\mathrm{s} \to \mathcal{P}\mathrm{ois}$. We can therefore apply Theorem 4.1.3 to $\mathcal{P}\mathrm{ois}$. Since $\mathcal{P}\mathrm{ois}$ is also twisted commutative, the twisted Lie algebra structure on $\mathrm{Prim}(\mathcal{P}\mathrm{ois})$ is trivial (see [Sto93]), and $\mathcal{P}\mathrm{ois}$ is isomorphic, as a twisted Hopf algebra, to the free twisted commutative algebra over the $\mathbb{S}$-module of its primitive elements.

It remains to prove that $\mathcal{L}\mathrm{ie}$ is the set of primitive elements in $\mathcal{P}\mathrm{ois}$. We conclude by a dimension argument based on the remark that, if the $\mathbb{S}$-module A is a sub-$\mathbb{S}$-module of B, and if $\dim_{\mathbf{k}} A(n) < \infty$ for all n, then, if the free twisted commutative algebras over A and B have the same dimension over $\mathbf{k}$ in each degree, it follows that $A = B$.

Recall from [PR04, Proposition 17] that $\mathcal{A}\mathrm{s}$ is, as a twisted Hopf algebra, the twisting enveloping algebra of $\mathcal{L}\mathrm{ie}$ (with the twisted Lie structure induced by the operadic structure of $\mathcal{L}\mathrm{ie}$). Due to Joyal's Poincaré–Birkhoff–Witt theorem, it follows that the dimension of the component of degree n of the free commutative twisted algebra over $\mathcal{L}\mathrm{ie}$ is equal to the dimension of $\mathcal{A}\mathrm{s}(n)$, that is, to $n!$. Besides, $\mathcal{P}\mathrm{ois}(n)$ is isomorphic to the regular representation of S_n as an S_n-module and, in particular, has dimension $n!$ as a vector space. Since $\mathcal{L}\mathrm{ie}$ is contained in $\mathrm{Prim}(\mathcal{P}\mathrm{ois})$, and since, according to our previous arguments, the dimensions of the graded components of the free twisted commutative algebras over $\mathcal{L}\mathrm{ie}$ and over $\mathrm{Prim}(\mathcal{P}\mathrm{ois})$ are equal, the theorem follows: $\mathcal{L}\mathrm{ie} = \mathrm{Prim}(\mathcal{P}\mathrm{ois})$. □

As a conclusion, let us mention that it would be very natural to apply the ideas and tools that we have developed in the present article in order to improve our understanding of the operads involved in the study of iterated loop spaces. These operads fit indeed in many respects in the framework of the present article—in particular of its last section. They actually did motivate the introduction of the notion of Hopf operad by Getzler and Jones. Besides, they generalize the Poisson operad. Further information on these operads and references on the classical work of Cohen, May et al. on the subject is available in [GJ90]. We postpone the Hopf-theoretical study of these operads since their definition requires a slightly different algebraic framework than the one chosen in the present article.

References

[Bar78] M.G. Barratt, Twisted Lie algebras, in: Geometric Applications of Homotopy Theory, Proc. Conf., II, Evanston, IL, 1977, in: Lecture Notes in Math., vol. 658, Springer-Verlag, Berlin, 1978, pp. 9–15.

[BLL98] F. Bergeron, G. Labelle, P. Leroux, Combinatorial Species and Tree-Like Structures, Encyclopedia Math. Appl., vol. 67, Cambridge University Press, Cambridge, 1998, translated from the 1994 French original by Margaret Readdy, with a foreword by Gian-Carlo Rota.

[Bro94] Kenneth S. Brown, Cohomology of Groups, Grad. Texts in Math., vol. 87, Springer-Verlag, New York, 1994, corrected reprint of the 1982 original.

[Cha03] Frédéric Chapoton, On a Hopf operad containing the Poisson operad, Algebr. Geom. Topol. 3 (2003) 1257–1273 (electronic).

[Dri86] V.G. Drinfel'd, Quantum groups, in: Proceedings of the International Congress in Mathematics, 1986, pp. 798–820.

[Foi05] Loïc Foissy, Bidendriform bialgebras, trees, and free quasi-symmetric functions, preprint, math.RA/0505207, 2005.

[Fre98] Benoit Fresse, Algèbre des descentes et cogroupes dans les algèbres sur une opérade, Bull. Soc. Math. France 126 (3) (1998) 407–433.

[Fre04] Benoit Fresse, Koszul duality of operads and homology of partition posets, in: Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory, in: Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 115–215.

[GJ90] Ezra Getzler, John D.S. Jones, A_∞-algebras and the cyclic bar complex, Illinois J. Math. 34 (2) (1990) 256–283.

[GJ94] Ezra Getzler, John D.S. Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, preprint, hep-th/9403055, 1994.

[Hol05] Ralf Holtkamp, On Hopf algebras structures over free operads, math.RA/0504531, 2005.

[Joy86] André Joyal, Foncteurs analytiques et espèces de structures, in: Combinatoire Énumérative, Montreal, Que., 1985/Quebec, Que., 1985, in: Lecture Notes in Math., vol. 1234, Springer-Verlag, Berlin, 1986, pp. 126–159.

[Lod04] Jean-Louis Loday, Scindement d'associativité et algèbres de Hopf, in: Actes des Journées Mathématiques à la Mémoire de Jean Leray, in: Sémin. Congr., vol. 9, Soc. Math. France, Paris, 2004, pp. 155–172.

[Mac95] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Math. Monogr., Clarendon Press/Oxford University Press, New York, 1995, with contributions by A. Zelevinsky, Oxford Science Publications.

[Mar96] Martin Markl, Distributive laws and Koszulness, Ann. Inst. Fourier (Grenoble) 46 (2) (1996) 307–323.

[MM65] John W. Milnor, John C. Moore, On the structure of Hopf algebras, Ann. of Math. (2) 81 (1965) 211–264.

[Moe02] I. Moerdijk, Monads on tensor categories, in: Category Theory, Coimbra, 1999, J. Pure Appl. Algebra 168 (2–3) (2002) 189–208.

[MSS02] Martin Markl, Steve Shnider, Jim Stasheff, Operads in Algebra, Topology and Physics, Math. Surveys Monogr., vol. 96, Amer. Math. Soc., Providence, RI, 2002.

[Pat94] Frédéric Patras, L'algèbre des descentes d'une bigèbre graduée, J. Algebra 170 (2) (1994) 547–566.

[PR04] Frédéric Patras, Christophe Reutenauer, On descent algebras and twisted bialgebras, Mosc. Math. J. 4 (1) (2004) 199–216, 311.

[Ree58] Rimhak Ree, Lie elements and an algebra associated with shuffles, Ann. of Math. 68 (1958) 210–220.

[Reu93] Christophe Reutenauer, Free Lie Algebras, London Math. Soc. Monogr. (N.S.), vol. 7, Oxford Sci. Publ., Clarendon Press/Oxford University Press, New York, 1993.

[Sto93] Christopher R. Stover, The equivalence of certain categories of twisted Lie and Hopf algebras over a commutative ring, J. Pure Appl. Algebra 86 (3) (1993) 289–326.

[Wig89] David Wigner, An identity in the free Lie algebra, Proc. Amer. Math. Soc. 106 (3) (1989) 639–640.