



A rigidity theorem for pre-Lie algebras

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Abstract

In this paper we prove a “Leray theorem” for pre-Lie algebras. We define a notion of “Hopf” pre-Lie algebra: it is a pre-Lie algebra together with a non-associative permutative coproduct Δ and a compatibility relation between the pre-Lie product and the coproduct Δ . A non-associative permutative algebra is a vector space together with a product satisfying the relation $(ab)c = (ac)b$. A non-associative permutative coalgebra is the dual notion. We prove that any connected “Hopf” pre-Lie algebra is a free pre-Lie algebra. It uses the description of pre-Lie algebras in terms of rooted trees developed by Chapoton and the author. We also interpret this theorem by way of cogroups in the category of pre-Lie algebras.

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0. Introduction

The classical Leray theorem [13,16] asserts that any graded connected commutative and cocommutative Hopf algebra is a free commutative algebra and free cocommutative coalgebra. Loday and Ronco [15] obtain a similar result for unital infinitesimal bialgebras: any such object (graded connected) is a free associative algebra as well as a free coassociative coalgebra. The same kind of result is also obtained by Foissy in [6] for dendriform, codendriform bialgebras. These three results are similar in the operadic

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framework: given the operad \mathcal{P} , where \mathcal{P} is the commutative operad or associative operad or dendriform operad, then any graded connected \mathcal{P} -algebra which is also a \mathcal{P} -coalgebra equipped with a relation between the coalgebra and algebra structures—also called “distributive law”—is rigid in the sense that it is free as a \mathcal{P} -algebra and as a \mathcal{P} -coalgebra. In this paper we obtain a result of that kind for pre-Lie algebras, although the originality of our result is that the costructure involved is not pre-Lie, but non-associative permutative. In fact, such a result can also be interpreted as a “Cartier–Milnor–Moore” theorem where the primitive part is a vector space without any additional structure and the enveloping algebra is the free \mathcal{P} -algebra functor. In that sense, such a result exists for Zinbiel algebras as a consequence of a theorem by Ronco [19] concerning the primitive elements of a free dendriform algebra.

Pre-Lie algebras have been of interest since the works of Vinberg [20] and Gerstenhaber [9]. In [3], using operad theory, pre-Lie algebras are described in terms of rooted trees, linking this structure to renormalisation theory à la Connes and Kreimer [4]. In this paper we describe another structure based on rooted trees, called non-associative permutative algebras (see e.g. [5]). A *non-associative permutative algebra* is a vector space together with a bilinear product satisfying the relation $(ab)c = (ac)b$. We state the following rigidity theorem.

Theorem. *Any pre-Lie algebra, together with a non-associative permutative connected coproduct satisfying the distributive law*

$$\Delta(a \circ b) = \Delta(a) \circ b + a \otimes b$$

is a free pre-Lie algebra and a free non-associative permutative coalgebra.

The proof of the theorem involves an idempotent in the vector space of endomorphism of L , which annihilates L^2 , the space of decomposable elements (see the [Fundamental Lemma 3.7](#)).

Although this paper can be read without referring to operads, the ideas coming from operad theory are always present, especially in the last section where we link our result to the theory of cogroups in the category of algebras over an operad as developed by Fresse in [7] and [8].

This paper is organized as follows: the first section is devoted to material concerning operads, rooted trees and pre-Lie algebras; in the second section we introduce non-associative permutative algebras and coalgebras; in the third section we state and prove the main theorem, assuming the fundamental lemma; the latter is proved in the fourth section; finally, the fifth section concerns cogroups, where we study the special case of associative algebras and pre-Lie algebras.

Notation. The ground field K is of characteristic 0. The symmetric group on n elements is denoted by Σ_n . For any vector space V , Σ_n acts on $V^{\otimes n}$, on the left by

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

For any subgroup G of Σ_n the subspace of $V^{\otimes n}$ of invariants under G is denoted $(V^{\otimes n})^G = \{x \in V^{\otimes n} \mid \sigma \cdot x = x, \forall \sigma \in G\}$.

1. Pre-Lie algebras and rooted trees

1.1. Operads

In this section, we review the material needed for this article. For a complement on algebraic operads we refer to Ginzburg and Kapranov [10] or Loday [14].

A Σ -module $\mathcal{M} = \{\mathcal{M}(n)\}_{n>0}$ is a collection of (right) Σ_n -modules. Any Σ_n -module M gives rise to a Σ -module \mathcal{M} by setting $\mathcal{M}(q) = 0$ if $q \neq n$ and $\mathcal{M}(n) = M$.

An *operad* is a right Σ -module $\{\mathcal{O}(n)\}_{n>0}$ such that $\mathcal{O}(1) = K$, together with composition products: for $1 \leq i \leq n$

$$\begin{array}{ccc} \circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) & \longrightarrow & \mathcal{O}(n+m-1) \\ p \otimes q & \longmapsto & p \circ_i q. \end{array}$$

These compositions are subject to associativity conditions, unitary conditions and equivariance conditions with respect to the action of the symmetric group.

Any vector space A yields an operad $\mathcal{E}nd_A$ with $\mathcal{E}nd_A(n) = \text{Hom}(A^{\otimes n}, A)$, where

$$\begin{aligned} (f \circ_i g)(a_1, \dots, a_{n+m-1}) \\ = f(a_1, \dots, a_{i-1}, g(a_i, \dots, a_{i+m-1}), a_{i+m}, \dots, a_{n+m-1}). \end{aligned}$$

An *algebra over an operad* \mathcal{O} , or *\mathcal{O} -algebra*, is a vector space A together with an operad morphism from \mathcal{O} to $\mathcal{E}nd_A$. This is equivalent to some data

$$ev_n : \mathcal{O}(n) \otimes_{\Sigma_n} A^{\otimes n} \rightarrow A$$

satisfying associativity conditions with respect to the compositions.

One of the most important points in the theory of algebraic operads is the following:

The free \mathcal{O} -algebra generated by V is the space

$$\mathcal{O}(V) = \bigoplus_{n \geq 1} \mathcal{O}(n) \otimes_{\Sigma_n} V^{\otimes n} \tag{1.1}$$

where the maps ev_n are described in terms of the \circ'_i s.

The following objects are defined as usual: given a Σ -module M , the *free operad generated by M* , denoted by $\mathcal{F}ree(M)$, exists and satisfies the usual universal property. The notion of an *ideal* of an operad exists: when modding out an operad by an ideal, one gets an operad. The operads with which we are concerned are *quadratic binary operads*, that is, they are of the form $\mathcal{F}ree(M)/\langle R \rangle$, where M is a Σ_2 -module, R is a sub- Σ_3 -module of $\mathcal{F}ree(M)(3)$, and $\langle R \rangle$ is the ideal of $\mathcal{F}ree(M)$ generated by R . The space $M = \mathcal{F}ree(M)(2)$ is the space of “operations”, whereas the space R is the space of “relations”. For instance, the operad that defines associative algebras is $\mathcal{F}ree(K[\Sigma_2])/\langle R_{As} \rangle$ where $K[\Sigma_2]$ is the regular representation of Σ_2 and R_{As} is the free Σ_3 -module generated by $\mu \circ_1 \mu - \mu \circ_2 \mu$, where μ is a generator of $K[\Sigma_2]$.

1.2. Rooted trees

A *rooted tree* is a nonempty connected graph without loop together with a distinguished vertex called *the root*. This root gives an orientation of the graph: edges are oriented

towards the root. The set of vertices of a tree T is denoted by $\text{Vert}(T)$. The orientation induces a partial order on $\text{Vert}(T)$, the root being the minimal element. The *degree* of a tree is the number of vertices. An n -labeled rooted tree is a rooted tree together with a bijection between $\text{Vert}(T)$ and the set $\{1, \dots, n\}$. An n -heap-ordered tree is an n -labeled rooted tree where the bijection respects the partial order on $\text{Vert}(T)$. The space $\mathcal{RT}(n)$ is the vector space spanned by the n -labeled rooted trees, and the space $\mathcal{HO}(n)$ is the one spanned by the n -heap-ordered trees. The right action of Σ_n on $\mathcal{RT}(n)$ is the action on the labeling. For a vector space V , denote by $\mathcal{RT}(V)$ the space $\sum_n \mathcal{RT}(n) \otimes_{\Sigma_n} V^{\otimes n}$. If V is equipped with a basis $\{e_\alpha\}_\alpha$ then the space $\mathcal{RT}(V)$ is the vector space spanned by the rooted trees labeled by the set $\{e_\alpha\}_\alpha$.

Following the notation of Connes and Kreimer [4], any tree T writes

$$T := B(r, T_1, \dots, T_k)$$

where r is the root (or the labeling of the root) and T_1, \dots, T_k are trees. The *arity* of T is k , the number of incoming edges of its root. Note that the list of the trees T_1, \dots, T_k is unordered.

1.3. Pre-Lie algebras

A (*right*) pre-Lie-algebra is a vector space L together with a product \circ satisfying the relation

$$(x \circ y) \circ z - x \circ (y \circ z) = (x \circ z) \circ y - x \circ (z \circ y). \quad (1.2)$$

These algebras also appeared under the name *right-symmetric algebras* or *Vinberg algebras* (if we deal with the left relation). The terminology of pre-Lie algebra is that of Gerstenhaber [9]. A pre-Lie algebra L yields a Lie algebra structure on L with the bracket $[a, b] = a \circ b - b \circ a$, which is denoted by L_{Lie} .

A theorem by Chapoton and the author links pre-Lie algebras to rooted trees as follows:

Theorem 1.4 ([3]). *The vector space $\mathcal{RT}(V)$ endowed with the product*

$$S \circ T = \sum_{v \in \text{Vert}(S)} S \circ_v T,$$

where $S \circ_v T$ is the tree obtained by grafting the root of T on the vertex v of S , is the free pre-Lie algebra generated by V .

The proof of this fact uses the description of pre-Lie algebras in terms of algebras over an operad as well as the relation (1.1). One can find some different proofs in Dzhumadil'daev and Löfwall [5] or in Guin and Oudom [11].

Remark 1.5. Using the pre-Lie product in $\mathcal{RT}(V)$, one has:

$$B(v, T_1, \dots, T_{n-1}, T_n) = B(v, T_1, \dots, T_{n-1}) \circ T_n - \sum_{i=1}^{n-1} B(v, T_1, \dots, T_i \circ T_n, \dots, T_{n-1}). \quad (1.3)$$

Hence a proposition concerned with the pre-Lie product in $\mathcal{RT}(V)$ can be proved by a double induction, on the degree and on the arity of a tree T .

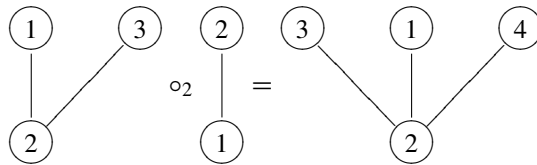
2. Non-associative permutative algebras and coalgebras

Definition 2.1. A non-associative permutative algebra is a vector space L equipped with a bilinear product satisfying the relation

$$(ab)c = (ac)b, \quad \forall a, b, c \in L.$$

Note that it is indeed the definition of a “right” non-associative permutative algebra, and we can define what is a “left” non-associative permutative algebra. Note also that a Novikov algebra is precisely a (left) pre-Lie algebra whose product satisfies the (right) non-associative permutative relation—see [17] for instance. An associative algebra that is a non-associative permutative algebra is a permutative algebra in the terminology of Chapoton in [2].

Definition 2.2. Let \mathcal{T} be the following operad: $\mathcal{T}(n)$ is the Σ_n -module of n -rooted trees $\mathcal{RT}(n)$; for trees $T \in \mathcal{RT}(n)$ and $S \in \mathcal{RT}(m)$, the composition $T \circ_i S$ is the rooted tree obtained by substituting the tree S for the vertex i in T . The outgoing edge of i , if it exists, becomes the outgoing edge of the root of S ; incoming edges of i are grafted onto the root of S . Then, it is easy to check that these compositions endow \mathcal{T} with a structure of an operad. Example:

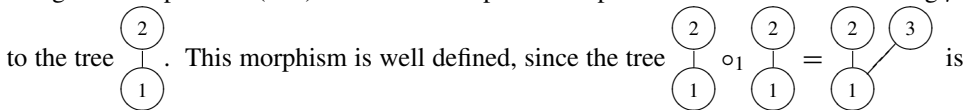


If we compare with the operad \mathcal{PL} based on rooted trees defining pre-Lie algebras in [3], this composition is a summand of the composition in \mathcal{PL} .

Note that the operad \mathcal{T} can also be defined on the set of n -labeled rooted trees: it is in fact an operad in the category of sets.

Proposition 2.3. The previous operad \mathcal{T} is the operad defining non-associative permutative algebras.

Proof. Following the notation of Section 1.1, the operad defining non-associative permutative algebras is the operad $\mathcal{NAP} = \mathcal{Free}(K[\Sigma_2]) / \langle R_{\text{nap}} \rangle$, where R_{nap} is the sub- Σ_3 -module of $\mathcal{Free}(K[\Sigma_2])(3)$ generated by the element $(\mu \circ_1 \mu) \cdot (Id - \tau_{23})$ with τ_{23} being the transposition (132). There is a morphism of operads from \mathcal{NAP} to \mathcal{T} sending μ



to the tree $\begin{matrix} \textcircled{2} \\ | \\ \textcircled{1} \end{matrix}$. This morphism is well defined, since the tree $\begin{matrix} \textcircled{2} \\ | \\ \textcircled{1} \end{matrix} \circ_1 \begin{matrix} \textcircled{2} \\ | \\ \textcircled{1} \end{matrix} = \begin{matrix} \textcircled{2} & \textcircled{3} \\ | & / \\ \textcircled{1} \end{matrix}$ is invariant under the permutation τ_{23} .

The inverse morphism is defined by induction on the number of vertices of T . Any tree can be written $T = B(i, T_1, \dots, T_n)$ where i is the root of T , thus $T = B(i, T_1) \circ_i B(i, T_2, \dots, T_n)$ (up to an appropriate permutation). It is easy to check that this decomposition does not depend on the choice of T_1 . Hence, by induction, it follows that we define a morphism of operads inverse to the previous one. \square

Corollary 2.4. *Let V be a vector space. The free non-associative permutative algebra generated by V is the vector space of rooted trees $\mathcal{RT}(V)$, endowed with the following product: let $T = B(i, T_1, \dots, T_n)$ and S be two trees in $\mathcal{RT}(V)$, then $T \cdot S = B(i, T_1, \dots, T_n, S)$.*

Proof. From relation (1.1) and Proposition 2.3, the free non-associative permutative algebra on V is

$$\mathcal{NAP}(V) = \bigoplus_{n \geq 1} \mathcal{RT}(n) \otimes_{\Sigma_n} V^{\otimes n}$$

where the product is given by the composition of the operad, i.e.

$$T \cdot S = \left(\begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \circ_2 S \right) \circ_1 T = \begin{array}{c} \textcircled{S} \\ | \\ \textcircled{1} \end{array} \circ_1 T,$$

which is the tree obtained by grafting S onto the root of T . \square

Remark 2.5. As pointed out before, since the operad \mathcal{T} can be defined in the category of sets, the corollary is also true when replacing non-associative permutative algebras with “right-commutative magma” in the terminology of Dzhumadil’daev and Löfwall [5]: these two results were proved in their paper, using different methods.

Definition 2.6. Let (C, Δ) be a vector space C together with a coproduct $\Delta : C \rightarrow C \otimes C$. The following defines a filtration on C :

$$\begin{aligned} \text{Prim } C &= C_1 = \{x \in C \mid \Delta(x) = 0\} \\ C_n &= \left\{ x \in C \mid \Delta(x) \in \sum_{i=1}^{n-1} C_i \otimes C_{n-i} \right\}. \end{aligned}$$

The vector space (C, Δ) is said to be *connected* if $C = \bigcup_{n \geq 1} C_n$. Note that any graded (C, Δ) , such that $C_0 = 0$, is connected.

Define $\Delta^k : C \rightarrow C^{\otimes(k+1)}$ by

$$\begin{aligned} \Delta^0 &= Id \\ \Delta^1 &= \Delta \\ \Delta^{k+1} &= (\Delta \otimes Id^{\otimes k}) \Delta^k = (\Delta^k \otimes Id) \Delta. \end{aligned}$$

We use Sweedler notation for Δ^k :

$$\Delta^k(x) = \sum x_{(1)} \otimes \dots \otimes x_{(k+1)}, \quad \forall x \in C.$$

Lemma 2.7. Let (C, Δ) be a vector space together with a coproduct; let P_n be the vector space of cooperations from C to $C^{\otimes(n+1)}$ built on Δ . More explicitly: $P_0 = \text{Span}\{Id\}$; $P_1 = \text{Span}\{\Delta\}$; $O \in P_n$ if and only if there exists $0 \leq m \leq n - 1$ and $O_1 \in P_m, O_2 \in P_{n-m-1}$ such that $O = (O_1 \otimes O_2)\Delta$.

If $x \in C_n$ then $O(x) = 0, \quad \forall O \in P_n$.

Proof. The lemma is trivial for $n = 1$. Let x be in C_n and $O = (O_1 \otimes O_2)\Delta$ in P_n , with $O_1 \in P_m$. Since $\Delta(x) = \sum x_i \otimes y_{n-i}$, with $x_i \in C_i$ and $y_{n-i} \in C_{n-i}$, we get by induction: $O_1(x_i) = 0$ if $m \geq i$ and $O_2(y_{n-i}) = 0$ if $n - m - 1 \geq n - i$, i.e. if $m \leq i - 1$; thus $(O_1 \otimes O_2)(\Delta(x)) = 0. \quad \square$

Definition 2.8. A non-associative permutative coalgebra is a vector space together with a coproduct $\Delta : C \rightarrow C \otimes C$ satisfying $(1 - \tau_{23})(\Delta \otimes Id)\Delta = 0$. This is the dual notion of a non-associative permutative algebra.

The following lemma is an immediate consequence of the definition of non-associative permutative coalgebras.

Lemma 2.9. If C is a non-associative permutative coalgebra then

$$\text{Im}(\Delta^k) \subset (C^{\otimes(k+1)})^{\Sigma_1 \times \Sigma_k}.$$

Theorem 2.10. Let V be a vector space. Then $\mathcal{RT}(V)$, together with the coproduct

$$\Delta(B(v, T_1, \dots, T_n)) = \sum_{i=1}^n B(v, T_1, \dots, \widehat{T}_i, \dots, T_n) \otimes T_i$$

is the free non-associative permutative connected coalgebra.

Proof. It is the dual statement of Corollary 2.4, since $\mathcal{RT}(V)$ with the coproduct is the graded dual of $\mathcal{RT}(V)$ considered as the free non-associative permutative algebra on V . It is graded by the number of vertices, and hence connected. \square

3. The space of rooted trees and the main theorem

Definition 3.1. Let L be a pre-Lie algebra and M be a vector space. M is a right L -module if there exists a map $\circ : M \otimes L \rightarrow M$ such that

$$(m \circ l_1) \circ l_2 - (m \circ l_2) \circ l_1 = m \circ [l_1, l_2].$$

Equivalently: M is a right L -module if and only if M is a L_{Lie} -module in the terminology of Lie algebras.

Examples: L is an L -module; $L^{\otimes n}$ is an L -module via derivation, as pointed out by Guin and Oudom in [11]:

$$(x_1 \otimes \dots \otimes x_n) \circ y = \sum_{i=1}^n x_1 \otimes \dots \otimes x_i \circ y \otimes \dots \otimes x_n.$$

Proposition 3.2. Consider the vector space $\mathcal{RT}(V)$ endowed with its pre-Lie product \circ and its non-associative permutative coalgebra product Δ . The following relation is satisfied

$$\begin{aligned}\Delta(x \circ y) &= x \otimes y + \Delta(x) \circ y \\ &= x \otimes y + x_{(1)} \circ y \otimes x_{(2)} + x_{(1)} \otimes x_{(2)} \circ y.\end{aligned}\tag{3.1}$$

Proof. For $S = B(v, S_1, \dots, S_n)$ and T in $\mathcal{RT}(V)$, one has

$$\begin{aligned}\Delta(S \circ T) &= \Delta(B(v, S_1, \dots, S_n, T)) + \sum_{i=1}^n \Delta(B(v, S_1, \dots, S_i \circ T, \dots, S_n)) \\ &= S \otimes T + \sum_{i=1}^n B(v, S_1, \dots, \widehat{S}_i, \dots, S_n, T) \otimes S_i \\ &\quad + \sum_{i \neq j} B(v, \dots, \widehat{S}_i, \dots, S_j \circ T, \dots, S_n) \otimes S_i \\ &\quad + \sum_i B(v, S_1, \dots, \widehat{S}_i, \dots, S_n) \otimes S_i \circ T \\ &= S \otimes T + \Delta(S) \circ T. \quad \square\end{aligned}$$

Corollary 3.3. Let H be a pre-Lie algebra, V a vector space, and $\phi : \mathcal{PL}(V) \rightarrow H$ a morphism of pre-Lie algebras. There exists a unique application $L : \mathcal{PL}(V) \rightarrow H \otimes H$ such that

$$\begin{aligned}L(v) &= 0 \quad \forall v \in V, \\ L(a \circ b) &= \phi(a) \otimes \phi(b) + L(a) \circ \phi(b) \quad \forall a, b \in \mathcal{PL}(V).\end{aligned}\tag{3.2}$$

Proof. The existence is given by $L = (\phi \otimes \phi)\Delta$ and the relation (3.1). To prove unicity it is sufficient to prove the following: if $L : \mathcal{PL}(V) \rightarrow H \otimes H$ satisfies $L(a \circ b) = L(a) \circ \phi(b)$ and $L(v) = 0$, then $L = 0$. This is proved by induction on the number of vertices of a tree T and the arity of T , thanks to the relation (1.3). \square

Main Theorem 3.4. Let (H, \circ_H, Δ_H) be a vector space together with a pre-Lie product and a non-associative permutative connected coproduct satisfying the relation $\Delta_H(a \circ_H b) = a \otimes b + \Delta_H(a) \circ_H b$. There is an isomorphism of pre-Lie algebras and of non-associative permutative coalgebras between H and $(\mathcal{RT}(\text{Prim } H), \circ, \Delta)$.

The proof of this theorem is similar to the proof of the rigidity of unital infinitesimal bialgebras of Loday–Ronco [15] and that of dendriform Hopf algebras of Foissy [6]. It relies mainly on the [Fundamental Lemma 3.7](#). To state this lemma we need the following definition.

Definition–Notation 3.5. Let (H, μ) be a pre-Lie algebra. We define linear operators $A_k : H^{\otimes k} \rightarrow H$ by induction on k :

$$A_1 = Id,$$

$$A_2 = \mu,$$

$$A_{k+1} = \sum_{l=1}^k \binom{k-1}{l-1} \mu(A_l \otimes A_{k+1-l}).$$

Remark 3.6. The operators A_k can be viewed as elements of $\mathcal{RT}(k)$ as follows. Define the scalar product on the basis of rooted trees by

$$\langle S, T \rangle = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if not.} \end{cases}$$

Let $\mathcal{HO}(k)$ be the space of k -heap-ordered trees. Then

$$A_k = \sum_{U \in \mathcal{HO}(k)} c(U)U$$

where $c(U)$ is defined by induction:

$$c(U) = \sum_{l=1}^k \binom{k-1}{l-1} \sum_{T \in \mathcal{HO}(l), T' \in \mathcal{HO}(k+1-l)} c(T)c(T')\langle T \circ T', U \rangle.$$

Fundamental Lemma 3.7. Let (H, \circ, Δ) be a pre-Lie algebra, non-associative permutative connected coalgebra satisfying the relation (3.1). The linear morphism

$$e : H \rightarrow H$$

$$x \mapsto x + \sum_{k \geq 1} \frac{(-1)^k}{k!} A_{k+1} \Delta^k(x) \tag{3.3}$$

satisfies the following properties:

- (1) e is a projector onto $\text{Prim}(H)$.
- (2) $e(x \circ y) = 0, \forall x, y \in H$.

Corollary 3.8. Let (H, μ) be a pre-Lie algebra together with a non-associative permutative connected coproduct satisfying relation (3.1). Then

$$H = \text{Prim } H \oplus H^2,$$

where $H^2 = \text{Im}(\mu)$.

Proof. By **Fundamental Lemma 3.7**: property (1) implies that, for all $x \in H$, one has $x = e(x) + x - e(x)$, where $e(x) \in \text{Prim } H$ and $x - e(x) = -\sum_{k > 0} \frac{(-1)^k}{k!} A_{k+1} \Delta^k(x) = \mu(y, z)$. Property (2) implies that $\text{Prim } H \cap H^2 = 0$. \square

Corollary 3.9. Let (H, μ) be a pre-Lie algebra together with a non-associative permutative connected coproduct satisfying relation (3.1). Then H is generated as a pre-Lie algebra by $\text{Prim}(H)$.

Proof. Following Definition 2.6, since H is connected, it decomposes as $H = \bigcup_{n \geq 1} H_n$; any $x \in H_n$ satisfies $\Delta^k(x) \in \sum_{(i_1, \dots, i_{k+1})} H_{i_1} \otimes \dots \otimes H_{i_{k+1}}$, where $i_j < n, \forall j$. The proof is performed by induction on n : for $n = 1$, it is true since $H_1 = \text{Prim } H$. If $x \in H_n$, then $x = e(x) - \sum_{k > 0} \frac{(-1)^k}{k!} A_{k+1} \Delta^k(x)$. But $e(x) \in \text{Prim } H$ and, by induction, each component in $\Delta^k(x)$ is generated by elements of $\text{Prim}(H)$, as is $A_{k+1} \Delta^k(x)$. \square

3.10. Proof of the main theorem

Set $V := \text{Prim}(H)$. Denote by $i : V \rightarrow H$ and $\iota : V \rightarrow \mathcal{RT}(V)$ the natural injections.

Since $(\mathcal{RT}(V), \circ)$ is the free pre-Lie algebra on V , there is a morphism of pre-Lie algebras $\phi : \mathcal{RT}(V) \rightarrow H$, such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & H \\ \downarrow \iota & \searrow \phi & \nearrow \\ \mathcal{RT}(V) & & \end{array}$$

commutes. The pre-Lie algebra H is generated by V by Corollary 3.9, then ϕ is surjective.

Consider the two linear maps $\Delta_H \phi$ and $(\phi \otimes \phi) \Delta$ from $\mathcal{RT}(V)$ to $H \otimes H$. An easy computation proves that these two maps satisfy conditions (3.2) of Corollary 3.3. Then, by unicity, these two maps coincide, thus ϕ is a morphism of non-associative permutative coalgebras.

Assume $\text{Ker } \phi \neq 0$ and let $a \in \text{Ker } \phi$ of minimal degree n . Since $(\phi \otimes \phi) \Delta(a) = \Delta_H \phi(a) = 0$, the element a lies in $\text{Ker}(\phi) \otimes \mathcal{RT}(V) + \mathcal{RT}(V) \otimes \text{Ker}(\phi)$. Since a is of minimal degree, it implies that $a = 0$, thus a contradiction. Hence ϕ is injective. \square

4. Proof of the Fundamental Lemma 3.7

Assume that (H, μ, Δ) satisfies the hypothesis of Fundamental Lemma 3.7. We define linear maps $U_{k+1} : H^{\otimes(k+1)} \rightarrow H^{\otimes 2}$, for $k > 0$ by $U_{k+1} = \sum_{l=1}^k \binom{k-1}{l-1} A_l \otimes A_{k+1-l}$, where the A_i s were defined in 3.5. Hence $A_k = \mu U_k$ for $k > 1$.

Proposition 4.1. Assume $k \geq 1$. On the vector space

$$Z_{k+1} = (H^{\otimes(k+1)})^{\Sigma_1 \times \Sigma_k} \cap \{x \in H^{\otimes(k+1)} \mid (\Delta \otimes Id^{\otimes k})(x) \in (H^{\otimes(k+2)})^{\Sigma_1 \times \Sigma_{k+1}}\}$$

one has the following:

$$\Delta A_{k+1} = kU_{k+1} + U_{k+2}(\Delta \otimes Id^{\otimes k}). \tag{4.1}$$

Proof. The proof is by induction on k . The relation (3.1) reads:

$$\Delta \mu = Id \otimes Id + (Id \otimes \mu)(\Delta \otimes Id) + (\mu \otimes Id)\tau_{23}(\Delta \otimes Id).$$

Since $A_1 = Id$, $A_2 = \mu$ and the domain is Z_2 , one has:

$$\Delta A_2 = U_2 + (A_1 \otimes A_2 + A_2 \otimes A_1)(\Delta \otimes Id) \text{ on } Z_2,$$

hence the relation (4.1) is satisfied for $k = 1$. For $k > 0$ one has

$$\begin{aligned} \Delta A_{k+1} &= \Delta \mu U_{k+1} \\ &= U_{k+1} + (Id \otimes \mu + (\mu \otimes Id)\tau_{23})(\Delta \otimes Id)U_{k+1}. \end{aligned}$$

By induction, one has

$$\begin{aligned} (\Delta \otimes Id)U_{k+1} &= \sum_{l=1}^k \binom{k-1}{l-1} \Delta A_l \otimes A_{k+1-l} \\ &= \sum_{l=2}^k \binom{k-1}{l-1} (l-1)U_l \otimes A_{k+1-l} \\ &\quad + \sum_{l=1}^k \binom{k-1}{l-1} U_{l+1}(\Delta \otimes Id^{\otimes(l-1)}) \otimes A_{k+1-l}. \end{aligned}$$

$(Id \otimes \mu + (\mu \otimes Id)\tau_{23})$ applied to the first summand gives $(k-1)U_{k+1}$. Indeed:

$$\begin{aligned} (Id \otimes \mu) \sum_{l=2}^k \binom{k-1}{l-1} (l-1)U_l \otimes A_{k+1-l} \\ &= (Id \otimes \mu) \sum_{l=2}^k \binom{k-1}{l-1} (l-1) \sum_{j=1}^{l-1} \binom{l-2}{j-1} A_j \otimes A_{l-j} \otimes A_{k+1-l} \\ &= \sum_{j=1}^{k-1} \sum_{1 \leq l-j \leq k-j} (k-1) \binom{k-2}{j-1} \binom{k-j-1}{l-j-1} A_j \otimes \mu(A_{l-j} \otimes A_{k+1-l}) \\ &= (k-1) \sum_{j=1}^{k-1} \binom{k-2}{j-1} A_j \otimes A_{k+1-j}, \end{aligned}$$

and

$$\begin{aligned} (\mu \otimes Id)\tau_{23} \sum_{l=2}^k \binom{k-1}{l-1} (l-1)U_l \otimes A_{k+1-l} \\ &= (\mu \otimes Id)\tau_{23} \sum_{l=2}^k \binom{k-1}{l-1} (l-1) \sum_{j=1}^{l-1} \binom{l-2}{j-1} A_j \otimes A_{l-j} \otimes A_{k+1-l} \\ &\stackrel{\text{on } \mathbb{Z}_{k+1}}{=} (\mu \otimes Id) \sum_{l=2}^k \binom{k-1}{l-1} (l-1) \sum_{j=1}^{l-1} \binom{l-2}{j-1} A_j \otimes A_{k+1-l} \otimes A_{l-j} \\ &\stackrel{(l=k+1+j-\alpha)}{=} \sum_{\alpha=2}^k \sum_{j=1}^{\alpha-1} \binom{k-1}{k+j-\alpha} (k+j-\alpha) \binom{k+j-\alpha-1}{j-1} \\ &\quad \times \mu(A_j \otimes A_{\alpha-j}) \otimes A_{k+1-\alpha} = \sum_{\alpha=2}^k (k-1) \binom{k-2}{\alpha-2} A_\alpha \otimes A_{k+1-\alpha}. \end{aligned}$$

Adding the two results, one gets $(k - 1)U_{k+1}$, thus

$$\begin{aligned} \Delta A_{k+1} &= kU_{k+1} + (Id \otimes \mu + (\mu \otimes Id)\tau_{23}) \\ &\quad \times \sum_{l=1}^k \binom{k-1}{l-1} U_{l+1}(\Delta \otimes Id^{\otimes(l-1)}) \otimes A_{k+1-l}. \end{aligned}$$

The computation of the second summand is similar to the previous one, thus on Z_{k+1} one has

$$\begin{aligned} (Id \otimes \mu + (\mu \otimes Id)\tau_{23}) \sum_{l=1}^k \binom{k-1}{l-1} U_{l+1}(\Delta \otimes Id^{\otimes(l-1)}) \otimes A_{k+1-l} \\ = U_{k+2}(\Delta \otimes Id^{\otimes k}). \quad \square \end{aligned}$$

4.2. Proof of property (1) of the fundamental lemma

For $x \in H$, since $\Delta^k(x) \in Z_{k+1}$ the formula (4.1) can be applied:

$$\begin{aligned} \Delta(e(x)) &= \Delta(x) + \sum_{k \geq 1} \frac{(-1)^k}{k!} \Delta A_{k+1} \Delta^k(x) \\ &\stackrel{\text{by (4.1)}}{=} \Delta(x) + \sum_{k \geq 1} \frac{(-1)^k}{k!} kU_{k+1} \Delta^k(x) + \sum_{k \geq 1} \frac{(-1)^k}{k!} U_{k+2} \Delta^{k+1}(x) \\ &= \Delta(x) - U_2 \Delta(x) = 0. \end{aligned}$$

Hence $\text{Im}(e) \subset \text{Prim}(H)$. Furthermore, if $y \in \text{Prim}(H)$ then $e(y) = y$, so e is an idempotent. \square

Lemma 4.3. *Let H be a pre-Lie algebra and a non-associative permutative coalgebra satisfying relation (3.1). We denote by \circ the product on H as well as the action of H on $H^{\otimes k}$ defined in 3.1. For $y \in H$, we define a map $\delta_y : H^{\otimes k} \rightarrow H^{\otimes(k+1)}$ by*

$$\delta_y(x_1 \otimes \cdots \otimes x_k) = \sum_{i=1}^k x_1 \otimes \cdots \otimes x_i \otimes y \otimes x_{i+1} \otimes \cdots \otimes x_k. \quad (4.2)$$

The following equation holds

$$\Delta^k(x \circ y) = \Delta^k(x) \circ y + \delta_y(\Delta^{k-1}(x)), \quad \forall k \geq 1. \quad (4.3)$$

Proof. The proof is by induction on k . For $k = 1$ it is the relation (3.1). For $k > 1$ one has

$$\begin{aligned} \Delta^{k+1}(x \circ y) &= (\Delta \otimes Id^{\otimes k})(\Delta^k(x \circ y)) \\ &= (\Delta \otimes Id^{\otimes k})(\Delta^k(x) \circ y + \delta_y(\Delta^{k-1}(x))) \\ &= \Delta^{k+1}(x) \circ y - x_{(1)} \circ y \otimes x_{(2)} \otimes \cdots \otimes x_{(k+2)} - x_{(1)} \otimes x_{(2)} \circ y \\ &\quad \otimes \cdots \otimes x_{(k+2)} + \Delta(x_{(1)} \circ y) \otimes x_{(2)} \otimes \cdots \otimes x_{(k+1)} \\ &\quad + \delta_y(\Delta^k(x)) - x_{(1)} \otimes y \otimes x_{(2)} \otimes \cdots \otimes x_{(k+1)} \\ &= \Delta^{k+1}(x) \circ y + \delta_y(\Delta^k(x)). \quad \square \end{aligned}$$

Proposition 4.4. *Let H be a pre-Lie algebra and a non-associative permutative coalgebra satisfying relation (3.1). The following equality holds for $k \geq 0$, $x \in (H^{\otimes(k+1)})^{\Sigma_1 \times \Sigma_k}$ and $y \in H$:*

$$(k + 1)A_{k+1}(x \circ y) = A_{k+2}\delta_y(x). \tag{4.4}$$

Proof. The proof is by induction on k . For $k = 0$, the equality reads $A_1(x \circ y) = x \circ y = \mu\delta_y(x) = A_2\delta_y(x)$. Denote by $m_y : H^{\otimes k} \rightarrow H^{\otimes k}$ the map that associates $x \circ y$ to $x \in H^{\otimes k}$. We sometimes denote m_y and δ_y by m_y^k and δ_y^k , if k is not fixed. For instance, we have the formulas, for $\bar{x} = x_1 \otimes \dots \otimes x_{k+1} \in H^{\otimes(k+1)}$, $y \in H$ and $1 \leq l \leq k + 1$:

$$\begin{aligned} \delta_y^{k+1}(\bar{x}) &= (\delta_y^{l-1} \otimes Id^{\otimes(k+2-l)})(\bar{x}) + x_1 \otimes \dots \otimes x_l \otimes y \otimes x_{l+1} \otimes \dots \otimes x_{k+1} \\ &\quad + (Id^{\otimes l} \otimes \delta_y^{k+1-l})(\bar{x}) \\ m_y^{k+1}(\bar{x}) &= (m_y^l \otimes Id^{\otimes(k+1-l)})(\bar{x}) + (Id^{\otimes l} \otimes m_y^{k+1-l})(\bar{x}). \end{aligned} \tag{4.5}$$

Hence the formula (4.4) reduces to $(k + 1)A_{k+1}m_y^{k+1} = A_{k+2}\delta_y^{k+1}$ on the invariant space $(H^{\otimes(k+1)})^{\Sigma_1 \times \Sigma_k}$:

$$\begin{aligned} A_{k+2}\delta_y^{k+1}(\bar{x}) &= \mu \left(\sum_{l=1}^{k+1} \binom{k}{l-1} A_l \otimes A_{k+2-l} \right) \delta_y^{k+1}(\bar{x}) \\ &\stackrel{\text{by (4.4)}}{=} \mu \left(\sum_{l=2}^{k+1} \binom{k}{l-1} A_l \delta_y^{l-1} \otimes A_{k+2-l} \right. \\ &\quad \left. + \sum_{l=1}^k \binom{k}{l-1} A_l \otimes A_{k+2-l} \delta_y^{k+1-l} \right) (\bar{x}) \\ &\quad + \mu \sum_{l=1}^{k+1} \binom{k}{l-1} (A_l \otimes A_{k+2-l})(x_1 \dots x_l \otimes y \otimes x_{l+1} \dots x_{k+1}) \\ &\stackrel{\text{by induction}}{=} \mu \left[\sum_{l=2}^{k+1} \binom{k}{l-1} (l-1)A_{l-1}m_y^{l-1} \otimes A_{k+2-l} \right. \\ &\quad \left. + \sum_{l=1}^k \binom{k}{l-1} (k+1-l)A_l \otimes A_{k+1-l}m_y^{k+1-l} \right] (\bar{x}) \\ &\quad + \mu \sum_{l=1}^{k+1} \binom{k}{l-1} (A_l \otimes A_{k+2-l})(x_1 \dots x_l \otimes y \otimes x_{l+1} \dots x_{k+1}). \end{aligned}$$

But the first two terms on the right-hand side, after a change of variables and thanks to the equality $l \binom{k}{l} = (k + 1 - l) \binom{k}{l-1} = k \binom{k-1}{l-1}$, can be written as:

$$\mu \sum_{l=1}^k k \binom{k-1}{l-1} (A_l m_y^l \otimes A_{k+1-l} + A_l \otimes A_{k+1-l} m_y^{k+1-l})(\bar{x}) = k A_{k+1} m_y(\bar{x}).$$

Again, by induction on k , we prove the following formula:

$$\sum_{l=1}^{k+1} \binom{k}{l-1} A_l(x_1 \dots x_l) \circ A_{k+2-l}(y \otimes x_{l+1} \dots x_{k+1}) = A_{k+1}m_y(\bar{x}). \quad (4.6)$$

If $k = 0$, this equation reads $x_1 \circ y = m_y(x_1)$. If $k = 1$, it reads $x_1 \circ (y \circ x_2) + (x_1 \circ x_2) \circ y = x_1 \circ (x_2 \circ y) + (x_1 \circ y) \circ x_2$, which is exactly the pre-Lie relation. For $k > 1$ one has

$$\begin{aligned} A_{k+1}m_y(\bar{x}) &= \mu \left(\sum_{l=1}^k \binom{k-1}{l-1} (A_l m_y \otimes A_{k+1-l} + A_l \otimes A_{k+1-l} m_y) \right) (\bar{x}) \\ &\stackrel{\text{by induction}}{=} \sum_{l=1}^k \sum_{j=1}^l \binom{k-1}{l-1} \binom{l-1}{j-1} (A_j(x_1 \dots x_j) \\ &\quad \circ A_{l+1-j}(y \otimes x_{j+1} \dots x_l)) \circ A_{k+1-l}(x_{l+1} \dots x_{k+1}) \\ &\quad + \sum_{j=1}^k \sum_{u=1}^{k+1-j} \binom{k-1}{j-1} \binom{k-j}{u-1} \underbrace{A_j(x_1 \dots x_j)}_R \\ &\quad \circ \underbrace{(A_u(x_{j+1} \dots x_{j+u})) \circ A_{k+2-j-u}(y \otimes x_{j+u+1} \dots x_{k+1})}_S. \end{aligned}$$

A change of variables in the first summand $((j, l) \mapsto (j, u)$ with $l+1-j = k+2-j-u$) gives a coefficient $\binom{k-1}{k-u} \binom{k-u}{j-1} = \binom{k-1}{j-1} \binom{k-j}{u-1}$, which is the same as the coefficient in the second summand. Furthermore, since $\bar{x} \in (H^{\otimes(k+1)})^{\Sigma_1 \times \Sigma_k}$, one has

$$\begin{aligned} x_1 \dots x_j \otimes x_{j+1} \dots x_{j+(k+1-j-u)} \otimes \dots \otimes x_{k+1} \\ = x_1 \dots x_j \otimes x_{j+u+1} \dots x_{k+1} \otimes x_{j+1} \dots x_{j+u}. \end{aligned}$$

The right-hand side of the equality has the form $(R \circ S) \circ T + R \circ (T \circ S)$, hence by the pre-Lie relation it can also be written $R \circ (S \circ T) + (R \circ T) \circ S$, which gives the following:

$$\begin{aligned} A_{k+1}m_y(\bar{x}) &= \sum_{j=1}^k \sum_{u=1}^{k+1-j} \binom{k-1}{j-1} \binom{k-j}{u-1} [A_j(x_1 \dots x_j) \\ &\quad \circ (A_{k+2-j-u}(y \otimes x_{j+u+1} \dots x_{k+1}) \circ A_u(x_{j+1} \dots x_{j+u})) \\ &\quad + (A_j(x_1 \dots x_j) \circ A_u(x_{j+1} \dots x_{j+u})) \\ &\quad \circ A_{k+2-j-u}(y \otimes x_{j+u+1} \dots x_{k+1})] \\ &\stackrel{\text{invariance of } \bar{x}}{=} \sum_{j=1}^k \sum_{u=1}^{k+1-j} \binom{k-1}{j-1} \binom{k-j}{u-1} [A_j(x_1 \dots x_j) \circ \\ &\quad (A_{k+2-j-u}(y \otimes x_{j+1} \dots x_{k+1-u}) \circ A_u(x_{k+2-u} \dots x_{k+1})) \\ &\quad + (A_j(x_1 \dots x_j) \circ A_u(x_{j+1} \dots x_{j+u})) \\ &\quad \circ A_{k+2-j-u}(y \otimes x_{j+u+1} \dots x_{k+1})]. \end{aligned}$$

Since $\binom{k-1}{j-1} \binom{k-j}{u-1} = \binom{k-j}{k+1-j-u} \binom{k-1}{j-1}$, the first summand gives

$$\sum_{j=1}^k \binom{k-1}{j-1} A_j(x_1 \dots x_j) \circ A_{k+2-j}(y \otimes x_{j+1} \dots x_{k+1}).$$

Since $\binom{k-1}{j-1} \binom{k-j}{u-1} = \binom{j+u-2}{j-1} \binom{k-1}{j+u-2}$, the second summand gives

$$\sum_{\alpha=2}^{k+1} \binom{k-1}{\alpha-2} A_\alpha(x_1 \dots x_\alpha) \circ A_{k+2-\alpha}(y \otimes x_{\alpha+1} \dots x_{k+1}).$$

Adding the two results, one gets the relation (4.6). \square

4.5. Proof of property (2) of the fundamental lemma

Let x, y be in H . Since $\Delta^k(x) \in (H^{\otimes(k+1)})^{\Sigma_1 \times \Sigma_k}$, one can apply formula (4.4):

$$\begin{aligned} e(x \circ y) &= x \circ y + \sum_{k>0} \frac{(-1)^k}{k!} A_{k+1} \Delta^k(x \circ y) \\ &\stackrel{(4.3)}{=} x \circ y + \sum_{k>0} \frac{(-1)^k}{k!} A_{k+1} (\delta_y(\Delta^{k-1}(x)) + \Delta^k(x) \circ y) \\ &\stackrel{(4.4)}{=} x \circ y + \sum_{k>0} \frac{(-1)^k}{k!} k A_k(\Delta^{k-1}(x) \circ y) + \sum_{k>0} \frac{(-1)^k}{k!} A_{k+1}(\Delta^k(x) \circ y) \\ &= x \circ y - A_1(\Delta^0(x) \circ y) = 0. \quad \square \end{aligned}$$

5. Using cogroups

In [7] and [8] Fresse proves that a cogroup in the category of connected graded \mathcal{P} -algebras, where \mathcal{P} is an operad is a free \mathcal{P} -algebra. In [18] Oudom generalises this theorem to comagma in this category. This is a generalisation of a theorem of Leray [13] for commutative algebras and of Bernstein [1] for associative algebras. This technique was also studied by Holtkamp, for instance in the context of \mathcal{P} -Hopf operads (see e.g. his habilitationsschrift [12]). The purpose of this section is to present the theorems of rigidity for associative algebras and for pre-Lie algebras by the way of abelian cogroups in some categories.

5.1. Coproduct for algebras over an operad

We recall the notation of Fresse. Let \mathcal{O} be an operad. The category of graded algebras over \mathcal{O} is endowed with a *coproduct* \vee : let A and B be two \mathcal{O} -algebras then $A \vee B = \mathcal{O}(A \oplus B)/rel$, where *rel* is generated by relations of the form

$$\mu \otimes (x_1 \otimes \dots \otimes x_n) - \mu(x_1, \dots, x_n)$$

for all $\mu \in \mathcal{O}(n)$ and $x_1 \otimes \cdots \otimes x_n \in A^{\otimes n} \cup B^{\otimes n}$. In particular, A and B are summands of $A \vee B$. There is a natural morphism $T : A \vee B \rightarrow B \vee A$. A cogroup in this category is a connected graded \mathcal{O} -algebra together with a map $\phi : A \rightarrow A \vee A$ such that ϕ is coassociative and the identity on each copy of A . A cogroup is abelian or cocommutative if $T\phi = \phi$. The theorem of Fresse that we use is the following.

Theorem ([8]). *Fix a ground field of characteristic 0. Let \mathcal{P} be a unital operad. If R is a connected graded \mathcal{P} -algebra equipped with a cogroup structure, then R is a free graded \mathcal{P} -algebra.*

5.2. Theorems of Berstein [1] and Loday–Ronco [15] revisited

Since we work in the context of operads, we are concerned with non-unital associative algebras. In this context, a *non-unital infinitesimal bialgebra* is a non-unital associative algebra together with a non-counital coassociative coproduct Δ satisfying the relation

$$\Delta(x \cdot y) = x \otimes y + x_{(1)} \otimes x_{(2)} \cdot y + x \cdot y_{(1)} \otimes y_{(2)}. \quad (5.1)$$

Theorem. *Let H be an object in the category of graded connected non-unital associative algebras. The following are equivalent:*

- (a) H is a non-unital infinitesimal bialgebra,
- (b) H is free,
- (c) H is an abelian cogroup.

The equivalence (a) and (b) is a theorem by Loday and Ronco [15] on the structure of unital infinitesimal bialgebras adapted to the non-unital case. The equivalence (b) and (c) is a consequence of a theorem by Berstein [1]. Indeed, the equivalence (a) and (c) can be proved independently. Here is the proof:

Recall that the coproduct in the category of associative algebras is given by $H_1 \star H_2 = H_1 \oplus H_2 \oplus \bigoplus_n \underbrace{H_1 \otimes H_2 \otimes \cdots}_{\text{alternative tensor } n \text{ times}} \oplus \underbrace{H_2 \otimes H_1 \otimes \cdots}_{n \text{ times}}$, hence

$$H \star H = \bigoplus_{n \geq 1} (1, H^{\otimes n}) + (2, H^{\otimes n}),$$

where the label 1 or 2 indicates the beginning of the alternative tensor. The product of two elements (i, a^n) and (j, b^m) with $a^n \in H^{\otimes n}$ and $b^m \in H^{\otimes m}$ lies in the copy i of H and is the concatenation of a and b if the last term of a^n lies in the copy $\neq j$ of H , and is $a_1^n \otimes \cdots \otimes a_{n-1}^n \otimes a_n^n \cdot b_1^m \otimes \cdots \otimes b_m^m$ otherwise. If H is an abelian cogroup in the category of associative algebras, then there is a map $\phi : H \rightarrow H \star H$ which is coassociative and is the identity on H : $\phi(h) = (1, h) + (2, h) + (1, h_1 \otimes h_2) + (2, h_1 \otimes h_2) + \cdots$. Define $\Delta : H \rightarrow H \otimes H$ by $\Delta(h) = h_1 \otimes h_2$ the projection π of ϕ onto the vector space $(1, H^{\otimes 2})$. The coassociativity of ϕ implies that of Δ . Since ϕ is a morphism of associative algebras, one has

$$\begin{aligned} \Delta(a \cdot b) &= \pi\phi(a \cdot b) \\ &= \pi[((1, a) + (1, a_1 \otimes a_2) + \cdots) \cdot ((1, b) + (2, b) + (1, b_1 \otimes b_2) + \cdots)] \\ &= a \otimes b + a \cdot b_1 \otimes b_2 + a_1 \otimes a_2 \cdot b, \end{aligned}$$

which is relation (5.1).

To prove the converse, it is easy to check that the map $\phi : H \rightarrow H \star H$ defined by $\phi(a) = \sum_{n \geq 1} (1, \Delta^{n-1}(a)) + (2, \Delta^{n-1}(a))$ is well defined (the algebra is graded connected), is coassociative, cocommutative and a morphism of algebras, thanks to the relation

$$\begin{aligned} \Delta^k(a \cdot b) &= \sum_{i=1}^k a_{(1)} \otimes \cdots \otimes a_{(i)} \otimes b_{(1)} \otimes \cdots \otimes b_{(k+1-i)} \\ &+ \sum_{i=1}^{k+1} a_{(1)} \otimes \cdots \otimes a_{(i)} \cdot b_{(1)} \otimes \cdots \otimes b_{(k+2-i)}. \quad \square \end{aligned}$$

5.3. Cogroups in the category of pre-Lie algebras

Let T be a rooted tree. A leaf of T is a vertex with no incoming edges. For $v \in \text{Vert}(T)$ and for some rooted trees S_1, \dots, S_n , the tree $T \circ_v \{S_1, \dots, S_n\}$ is the tree obtained by grafting the trees S_i onto the vertex v of T .

Denote by \mathcal{PL} the operad defining pre-Lie algebras. Recall that $\mathcal{PL}(V)$ denotes the free pre-Lie algebra generated by V . Let L_1 and L_2 be two pre-Lie algebras. The coproduct of L_1 and L_2 is $L_1 \vee L_2 = \mathcal{PL}(L_1 \otimes L_2)/rel$, where the equivalence relation is that defined in 5.1. It is not difficult to prove the following lemma.

Lemma 5.3.1. *Let L be a pre-Lie algebra, and $(e_\alpha)_\alpha$ a basis of its underlying vector space. Then a basis of $L \vee L$ is given by rooted trees labeled by the set $(i, e_\alpha)_{i \in \{1,2\}, \alpha}$ satisfying the “leaf condition”: the labeling i of any leaf is different from that of its adjacent vertex.*

Furthermore, its pre-Lie structure is given by the following: let T, S be two trees. If $\text{deg}(S) > 1$, then $T \circ S = \sum_{v \in \text{Vert}(T)} T \circ_v \{S\}$. If $\text{deg}(S) = 1$, then assume S is a single vertex labeled by $(1, e_\alpha)$ for instance. If $v \in \text{Vert}(T)$ is labeled by 2, then set $T \hat{\circ}_v S = T \circ_v \{S\}$. If v is labeled by 1, then $T \circ_v \{S\}$ does not respect the “leaf condition”. Denote by T_1^v, \dots, T_k^v the incoming subtrees of T at the vertex v ; denote by $T_{v \leftarrow w}$ the tree obtained by replacing v by w . If $v = (1, e_\beta)$, then we define

$$T \hat{\circ}_v (1, e_\alpha) = T_{v \leftarrow (1, e_\beta \circ e_\alpha)} - \sum_{I, J} T \circ_v \{T_I, (1, e_\alpha) \circ T_J\},$$

where the sum is over all partition $I \cup J$ of $\{1, \dots, k\}$ such that $J \neq \emptyset$. Then, for $\text{deg}(S) = 1$, $T \circ S = \sum_{v \in \text{Vert}(T)} T \hat{\circ}_v S$.

Lemma 5.3.2. *If L is a pre-Lie algebra together with a cogroup structure $\phi : L \rightarrow L \vee L$, then L is endowed with a non-associative permutative coproduct Δ satisfying the relation (3.1).*

Proof. The relation $(Id - \tau_{23})(\Delta \otimes Id)\Delta$ is a consequence of the associativity of ϕ . Relation (3.1) is a consequence of ϕ being a morphism of pre-Lie algebras. \square

Theorem 5.3.3. *Let L be an object in the category of graded connected pre-Lie algebras. The following are equivalent:*

- (a) L is a non-associative permutative coalgebra satisfying relation (3.1).

- (b) L is free.
 (c) L is an abelian cogroup.

Proof. The equivalence (a) and (b) is the purpose of our paper. The equivalence (b) and (c) is the theorem of Fresse applied to the operad pre-Lie. (c) implies (a) from the previous lemma. \square

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