Monte Carlo methods for option pricing

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We consider a d-dimensional Brownian diffusion $(X_t)_{t\in[0,T]}$ starting at $x \in \mathbb{R}^d$, $X_0 = x$, solution to the Stochastic Differential Equation (SDE) :

$$(E) \qquad dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $b : [0, T] \times \mathbb{R}^d \to \mathbb{M}(d \times q)$ are continuous functions and $(W_t)_{t\in[0,T]}$ denotes a q-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Example

Black-Scholes $(q = 1,\ d = 1,\ b(t, x) = rx,\ \sigma(t, x) = \sigma x)$

$$dS_t = S_t(rdt + \sigma dWt).$$

Local volatility models $(q = 1,\ d = 1)$

$$dS_t = rS_tdt + \sigma(t, S_t)S_tdWt.$$

Stochastic volatility models $(q = 2,\ n = 2,\ \rho \in [0, 1])$

$$\begin{cases} dS_t &= b_1(t, S_t)dt + \sigma_t S_t dW_t^1 \\ d\sigma_t &= b_2(t, \sigma_t)dt + \rho a_1(t, \sigma_t)dW_t^1 + \sqrt{1-\rho^2}a_2(t, \sigma_t)dW_t^2. \end{cases}$$

Definition 1 *A strong solution to the SDE (E) is a process $(X_t)_{t\in[0,T]}$ adapted to the completed filtration generated by the Brownian motion satisfying*

- $\int_0^T |b(t, X_t)| + |\sigma(t, X_t)|^2 dt < 1 a.s$
- $\forall t \in [0, T], \quad X_t = x + \int_0^t b(s, Xs)ds + \int_0^t \sigma(s, Xs)dW_s \quad a.s.$

Under usual assumptions for example we assume b and σ are Lipschitz continuous in x uniformly with respect to $t \in [0, T]$ with linear growth we have a unique strong solution to the SDE (E). This solution has continuous paths and is adapted to the completed filtration generated by the Brownian motion, we denote it by $(\mathcal{F}_t)_{t\in[0,T]}$. More precisely we have.

Theorem 1 *Under the following assumption*

$$\mathcal{H}_1 \begin{cases} \exists K > 0, \forall t \in [0, T], \forall (x, y) \in (\mathbb{R}^d)^2 \\ |b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \le K|x - y| \\ |b(t, x)| + |\sigma(t, x)| \le K(1 + |x|) \end{cases}$$

the SDE (E) has a unique strong solution $(X_t)_{t\in[0,T]}$ satisfying $\mathbb{E}\sup_{t\in[0,T]} |X_t|^2$.

Note that the assumption $\mathcal{H}_1$ can be relaxed, some one can think to the rate model of Cox-Ingersoll-Ross (CIR) : $dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dWt$ with $(a, b, \sigma, r_0) \in \mathbb{R}_+$. In practice, for example in finance, we often have to calculate quantities of type $\mathbb{E}f(X_T)$ where $(X_t)_{t\in[0,T]}$ is solution to (E). To approximate $\mathbb{E}f(X_T)$ by a Monte Carlo method, we need first to approximate $(X_t)_{t\in[0,T]}$ by a process $(X^n_t)_{t\in[0,T]}$ such that

$$\mathbb{E}f(X_T) \approx \mathbb{E}f(X^n_T) \approx \frac{1}{M}\sum_{i=1}^{M} f(X^n_{i,T}).$$

To do this, we introduce first the Euler scheme.

1 Euler scheme

The stepwise constant Euler scheme is defined by, $\bar{X}_0 = x$ and

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + \frac{T}{N}b(t_k, \bar{X}_{t_k}) + \sigma(t_k, \bar{X}_{t_k})\Delta W_{k+1}, k = 0, \cdots, N-1$$

where $t_k = \frac{kT}{N}$, $k = 0, \cdots, N$ and $\Delta W_k = W_{t_k} - W_{t_{k-1}}$, $k = 1, \cdots, N$ denotes increments of the Brownian motion between times t_k and t_{k-1}. Since ΔW_k, $k = 1, \cdots, N$, are independent random variables with the same distribution $\mathcal{N}\left(0, \frac{T}{N}I_q\right)$ the Euler scheme is easily implementable. The stepwise constant Euler scheme consists by approximating X_t by $\bar{X}_{t_k}$ for $t \in [t_k, t_{k+1}[$. It is also convenient from a theoretical point of view to introduce the continuous Euler scheme defined by

$$\bar{X}_t = \bar{X}_{t_k} + (t - t_k)b(t_k, \bar{X}_{t_k}) + \sigma(t_k, \bar{X}_{t_k})(W_t - W_{t_k}), t \in [t_k, t_{k+1}[, k = 0, \cdots, N-1.$$

We consider the following assumption

$$\mathcal{H}_2 \left\{ \begin{array}{c} \exists \alpha > 0, \exists K > 0, \forall (s,t) \in [0,T]^2, \forall x \in \mathbb{R}^d \\ |b(t,x) - b(s,x)| + |\sigma(t,x) - \sigma(s,x)| \leq K(1 + |x|)|t - s|^{\alpha} \end{array} \right.$$

Note that this assumption is obviously satisfied for an homogeneous stochastic differential equation.

Theorem 2 *Under assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$ we have*

$$\forall p \geq 1, \exists C_p > 0, \forall N \in \mathbb{N}^*, \quad \mathbb{E}\left(sup_{t\in[0,T]}|X_t - \bar{X}_t|^{2p}\right) \leq \frac{C_p}{N^{2\beta p}}$$

with $\beta = \min(\alpha, \frac{1}{2})$. Further, for $\gamma < \beta$

$$N^{\gamma} sup_{t\in[0,T]}|X_t - \bar{X}_t| \longrightarrow 0 \quad a.s. \text{ when } N \to \infty.$$

Hence, for an homogeneous stochastic differential equation we have the following result.

Corollary 1 *If (E) is an homogeneous SDE then under assumption $\mathcal{H}_1$ we have*

$$\forall p \geq 1, \exists C_p > 0, \forall N \in \mathbb{N}^*, \quad \mathbb{E}\left(sup_{t\in[0,T]}|X_t - \bar{X}_t|^{2p}\right) \leq \frac{C_p}{N^p}$$

and for $\gamma < \frac{1}{2}$

$$N^\gamma sup_{t\in[0,T]}|X_t - \bar{X}_t| \longrightarrow 0 \quad a.s.\ when\ N \to \infty.$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz continuous function,

$$\left|\mathbb{E}f(X_t) - \mathbb{E}f(\bar{X}_t)\right| \leq K\sqrt{\mathbb{E}|X_t - \bar{X}_t|^2}.$$

Corollary 2 *We obtain the first result for weak convergence.*

- *Under assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$, if $\alpha > \frac{1}{2}$*

$$\left|\mathbb{E}f(X_t) - \mathbb{E}f(\bar{X}_t)\right| \leq \frac{C}{N^{\frac{1}{2}}}.$$

- *If (E) is an homogeneous SDE then under assumption $\mathcal{H}_1$ we have*

$$\left|\mathbb{E}f(X_t) - \mathbb{E}f(\bar{X}_t)\right| \leq \frac{C}{N^{\frac{1}{2}}}.$$

Example : European call or put option in unidimensional framework $d = 1$.

In the same manner, let $g : (\mathbb{R}^d)^{n+1} \longrightarrow \mathbb{R}$ be a Lipschitz continuous function and $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$,

Corollary 3

- *Under assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$, if $\alpha > \frac{1}{2}$*

$$\left|\mathbb{E}g(X_{t_0}, \cdots, X_{t_n}) - \mathbb{E}g(\bar{X}_{t_0}, \cdots, \bar{X}_{t_n})\right| \leq \frac{C}{N^{\frac{1}{2}}}.$$

- *If (E) is an homogeneous SDE then under assumption $\mathcal{H}_1$ we have*

$$\left|\mathbb{E}g(X_{t_0}, \cdots, X_{t_n}) - \mathbb{E}g(\bar{X}_{t_0}, \cdots, \bar{X}_{t_n})\right| \leq \frac{C}{N^{\frac{1}{2}}}.$$

Example : Asiatic call or put option in unidimensional framework $d = 1$.

We study now the convergence rate of $\left|\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T)\right|$.

Notation We introduce the following notation.

- We denote by $C_b^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^p)$ the set of functions in $C^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^p)$ with bounded derivatives for all order.
- We denote by $C_{pol}^\infty(\mathbb{R}^d;\mathbb{R})$ the set of functions in $C^\infty(\mathbb{R}^d;\mathbb{R})$ with polynomial growth derivatives for all order. Let $F \in C_{pol}^\infty(\mathbb{R}^d;\mathbb{R})$, we have $\forall a = (a_1, \cdots, a_d) \in \mathbb{N}^d, \exists p_a \in \mathbb{N}, \exists C_a > 0, \forall x \in \mathbb{R}^d$,

$$\left|\frac{\partial^{(a_1+\ldots+a_d)}F}{\partial x_1^{a_1}\cdots\partial x_d^{a_d}}(x)\right| \leq C_a(1 + |x|^{p_a}).$$

Theorem 3 *(Talay-Tubaro 1990) Let $b \in C_b^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in C_b^\infty([0,T] \times \mathbb{R}^d; \mathbb{R}^{dq})$. If $f \in C_{pol}^\infty(\mathbb{R}^d; \mathbb{R})$ then $\forall k \in \mathbb{N}^*$ we have*

$$\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T) = \sum_{i=1}^{k} \frac{C_i}{N^i} + O(\frac{1}{N^{k+1}}).$$

where constants $C_1, \cdots, C_q$ depend only on the function f.

Remark : This result can not be used for European call or put option.

Theorem 4 *(Bally-Talay 1995) We consider an homogeneous SDE (E). Let $b \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in C_b^\infty(\mathbb{R}^d; \mathbb{R}^{dq})$ such that*

$$\exists A > 0, \forall \xi \in \mathbb{R}^d, \forall x \in \mathbb{R}^d, \quad \xi^{tr} \sigma(x) \xi \sigma^{tr}(x) \geq A|\xi|^2.$$

If f is mesreable then

$$\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T) = \frac{C_1}{N} + O(\frac{1}{N^2}).$$

where C depends only on the function f.

Remark : On one hand, this result can be used for European call or put option. On onther hand, this result provide an other algorithm with order $\frac{1}{N^2}$ knowen by Romberg Extrapolation

$$\mathbb{E}f(X_T) - \mathbb{E}\left[2f(\bar{X}_T^{2N)} - f(\bar{X}_T^N)\right] = O(\frac{1}{N^2}).$$

2 Milshtein scheme

In this section we will consider an homogeneous diffusion and we will take $d = q = 1$. The idea is to improve the approximation of the stochastic integral using

$$\begin{aligned} \sigma(X_s) &\approx \sigma(X_{t_k}) + \sigma'(X_{t_k})(X_s - X_{t_k}) \\ &\approx \sigma(X_{t_k}) + \sigma'(X_{t_k})\sigma(X_{t_k})(W_s - W_{t_k}) \end{aligned}$$

The stepwise constant Milshtein scheme is defined by, $\tilde{X}_0 = x$ and

$$\tilde{X}_{t_{k+1}} = \tilde{X}_{t_k} + \frac{T}{N} b(\tilde{X}_{t_k}) + \sigma(\tilde{X}_{t_k})\Delta W_{k+1} + \frac{1}{2}\sigma'(X_{t_k})\sigma(X_{t_k})((\Delta W_{k+1})^2 - \frac{T}{N})$$

for $k = 0, \cdots, N-1$. We define similarly the continuous version.

Theorem 5 *If (E) is an homogeneous SDE, if σ and b are C^2 with bounded derivatives then*

$$\forall p \geq 1, \exists C_p > 0, \forall N \in \mathbb{N}^*, \quad \mathbb{E}\left(sup_{t\in[0,T]}|X_t - \tilde{X}_t|^{2p}\right) \leq \frac{C_p}{N^{2p}}$$

and for $\gamma < \frac{1}{2}$

$$N^\gamma sup_{t\in[0,T]}|X_t - \bar{X}_t| \longrightarrow 0 \quad a.s.\ when\ N \to \infty.$$

3 Numerical illustrations

we will carry out some simulations on the Black & Scholes model by using different methods introduced in our lesson. The price of the risky asset in the model is given by the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = s_0 > 0, , t \in [0, T]$$

with $(W_t)_{0 \le t \le T}$ is a standard Brownian motion. We denote by $(\mathcal{F}_t)_{0 \le t \le T}$ its canonical filtration.

- The SDE above has an explicit solution given by

$$S_t = s_0 exp\left((r - \frac{\sigma^2}{2})t + \sigma W_t \right).$$

  We denote $(S_t^{0,s_0})_{0 \le t \le T}$ the solution starting at s_0, $S_0 = s_0$. An European option may be exercised only at the expiration date of the option T and it is well defined by the payoff $g(S_T)$ where g is a real valued function. The price of the option called the prime at time $t \in [0, T]$ is given by

$$\mathbb{E}\left(e^{-r(T-t)} g(S_T^{0,s_0} | \mathcal{F}_t)\right).$$

- For the call option the payoff $g(x) = (x - K)_+$ and the prime at time $t \in [0, T]$ is given by $V(t, S_t^{0,s_0})$ with

$$V(t, x) = xN(d_1) - Ke^{-r(T-t)} N(d_2),$$

  where

$$d_1 = \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \ \ and \ \ d_2 = d_1 - \sigma\sqrt{T - t}$$

  Similarly, we have a closed formula for the put option with payoff $g(x) = (K - x)_+$ and $V(t, x) = Ke^{-r(T-t)} N(-d_2) - xN(-d_1)$. The aim of our practical work is to illustrate the different results given for the Euler and the Milshtein schemes on the Black & Scholes model. We denote by T the expiration date, $\delta = T/N$ the time step of discretization and $t_k = kT/N$, $k \in \{0, \cdots, N\}$, the times of discretization.

3.1 Strong convergence

Write in terms of the asset price S_{t_k} at time t_k and the increment $\Delta W_{k+1} = W_{t_{k+1}} - W_{t_k}$ the asset price $S_{t_{k+1}}$ at time t_{k+1}. Similarly, we write in terms of $S_{t_k}^e$ the Euler scheme at time t_k (respectively $S_{t_k}^m$) the Milshtein scheme) and the increment $\Delta W_{k+1} = W_{t_{k+1}} - W_{t_k}$ the approximation $S_{t_{k+1}}^e$ (respectively $S_{t_{k+1}}^m$))at time t_{k+1}. Study the L^2-error $\mathbb{E}(S_T - S_T^e)^2$ and $\mathbb{E}(S_T - S_T^m)^2$ as a function of N.

```
//parameters
T=1;//expiration date
sig=0.2;//volatility
r=0.05;//interest rate
S0=100;//initial value of the underlying asset
N=2;//initial number of discretization steps
```

```
M=10000;//number of Monte Carlo simulation
erreul=[];//vector of Euler scheme's strong errors
errmil=[];//vector of  Milshtein scheme's strong errors
liceul=[];//radius of the CI of the Euler scheme
licmil=[];//radius of the CI of the Milshtein scheme
Npas=[];// vector of number of discretization steps
for j=1:5, //loop on the number of discretization steps
//Useful parameters corresponding to the discretization step N
//////////////////////////////////////////////////
//Complete with useful parameters
//////////////////////////////////////////////////
//storage variables
someul=0;
careul=0;
sommil=0;
carmil=0;
for i=1:M, //Monte Carlo simulations
S=S0;
Se=S0;
Sm=S0;
for k=1:N, //loop on  the discretization steps
g=rand(1,'g');//generate a standard Gaussian
////////////////////////////////////////////////////////////
//Complete with the evolution of the underlying asset and its both  approximations
////////////////////////////////////////////////////////////
end;
someul=someul+(S-Se)^2;
careul=careul+(S-Se)^4;
sommil=sommil+(S-Sm)^2;
carmil=carmil+(S-Sm)^4;
end;
erreul=[erreul,someul/M];
liceul=[liceul,1.96*sqrt((careul/M-(someul/M)^2)/M)];
errmil=[errmil,sommil/M];
licmil=[licmil,1.96*sqrt((carmil/M-(sommil/M)^2)/M)];
Npas=[Npas,N];
N=N*2;//multiplying the number of steps N by 2
end;
//Display vectors of errors and radius of CI
erreul
liceul
errmil
licmil
//plot of (1/N,erreul) and (1/(N*N),errmil)
```

```
xbasc();
subplot(2,1,1);
plot2d(1../Npas',[erreul;erreul-liceul;erreul+liceul]');
subplot(2,1,2);
plot2d(1../Npas^2',[errmil;errmil-licmil;errmil+licmil]');
```

3.2 Weak convergence

Our aim now is to study in terms the discretization parameter N the weak convergence of both Euler scheme and Milshtein scheme for the price of European put with expiration date T and strike K. In order to do this we will compute $\mathbb{E}(e^{-rT}(K-S_T^e)_+) - \mathbb{E}(e^{-rT}(K-S_T)_+)$ and $\mathbb{E}(e^{-rT}(K-S_T^m)_+) - \mathbb{E}(e^{-rT}(K-S_T)_+)$ where the simulation of the both expectations are given by the same trajectories of the brownian motion in other terms we will use the same gaussian increments $\Delta W_{k+1} = W_{t_{k+1}} - W_{t_k}$.

```
//parameters
T=1;//expiration date
sig=0.2;//volatility
r=0.05;//interest rate
S0=100;//initial value of the underlying asset
K=110;// Put strike
N=2;//initial number of discretization steps

M=10000;//number of Monte Carlo simulation
//Black-Scholes formula for put price
d1=(log(S0/K)+r*T)/(sig*sqrt(T))+sig*sqrt(T)/2;
d2=d1-sig*sqrt(T);
BS=K*exp(-r*T)*cdfnor("PQ",-d2,0,1)-S0*cdfnor("PQ",-d1,0,1);
//payoff function of the put
function y=put(S,K)
y=0;
if (S<K)
y=K-S;
end;
endfunction;
erreul=[];//vector of Euler scheme's weak errors
liceul=[];//radius of the CI of the Euler scheme
errmil=[];//vector of  Milshtein scheme's weak errors
licmil=[];//radius of the CI of the Milshtein scheme
//conterreul=[];//vecteur des erreurs faibles Euler avec variable de contrôle
//contliceul=[];//largeur des intervalles de confiance 95%
//conterrmil=[];//vecteur des erreurs faibles Milshtein avec variable de contrôle
//contlicmil=[];//largeur des intervalles de confiance 95%
```

```
Npas=[];// vector of number of discretization steps
for j=1:5,//loop on the number of discretization steps
//storage variables
puteul=0;
careul=0;
putmil=0;
carmil=0;
puteul=0;
//contputeul=0;
//contcareul=0;
//contputmil=0;
//contcarmil=0;

//Useful parameters corresponding to the discretization step N
//////////////////////////////////////////////////
//Complete with useful parameters
//////////////////////////////////////////////////
for i=1:M,//Monte Carlo simulations
S=S0;
Se=S0;
Sm=S0;
for k=1:N,//loop on  the discretization steps
g=rand(1,'g');//generate a standard Gaussian
////////////////////////////////////////////////////////////
//Complete with the evolution of the underlying asset and its both  approximations
////////////////////////////////////////////////////////////
end;
//contribution of the trajectoir of S
paymc=put(S,K);
//contribution of S^e trajectoiry
payeul=put(Se,K);
puteul=puteul+payeul;
careul=careul+payeul^2;
////////////////////////////////////////////////////////////
//To be completed by storing in contputeul (resp. contcareul)
//the sum (resp. sum of squares) differences between payoff
//S and its approximation Se
////////////////////////////////////////////////////////////
//contribution of S^m trajectoiry
////////////////////////////////////////////////////////////
//The same calculations for  Milshtein in place of Euler
////////////////////////////////////////////////////////////
end;
erreul=[erreul,exp(-r*T)*puteul/M-BS];
```

```
liceul=[liceul,1.96*exp(-r*T)*sqrt((careul/M-(puteul/M)^2)/M)];
errmil=[errmil,exp(-r*T)*putmil/M-BS];
licmil=[licmil,1.96*exp(-r*T)*sqrt((carmil/M-(putmil/M)^2)/M)];
conterreul=[conterreul,exp(-r*T)*contputeul/M];

contliceul=[contliceul,1.96*exp(-r*T)*sqrt((contcareul/M-(contputeul/ ...
M)^2)/M)];
conterrmil=[conterrmil,exp(-r*T)*contputmil/M];
contlicmil=[contlicmil,1.96*exp(-r*T)*sqrt((contcarmil/M-(contputmil/ ...
M)^2)/M)];
Npas=[Npas,N];
N=N*2;//multiplication du nombre N de pas par 2
end;
erreul
liceul
errmil
licmil
conterreul
contliceul
conterrmil
contlicmil
//plot (1/N,conterreul) and (1/N,conterrmil)
xbasc();
subplot(2,1,1);
plot2d(1../Npas',[conterreul;conterreul-contliceul;conterreul+contliceul]');
subplot(2,1,2);
plot2d(1../Npas',[conterrmil;conterrmil-contlicmil;conterrmil+contlicmil]');
```