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On Max-Multiscaling Distributions as Extended Max-Semistable Ones

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ABSTRACT

We introduce max-multiscaling distributions as solutions to a functional equation which, in a natural way, extends the one fulfilled by max-semistable distributions. We establish that strictly max-multiscaling distributions are products of at most two max-semistable distributions. Next, we show how to obtain these solutions as limit laws of normalized maximum of suitable independent sequences of random variables when sample size has geometric growth.

Key Words: Max-stability; Max-semistability; Choquet–Deny functional equation; Limit laws.

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1. INTRODUCTION

This work deals with an extension of max-semistable (in short *MSS*) laws which are invariant under affine normalizations: the so-called max-multiscaling (*MM*) distributions.

An important building block of the *MM* models is the class of strictly *MM* random variables of type I. These are identified with the ones with support $[0, \infty)$, whose distribution function (df), say F, satisfies a functional equation of the form

$$F(v) = \prod_{i=1}^{m} F(v/c_i)^{\gamma_i} \tag{1}$$

for some $m > 1$, $c_i \in (0, \infty)$, $\gamma_i > 0$, $i = 1, \ldots, m$. This equation generalizes the one characterizing strictly max-semistable distributions of type I which is obtained when $m = 1$.

Our physical motivations are the following ones. Just like for sums, maximum infinitely divisible (*MID*) variables are either of the compound Poisson type or suitable weak limits of these.[1] Indeed, let V^+ be a random variable with support $[0, \infty)$. When $v > 0$, its df F reads $F(v) = \exp\{-\bar{\pi}(v)\}$, where $\bar{\pi}\ (\cdot)$ is a right-continuous, non-increasing function satisfying $\bar{\pi}(v) \to_{v\uparrow\infty} 0$. Let $(\chi_p(\epsilon),\, p \geq 1)$ be an independent and identically distributed (iid) sequence satisfying

$$\mathbb{P}(\chi_1(\epsilon) > v) = \frac{\bar{\pi}(v)}{\bar{\pi}(\epsilon)}, \quad v > \epsilon.$$

Let $\mathscr{P}(\lambda(\epsilon))$ be a Poisson random variable with intensity $\lambda(\epsilon) > 0$, independent of $(\chi_p(\epsilon), p \geq 1)$ with $\frac{\lambda(\epsilon)}{\bar{\pi}(\epsilon)} \to_{\epsilon\downarrow 0} 1$. We then have the weak approximation

$$\max_{p=1,\ldots,\mathscr{P}(\lambda(\epsilon))} \chi_p(\epsilon) \underset{\epsilon\downarrow 0}{\overset{d}{\to}} V^+.$$

If $\bar{\pi} \to_{v\downarrow 0} \infty, V^+$ is not in the compound Poisson class and $\lambda(\epsilon) \to_{\epsilon\downarrow 0} \infty$, suggesting that "micro-events" $(\chi_p(\epsilon), p \geq 1)$ with small amplitudes are extremely frequent. The limiting observable V^+ now physically interprets as the maximum of a Poisson number of such micro-events.

Suppose F satisfies functional Eq. (1) with $m = 1$, $\rho := \gamma_1 > 1$ and $c := c_1 \in (0, 1)$. Such an *MSS* "observable" V^+, as defined by $V^+ \overset{d}{\approx} \max_{p=1,\ldots,\mathscr{P}(\lambda(\epsilon))} \chi_p(\epsilon)$, might as well result from more frequent micro-events but with smaller reduced amplitudes. Indeed, in this case, V^+ also satisfies $V^+ \overset{d}{\approx} \max_{p=1\ldots\mathscr{P}(\rho\lambda(\epsilon))} c\chi_p(\epsilon)$. This translates an amplitude and scale invariance principle for the observable.

Let us now consider the case of a strictly *MM* observable V^+ of type I. Suppose that there are $m > 1$ possible types of sources from which independent observations of the same random phenomenon V^+ can come from. The functional Eq. (1) suggests that V^+ also satisfies

$$V^+ \overset{d}{\approx} \max_{i=1,\ldots,m}\ \max_{p=1\ldots\mathscr{P}_i(\gamma_i\lambda(\epsilon))} c_i\chi_{i,p}(\epsilon).$$

Here, $(\chi_{i,p}(\epsilon), p \geq 1, i = 1, \ldots, m)$ is an *iid* sequence with $\chi_{i,p}(\epsilon) \stackrel{d}{=} \chi_1(\epsilon)$, independent of the independent Poisson sequence $\mathscr{P}_i(\gamma_i \lambda(\epsilon))$, with intensities $\gamma_i \lambda(\epsilon)$. This means that the observation might as well result from the aggregation of $m > 1$ independent observations of statistically similar events, each with its specific intensity and scale.

Central to the strictly type I solutions to functional Eq. (1) is the structure function

$$\tau(q) = \sum_{i=1}^{m} \gamma_i c_i^q, \quad q \in \mathbb{R} \tag{2}$$

and the number $|\mathscr{S}^+|$, with $\mathscr{S}^+ = \{\alpha > 0 : \tau(\alpha) = 1\}$. We have $|\mathscr{S}^+| \in \{0, 1, 2\}$ and if $|\mathscr{S}^+| \neq 0$ the closed form solutions to (1) are exhibited. More precisely, if $|\mathscr{S}^+| = 1$ one recovers the Fréchet *MSS* class (in the lattice case $c_i = c^{r_i}, c \in (0, 1), r_i \in \mathbb{Z}$) and the Fréchet max-stable (*MS*) class in the non-lattice case. If $|\mathscr{S}^+| = 2$, one gets the full type I *MM* distributions. It will be shown in Corollary 2 that the df of such distributions is representable as the product of two semi-stable dfs. More precisely, if $|\mathscr{S}^+| = 2$ with two exponents $\alpha_1 < \alpha_2$, we shall show that solutions to (1) are of the form

$$F(v) = \exp\left\{ -\left[v^{-\alpha_1} e^{\alpha_1 \nu_1(\log v)} + v^{-\alpha_2} e^{\alpha_2 \nu_2(\log v)} \right] \right\}, \quad v > 0$$

where $\nu_k(x)$, $k = 1, 2$ are essentially periodic functions with the same periods $x_i = -\log c_i$, $i = 1, \ldots, m$.

In this case, $-\log F(v) \sim_{v\uparrow+\infty} v^{-\alpha_1} e^{\alpha_1 \nu_1(\log v)}$ and $-\log F(v) \sim_{v\downarrow 0} v^{-\alpha_2} e^{\alpha_2 \nu_2(\log v)}$ and empirical evidence of max-multiscaling phenomenon results from

$$\begin{aligned} -\log(-\log F_n(v)) &\approx \alpha_1(\log v - \nu_1(\log v)) \quad \text{for large } v \\ -\log(-\log F_n(v)) &\approx \alpha_2(\log v - \nu_2(\log v)) \quad \text{for small } v. \end{aligned}$$

Here, $F_n := \frac{1}{n} \sum_{m=1}^{n} 1(v_m \leq v)$ is the empirical distribution function where $(v_m, m = 1, \ldots, n)$ are *iid* random variables with distribution F. A plot of $-\log(-\log F_n(v))$ against $\log v$ should exhibit oscillations around a linear trend with positive slope α_1 at $v = +\infty$ and positive slope $\alpha_2 > \alpha_1$ at $v = 0$.

Discrete scale invariance (with a single scaling exponent) and selfsimilarity are naturally associated to critical phenomena in Physics.[26] Laws with non-unique scaling exponent are currently called multiscaling in the physics literature and are relevant in fully developed turbulence and disordered systems. Precise relationship between such intensity and scale invariance with the concept of (semi-)selfsimilarity will be briefly discussed in Subsec. 3.3.

In more details, this work is organized as follows.

In Sec. 2, the main features of *MSS* distributions are recalled[5–8,16,17]. These constitute a first-step extension of the max-stable (*MS*) Fréchet–Weibull–Gumbel trio. Some of their remarkable properties are briefly discussed, in particular their occurrence as limit laws.

In Sec. 3, we introduce the class of *MM* distributions. They can be defined as the fixed point of some transformation. More precisely, strictly max-multiscaling random variables of type I are identified with the ones with support $[0, \infty)$, whose df, say F, satisfies a functional equation of the form (1). Strictly max-multiscaling random variables of type II are the ones satisfying (1) with support $(-\infty, 0]$. Max-multiscaling random variable of type III are the ones with support $\mathbb{R}$ satisfying $F(x) = \prod_{i=1}^{m} F(x+\beta_i)^{\gamma_i}$, for some $m > 1$, $\beta_i \in \mathbb{R}$, $\gamma_i > 0$, $i = 1, \ldots, m$. Type II may be seen as negative reciprocal inverses of type I and type III as log type I. Introducing shifts, broad sense *MM* (of type I and II) distributions can be obtained. They are the ones solutions to $F(v) = \prod_{i=1}^{m} F(v/c_i + \beta_i)^{\gamma_i}$ with $\beta_i = \tilde{x}(1 - 1/c_i)$, $i = 1, \ldots, m$, for some real number $\tilde{x}$, an endpoint of the support.

Our representation results of *MM* distributions of all types are summarized in Theorem 1, Corollary 2 at the beginning of Sec. 3. After a preliminary study of strictly max-multiscaling models of type I in Subsec. 3.1, we complete the proof of this theorem in Subsec. 3.2.

Subsec. 3.3 deals with the concept of (semi-) selfsimilarity in relation with *MM* models.

In Sec. 4, we finally show that *MM* distributions may be seen as limit laws for some renormalized maximum of independent random variables, when sample size grows geometrically with some parameter r. We proceed progressively by giving two examples. A primary deterministic illustration of geometrical growth is when the sample size $n(r) \sim K\rho^r$ for some constants $K > 0$, $\rho > 1$. This number is generated by a deterministic version of the Crump and Mode branching process.[2] In this case, the *MM* limiting laws are in the class of *MSS* distributions. The obtained result is displayed in Proposition 7; it is in accordance with known results on max-domain of attraction for *MSS* distributions as described in Ref.[16]. Assuming a random Poisson sample size $N(r)$ with geometrically growing intensity, an example of a normalized sequence with *MM* distributions as limit laws is finally supplied. The main result on this construction is displayed in Theorem 8.

2. MAX-SEMISTABLE MODELS

We shall first recall a concept whose generality is larger than the well-known one of max-stability of the Fréchet–Weibull–Gumbel models, i.e., of limit laws of classical extreme value theory: namely the one of max-semistability. Max-semistable laws are identified with the ones whose df satisfies a functional equation of the form

$$F(v) = F(v/c + \beta)^{\rho} \tag{3}$$

for some $c > 0$, $\beta \in \mathbb{R}$, $\rho > 0$. They constitute the ''max version'' of the notion of semistability for sums first introduced by Paul Lévy in 1937 (see Ref.[12], p. 45 for a survey on this point).

Max-Semistable Model (Type I): "Extended" Fréchet

Let $\rho > 1$ and $c \in (0, 1)$. First, consider the class of positive random variable V^+ whose df satisfies the simpler functional equation

$$F_{V^+}(v) = F_{V^+}(v/c)^{\rho}. \tag{4}$$

These will be called strictly *MSS* variables of type I. The class of solutions of (4) are found to be (see Ref.[8])

$$F_{V^+}(v) = \exp\{-v^{-\alpha} s(\log v)\}, \quad v > 0, \tag{5}$$

where (i) $\alpha = -\log_c \rho > 0$, (ii) $s(x)$ is a non-negative function such that $s(x) := e^{\alpha\nu(x)}$, for some right-continuous bounded periodic function ν with period $-\log c$, satisfying the additional condition that $x - \nu(x)$ is non-decreasing. In sharp contrast with Fréchet's model, the scale parameter is not constant. Actually, the variable V^+ is not representative of all the class of *MSS* variables of type I. Those obtained after a shift of V^+ are also in this class. Indeed, let $x^+ \in \mathbb{R}$; consider the shifted variable $X^+ := V^+ + x^+$. The shifted variable now satisfies the functional equation of the form (3), with $\beta = x^+(1 - 1/c)$ whose solution is

$$F_{X^+}(x) = \exp\{-(x - x^+)^{-\alpha} s(\log(x - x^+))\}, \quad x > x^+. \tag{6}$$

If in (6) the scale function $s(x) = s > 0$, a constant, we recover the Fréchet laws.

Max-Semistable Model (Type II): "Extended" Weibull. Letting $V^- := -1/V^+ < 0$, its *df* is, from (5)

$$F_{V^-}(v) = F_{V^+}(-1/v) = \exp\{-(-v)^{\alpha} s(-\log(-v))\}, \quad v < 0. \tag{7}$$

It is again *MSS* and negative in that it now satisfies the functional equation (of the form (4)): $F_{V^-}(v) = F_{V^-}(vc)^{\rho}$. To get all the *MSS* type II class, we need too shift V^-. The shifted variable $X^- := V^- + x^-$, $x^- \in \mathbb{R}$, now satisfies the functional equation of the form (3) and

$$F_{X^-}(x) = \exp\{-(x^- - x)^{\alpha} s(-\log(x^- - x))\}, \quad x < x^-. \tag{8}$$

This model is the *MSS* distribution of type II, extending the one of Weibull laws.

The Max-Semistable Model (Type III): "Extended" Gumbel. Let us introduce the variable $X = \log V^+$, i.e., the logarithm of the *MSS* variables V^+ defined in (5). From (5) the *df* for the variable X is found to be

$$F_X(x) = \exp\{-s(x)e^{-\alpha x}\}, \quad x \in \mathbb{R}. \tag{9}$$

We identify this *df* as the one of an extended Gumbel distribution. With $\rho > 1$ and $\beta := -\log c > 0$, we note that X is a real-valued random variable, whose *df* is the solution to the functional equation of the form (3). In this case: $\alpha = \frac{(\log \rho)}{(\beta)}$.

Note that every *MSS* law, as solution of (3), is of one of these three types.[5] Let us now supply an additional remark, underlining the importance of *MSS* models in Statistics.[14]

2.1. Max-Semistable Models as Limit Laws in Statistics

The Fréchet–Weibull–Gumbel models are known to constitute the limit distributions for centered and normalized *iid* maxima of n-samples: in this context, they are to maxima what Lévy-stable laws are to sums. Their maximum domains of attraction (MDA) and the centering ($x_n \in \mathbb{R}$) and normalizing ($\sigma_n > 0$) constants are available for example in Ref.[3,23]. Let X be any of these variables. The max-stability property reads: for any $n \geq 1$, with $X_m \stackrel{d}{=} X, m = 1, \ldots, n, iid$ random variables, the following identity in distribution holds

$$X \stackrel{d}{=} \max_{m=1,\ldots,n} \frac{X_m - x_n}{\sigma_n}.$$

Let now X be any MSS variable with df either given by (6–9). These variables also appear as all possible non-degenerate limit laws of the distributions $\max_{m=1,\ldots,\rho_n}(\mathscr{X}_m - x_n)/\sigma_n$, where $\mathscr{X}_m \stackrel{d}{=} \mathscr{X}, m \geq 1$ is an *iid* sequence, $x_n \in \mathbb{R}$, $\sigma_n > 0$ and $\rho_n > 0$ (see also Refs.[5,7,16]). The integer-valued sequences ρ_n is assumed to satisfy the geometrical growth properties: $\lim_{n\uparrow+\infty} \rho_n = +\infty$ and $\lim_{n\uparrow\infty} \rho_{n+1}/\rho_n = \rho \geq 1$. The variable $\mathscr{X}$ is said belong to the MDA of X. Note that X itself belongs to its own MDA.

Max-stable variables are MSS and can be obtained when $\rho = 1$ (see Ref.[16], Theorem 7).

3. THE MAX-MULTISCALING MODELS

The following main theorem yields the max-multiscaling distributions of the different types. We consider the general problem of characterizing distribution functions satisfying the functional equation

$$F(x) = \prod_{i=1}^{m} F(x/c_i + \beta_i)^{\gamma_i} \tag{10}$$

for some $m \geq 1, c_i > 0, \gamma_i > 0, \beta_i \in \mathbb{R}, i = 1, \ldots, m$. We have

Theorem 1. *1/ If $c_i \neq 1, \beta_i = \tilde{x}(1 - 1/c_i)$ for some $\tilde{x} \in \mathbb{R}, i = 1, \ldots, m$, then necessarily the solution's support is $[\tilde{x}, \infty)$ or $(-\infty, \tilde{x}]$:*

(i) *The solution's support is $[\tilde{x}, \infty)$: with $\mathscr{S}_1^+ = \{\alpha > 0 : \sum_{i=1}^m \gamma_i c_i^\alpha = 1\}$, we have $|\mathscr{S}_1^+| \in \{0, 1, 2\}$. We get non-degenerate solutions of type* I *if and only if $|\mathscr{S}_1^+| \neq 0$, with*

$$F(x) = \exp\left\{-\sum_{\alpha_k \in \mathscr{S}_1^+} (x - \tilde{x})^{-\alpha_k} e^{\alpha\nu_k(\log(x-\tilde{x}))}\right\}, \quad x \geq \tilde{x}. \tag{11}$$

(ii) *The solution's support is $(-\infty, \tilde{x}]$: with $\mathscr{S}_2^+ = \{\alpha > 0 : \sum_{i=1}^m \gamma_i c_i^{-\alpha} = 1\}$, we have $|\mathscr{S}_2^+| \in \{0, 1, 2\}$. We get non-degenerate solutions of type* II *if and* *only if* $|\mathscr{S}_2^+| \neq 0$, *with*

$$F(x) = \exp\left\{-\sum_{\alpha_k \in \mathscr{S}_2^+} (\tilde{x} - x)^{\alpha_k} e^{\alpha_k \nu_k (\log 1/(\tilde{x}-x))}\right\}, \quad x \leq \tilde{x}. \tag{12}$$

In Eqs. (11) *and* (12), *the functions* ν_k *are right-continuous bounded periodic, with periods* $-\log c_i, i = 1, \ldots, m$, *such that* $x - \nu_k(x)$ *are non-decreasing for each* $k \in \{1, 2\}$.

2/ *If* $c_i = 1$, $\beta_i \neq 0$, $i = 1, \ldots, m$, *necessarily the solution's support is* $\mathbb{R}$.

With $\mathscr{S}_3^+ = \{\alpha > 0 : \sum_{i=1}^m \gamma_i e^{-\beta_i \alpha} = 1\}$, *we have* $|\mathscr{S}_3^+| \in \{0, 1, 2\}$. *We get non-degenerate solutions of type* III *if and only if* $|\mathscr{S}_3^+| \neq 0$, *with*

$$F(x) = \exp\left\{-\sum_{\alpha_k \in \mathscr{S}_3^+} e^{-\alpha_k (x - \nu_k(x))}\right\}, \quad x \in \mathbb{R}. \tag{13}$$

In (13), *the functions* ν_k *are right-continuous bounded periodic, with periods* $\beta_i, i = 1, \ldots, m$, *such that* $x - \nu_k(x)$ *are non-decreasing for each* $k \in \{1, 2\}$.

Remark 1. (i) In type I and II distributions found in Theorem 1, part 1/, we do not loose generality by imposing the condition $c_i \neq 1, i = 1, \ldots, m$. Indeed, let $I := \{i \in \{1, \ldots, m\} : c_i = 1\}$ and suppose $1 \leq |I| < m$ in (1). From this equation, it holds that $F(x) \leq F(x)^\gamma$ for all x where $\gamma := \sum_{i \in I} \gamma_i$. This forces $\gamma < 1$. Equation (1) may be written as $F(x) = \prod_{i \in \{1,\ldots,m\} \setminus I} F(x/c_i)^{\gamma_i/(1-\gamma)}$ where none of the remaining c_i are equal to 1. Similarly, concerning type III, the condition $\beta_i \neq 0$ could be released.

(ii) The distributions found in Theorem 1 are invariant under scaling only or under translations only or under scaling and translations when the shift β_i takes the particular form $\beta_i = \tilde{x}(1 - 1/c_i)$ for some $\tilde{x} \in \mathbb{R}$. It is not claimed that the full solutions to (10) have been characterized. Considering solutions invariant under simultaneous change of scale and location in general, that is for general c_i and β_i (not of the above form), is still an open problem to the authors' knowledge.

(iii) In case 1/, the support is either $[\tilde{x}, \infty)$ or $(-\infty, \tilde{x}]$. By simple change of variables of the type used in the presentation of *MSS* distributions, one easily obtains random variable with support $[0, \infty)$ whose df is solution to (1). In case 2/, the solution's support is $\mathbb{R}$ and reasoning in a similar way with the transformation $\exp\{x\}$, we also get a random variable with support $[0, \infty)$ whose df is solution to (1). The proof of the theorem will follow from results concerning the solutions to (1) satisfying $\inf\{v : F(v) > 0\} = 0$. This will be done in Subsecs. 3.1 and 3.2.

Exploiting the periodic properties of the functions ν_k in Theorem 1, the solution depends on the commensurability of the sequence $(-\log c_i, i = 1, \ldots, m)$.

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More precisely, we obtain

Corollary 2. *Under the hypothesis of Theorem* 1, *it holds for MM distributions of type* I (*respectively type* II *and type* III):

(i) *If* $(-\log c_i,\ i = 1, \ldots, m)$ *are commensurable with common period* $-\log c$, *then*:

  (a) $|\mathscr{S}^+| = 1$ *and the solution is a MSS distribution of type* I (*respectively* II *and* III).
  (b) $|\mathscr{S}^+| = 2$ *and the solution is the maximum of two independent MSS type* I (*respectively* II *and* III) *distributed random variables.*

(ii) *If* $(-\log c_i, i = 1, \ldots, m)$ *are non-commensurable, then:*

  (a) $|\mathscr{S}^+| = 1$ *and the solution is a Fréchet* (*respectively Weibull, Gumbel*) *distribution.*
  (b) $|\mathscr{S}^+| = 2$ *and the solution is the maximum of two independent Fréchet* (*respectively Weibull, Gumbel*) *distributed random variables.*

Proof. We note that the periodicity condition for ν_k is equivalent to

$$\nu_k(x) = \nu_k\left(x + \sum_{i=1}^{m} p_i x_i\right) \tag{14}$$

with $x_i = -\log c_i$ and for all $p_i \in \mathbb{Z},\ i = 1, \ldots, m$. Two different cases then arise

- Lattice case: $x_i = -r_i \log c,\ i = 1, \ldots, m,\ c \in (0, 1),\ r_i \in \mathbb{Z}$ and $\gcd(r_i) = 1$. The functions ν_k are identified with periodic functions with period $-\log c$.

If $|\mathscr{S}_1| = 1$, we recognize the distribution function of a *MSS* law. In the case $|\mathscr{S}_1| = 2$, F factorizes into the distributions of two independent random variables (X_1, X_2) where each X_k is a *MSS* distributed random variable.

- Non-lattice: the periods x_i are non-commensurable: only the constants $\nu(x) = \nu$ are right-continuous bounded solutions of (14), since the set $\sum_{i=1}^{m} x_i\mathbb{Z}$ is dense in $\mathbb{R}$. Now, like in the lattice case, the solutions are either the max-stable $df\,(|\mathscr{S}_1| = 1)$ or the product of two such max-stable $dfs\,(|\mathscr{S}_1| = 2)$. This completes the proof.

3.1. Strictly Max-Multiscaling Laws of Type I

Consider therefore the functional Eq. (1) with $c_i \in (0, \infty)\backslash\{1\}$. Proposition below yields a formal solution to Eq. (1); by formal solution, we mean here a solution not necessarily in the class of distribution functions.

Proposition 3. *Consider a random variable V^+ whose df, say $F_{V^+}(v)$, is solution of the functional Eq.* (1). *Assume* $\inf\{v : F_{V^+}(v) > 0\} = 0$. *Let* $\mathscr{S} = \{\alpha \in \mathbb{R} : \sum_{i=1}^m \gamma_i c_i^\alpha = 1\}$. *We have* $|\mathscr{S}| \in \{0, 1, 2\}$ *and* $F_{V^+}(v)$ *reads*

$$F_{V^+}(v) = \exp\left\{-\sum_{\alpha_k \in \mathscr{S}} v^{-\alpha_k} s_k(\log v)\right\}, \tag{15}$$

with the convention that the sum over the empty set is null. Here, $s_k(.)$ *are non-negative periodic functions with periods* $-\log c_i, i = 1, \dots, m$.

Proof. Under our hypothesis, $1 > F_{V^+}(v) > 0$ for $\infty > v > 0$. Upon reasoning now with the positive function $H(x) := -\log F_{V^+}(e^x)$, this functional equation takes the simpler convolution form

$$H(x) = \sum_{i=1}^m \gamma_i H(x + x_i) \tag{16}$$

with $x_i = -\log c_i$. To solve this last equation, we shall now need to introduce the structure function (using the terminology of Refs.[9,13]):

$$\tau(q) := \sum_{i=1}^m \gamma_i c_i^q, \quad q \in \mathbb{R}. \tag{17}$$

Under our hypothesis, it is positive and convex. As a result, the equation: $\tau(\alpha) = 1$ admits none, one or two solutions in $\mathbb{R}$. Now, we are in the position to apply the Lau–Rao–Shanbhag theorem (see p. 38 Corollary 2.3.2 of Theorem 2.3.1, p. 36 of Ref.[20]) which states that the multiscaling function H takes the form

$$H(x) = \sum_{\alpha_k \in \mathscr{S}} e^{-\alpha_k x} s_k(x). \tag{18}$$

Here, the functions s_k are positive and periodic, that is satisfying $s_k(x) = s_k(x + x_i)$, for all $i = 1, \dots, m$. In terms of the df itself, we thus get the announced formal solution of (1).

As F_{V^+} must be the df of some random variable V^+, additional conditions have to be imposed. First, for $\alpha > 0$, we shall denote by

$$\mathscr{E}_\alpha = \{s : \mathbb{R} \to (0, \infty) \text{ such that } s(x) = e^{\alpha\nu(x)}\},$$

for some right-continuous bounded periodic function ν with periods $x_i = -\log c_i$ for $i = 1, \dots, m$, such that $x - \nu(x)$ is non-decreasing function.

Proposition 4. *Let V^+ be a non-degenerate random variable whose df,* F_{V^+}, *is given by Eq.* (15) *satisfying* $\inf\{v : F_{V^+}(v) > 0\} = 0$. *Let* $\mathscr{S}^+ := \{\alpha > 0 : \sum_{i=1}^m \gamma_i c_i^\alpha = 1\}$. *Then* $|\mathscr{S}^+| \in \{1, 2\}$ *and necessarily* $\alpha_k \in \mathscr{S}^+$ *and* $s_k \in \mathscr{E}_{\alpha k}$. *In other words, we have*

$$F_{V^+}(v) = \exp\left\{-\sum_{\alpha_k \in \mathscr{S}^+} v^{-\alpha_k} s_k(\log v)\right\}.$$

Proof. Consider the formal solution (15).

(i) If $\mathscr{S} = \emptyset$, we get the degenerate solution $V^+ = 0$.
(ii) If $|\mathscr{S}| = 1$, letting $\mathscr{S} = \{\alpha\}$, we get

$$F_{V^+}(v) = \exp\{-v^{-\alpha} s(\log v)\}, \quad v > 0. \tag{19}$$

As F_{V^+} must be the df of some random variable V^+, additional conditions have to be imposed. First the non-negative function $s(.)$ must be bounded. Indeed, $s(\log v) = -v^{\alpha} \log F_{V^+}(v)$; now, as $-v^{\alpha} \log F_{V^+}(v)$ is locally bounded and as $s(x)$ is periodic, necessarily $s(x)$ is bounded. Next, as $F_{V^+}(0) = 0$ and $F_{V^+}(\infty) = 1$, necessarily, α must be a positive number.

Finally, the hazard function $v^{-\alpha} s(\log v)$ should be non-increasing with v. In other words, if we let $s(x) := e^{\alpha\nu(x)}$, for some periodic function ν with periods $x_i = -\log c_i$, for $i = 1, \ldots, m$, then: $x - \nu(x)$ has to be a non-decreasing function. Finally, it is necessary that $\nu(x)$ be right-continuous. Thus, we need to impose $\alpha > 0$ and $s \in \mathscr{E}_{\alpha}$.

(iii) If $|\mathscr{S}| = 2$, letting $\mathscr{S} = \{\alpha_1, \alpha_2\}$ with, say $\alpha_1 < \alpha_2$, then we have

$$F_{V^+}(v) = \exp\{-[v^{-\alpha_1} s_1(\log v) + v^{-\alpha_2} s_2(\log v)]\}, \quad v > 0 \tag{20}$$

where $s_k(x) := e^{\alpha_k \nu_k(x)}, k = 1, 2$ are positive and periodic with the same periods $x_i = -\log c_i, i = 1, \ldots, m$. As F_{V^+} must be the df of some random variable V^+, additional conditions have to be imposed. First each non-negative function s_k must be bounded; indeed,

$$s_k(\log v) \leq -v^{\alpha_k} \log F_{V^+}(v).$$

As $-v^{\alpha_k} \log F_{V^+}(v)$ is locally bounded and as s_k is periodic, necessarily $\|s_k\|_{\infty} < \infty$. Using the same arguments, as for the case $|\mathscr{S}| = 1$, α_1 and α_2 must be positive numbers.

Finally, $e^{-\alpha_1 x} s_1(x) + e^{-\alpha_2 x} s_2(x)$ should be non-increasing in such a way that the hazard function $-\log F_{V^+}(v)$ be non-increasing with v. In other words, if we let $s_k(x) := e^{\alpha_k \nu_k(x)}$, for some periodic functions ν_k with periods $x_i = -\log c_i$, for $i = 1, \ldots, m$, then

$$H(x) := e^{-\alpha_1(x - \nu_1(x))} + e^{-\alpha_2(x - \nu_2(x))}$$

should be a non-increasing function. Let $H_k(x) := e^{-\alpha_k(x - \nu_k(x))}, k = 1, 2$, we shall show that both $H_1(x)$ and $H_2(x)$ should in fact be non-increasing.

Indeed, let $z_2 > z_1$. It is thus necessary that $H(z_2) \leq H(z_1)$. Now, as $0 < \alpha_1 < \alpha_2$ and from the boundedness of s_k: $H(z_2) = H_1(z_2)(1 + \epsilon(z_2))$, with $\epsilon(x) = H_2(x)/H_1(x) \to_{x\uparrow+\infty} 0$. Finally, for any $i \in \{1, \ldots, m\}$ and for any $n \in \mathbb{Z}$, the condition $H(z_2) \leq H(z_1)$ also reads

$$\frac{H_1(z_2 + nx_i)}{H_1(z_1 + nx_i)} \leq \frac{1 + \epsilon(z_1 + nx_i)}{1 + \epsilon(z_2 + nx_i)}.$$

But, from the expression of H_1 and the periodicity of $\nu_1(x)$, we get

$$\frac{H_1(z_2+nx_i)}{H_1(z_1+nx_i)}=\frac{H_1(z_2)}{H_1(z_1)}.$$

Hence

$$\frac{H_1(z_2)}{H_1(z_1)}\le\frac{1+\epsilon(z_1+nx_i)}{1+\epsilon(z_2+nx_i)}\underset{nx_i\uparrow\infty}{\longrightarrow}1.$$

As a result: $H_1(z_2)\le H_1(z_1)$. In a similar way, $nx_i\downarrow -\infty$, one can establish that $H_2(z_2)\le H_2(z_1)$. Finally, it is necessary that $\nu_k(x)$ be right-continuous and $\alpha_k>0$ and $s_k\in\mathscr{E}_{\alpha_k}$, $k=1,2$.

Putting all this material together, we have

Proposition 5. *Consider a strictly type* I *max-multiscaling model of a non-degenerate random variable* V^+. *Assume* $\inf\{v: F_{V^+}(v)>0\}=0$. *Its df, as a solution of* (1), *reads*

$$F_{V^+}(v)=\exp\left\{-\sum_{\alpha_k\in\mathscr{S}^+}v^{-\alpha_k}s_k(\log v)\right\}$$

where $\mathscr{S}^+=\{\alpha>0:\sum_{i=1}^m\gamma_i c_i^\alpha=1\}$ *and for each* $\alpha_k\in\mathscr{S}^+$, $s_k\in\mathscr{E}_{\alpha_k}$.

Proof. This easily obtained by combining Proposition 3 and Proposition 4.

Let us give a simple illustrative example corresponding to the case $|\mathscr{S}^+|=2$.

Example 1. Let $m=2$. Let $c\in(0,1)$ and $c_1=c$, $c_2=c^{-1}$, in such a way that $c_1<1<c_2$. Let $\gamma_1=1$ and $\gamma_2:=\gamma>0$. Under these hypothesis, it may be checked that the equation $\tau(\alpha)=1$ admits two positive solutions if and only if $0<\gamma<1/4$ in which case these solutions are

$$0<\alpha_1=-\log_c\left(\frac{1-\sqrt{1-4\gamma}}{2\gamma}\right)<\alpha_2=-\log_c\left(\frac{1+\sqrt{1-4\gamma}}{2\gamma}\right).$$

Thus $V^+=\max(V_1^+,V_2^+)$ where (V_1^+,V_2^+) are independent and whose dfs satisfy $F_{V_1^+}(v)=F_{V_1^+}(v/c)^{\rho 1}$ and $F_{V_2^+}(v)=F_{V_2^+}(v/c)^{\rho 2}$, with $(\rho_1>1,\rho_2>1)$ defined by $\rho_1=c^{-\alpha_1}=\frac{1-\sqrt{1-4\gamma}}{2\gamma}$ and $\rho_2=c^{-\alpha_2}=\frac{1+\sqrt{1-4\gamma}}{2\gamma}$.

As it was emphasized in Proposition 5 the number of positive solutions to $\tau(\alpha)=1$ is a central point in characterizing the solutions of the functional Eq. (1).

We derive below, from an elementary study of the structure function $\tau(q)$, the following corollary:

Corollary 6. (i) *Let* $c_i \in (0,1)$, $i = 1, \ldots, m$. *If* $\sum \gamma_i \leq 1$ *then* $|\mathscr{S}^+| = 0$. *Else,* $\sum \gamma_i > 1$ *and* $|\mathscr{S}^+| = 1$. *In this last case, if* $c_i = c^{r_i}$ *for some* $c \in (0,1)$ *and* $r_i \in \mathbb{N}$, *then solution to* (1) *is a strictly MSS type* I *distribution*; *else, it is a Fréchet distribution.*

(ii) *Let* $c_i > 1$, $i = 1, \ldots, m$. *If* $\sum \gamma_i \geq 1$ *then* $|\mathscr{S}^+| = 0$. *Else,* $\sum \gamma_i < 1$ *and* $|\mathscr{S}^+| = 1$. *In this last case, if* $c_i = c^{r_i}$ *for some* $c > 1$ *and* $r_i \in \mathbb{N}$, *then solution to* (1) *is a strictly MSS type* I *distribution*; *else, it is a Fréchet distribution.*

(iii) *If* $\min_{i=1,\ldots,m}(c_i) < 1 < \max_{i=1,\ldots,m}(c_i)$. *If* $\sum \gamma_i < 1$ *then* $|\mathscr{S}^+| = 1$. *If* $\sum \gamma_i = 1$ *then* $|\mathscr{S}^+| \in \{0,1\}$ *and if* $\sum \gamma_i > 1$, *then* $|\mathscr{S}^+| \in \{0,2\}$.

Proof. (i) When $c_i \in (0,1), i = 1, \ldots, m$, the convex function $\tau(q)$ satisfies $\lim_{q\uparrow+\infty} \tau(q) = 0, \lim_{q\downarrow-\infty} \tau(q) = +\infty$ and $\tau(0) = \sum \gamma_i$. The result can easily be obtained from this.

(ii) In this case we have $\lim_{q\uparrow+\infty} \tau(q) = +\infty, \lim_{q\downarrow-\infty} \tau(q) = 0$.

(iii) In this case, $\tau(q) \to +\infty$ when $q \to \pm\infty$.

3.2. Proof of Theorem 1

We start studying the support of the solution to (1).

The Solution's Support. Let $F(v)$ be the df of V and suppose that $F(v)$ solves (1) with $c_i \neq 1$, $i = 1, \ldots, m$. Let $a := \inf\{v : F(v) > 0\} \geq -\infty$. If a is finite, necessarily $a = 0$ because, from (1), we have $a = \max_{i=1,\ldots,m}(c_i a)$. If $a = -\infty$, then $F(v) > 0$ on the interval $[0, \infty)$ and reasoning on the restriction of F to $[0, \infty)$, going along the same lines as in the proof of Proposition 3, the restriction of F to $[0, \infty)$ leads to $F(0) = 0$, which is absurd, or to $F(v) = 1$ for all $v > 0$. In the latter case, reasoning similarly, if $b := \sup\{v : F(v) < 1\}$, then $b = \max_{i=1,\ldots,m}(c_i b)$ and $b = 0$.

Broad Sense Max-Multiscaling Laws of Type I. Consider a random variable V^+ with support $[0, \infty)$ whose df satisfies (1) with all $c_i \neq 1$. Let $X^+ := V^+ + x^+, x^+ \in \mathbb{R}$, be the shifted variable. Equivalently, its df satisfies the functional Eq. (10), whose solution is given by (11) with $\tilde{x} = x^+$ and $\beta_i = x^+(1 - 1/c_i)$. This is part (i) of case 1/ in Theorem 1.

Multiscaling Distributions of Type II. Consider a random variable V^- with support $(-\infty, 0]$ whose df satisfies (1) with all $c_i \neq 1$. Let $V^+ := -1/V^-$ with support $[0, \infty)$. Its df satisfies the functional equation $F_{V^+}(v) = \prod_{i=1}^m F_{V^+}(vc_i)^{\gamma_i}$ of the type (1). Let $X^- := V^- + x^-$, $x^- \in \mathbb{R}$, be the shifted variable. Equivalently, its df satisfies functional Eq. (10) whose solution is given by (12), with $\tilde{x} = x^-$ and $\beta_i = x^-(1 - 1/c_i)$. This completes the proof of case 1/ part (ii) in Theorem 1.

Multiscaling Laws of Type III. Consider a random variable X with support $\mathbb{R}$ whose *df* satisfies (10) with all $c_i = 1$ and $\beta_i \neq 0$. Let $V^+ = \exp\{X\}$, with support $[0, \infty)$. Equivalently, its *df* satisfies the functional equation $F_{V^+}(v) = \prod_{i=1}^{m} F_{V^+}(v/c_i)^{\gamma_i}$ of the type (1) with $c_i = \exp\{-\beta_i\}$. This completes the proof of case 2/ in Theorem 1.

3.3. Max-Multiscaling and (Semi-) Selfsimilarity

Let X be some random variable with *df* F. Consider the external process $\{X(t), t > 0\}$ associated to F (see Refs.[3,24]). The process $\{X(t), t > 0\}$ is selfsimilar,[25] with scaling exponent H and translational exponent $\tilde{x}$ if, for any $\varsigma > 0$

$$\{X(\varsigma t),\, t > 0\} \stackrel{d}{=} \{\tilde{x} + \varsigma^H (X(t) - \tilde{x}),\, t > 0\}. \tag{21}$$

One may easily check that the extremal processes associated to Fréchet–Weibull random variables are selfsimilar; Eq. (21) is fulfilled, with $(H = 1/\alpha, \tilde{x} = x^+)$ and $(H = -1/\alpha, \tilde{x} = x^-)$, respectively.[18] We say that $\{X(t), t > 0\}$ is translational selfsimilar[25] with exponent H, if, for any ς

$$\{X(\varsigma t),\, t > 0\} \stackrel{d}{=} \{X(t) + H \log \varsigma,\, t > 0\}. \tag{22}$$

One may check that the extremal Gumbel process is translational selfsimilar with exponent $H = 1/\alpha$.

We now consider the extremal process associated to max-semistable distributions of types I and II. For some $\rho > 1$ and $c \in (0, 1)$ related through $\rho c^\alpha = 1, \alpha > 0$, we get[19]

$$\{X(\rho^n t),\, t > 0\} \stackrel{d}{=} \{\tilde{x} + \rho^{nH} (X(t) - \tilde{x}),\, t > 0\} \tag{23}$$

for any $n \in \mathbb{Z}$, respectively, with $(H = 1/\alpha, \tilde{x} = x^+)$ and $(H = -1/\alpha, \tilde{x} = x^-)$ for types I and II. This is the definition of semi-selfsimilarity from Ref.[25]. Equation (21) is fulfilled only for those ς of the particular form $\varsigma = \rho^n$, $n \in \mathbb{Z}$. Translational semi-selfsimilarity is defined similarly,[25] consider the extremal process associated to max-semistable distributions of type III. It satisfies Eq. (22) with $H = 1/\alpha$, only at points $\varsigma = \rho^n, n \in \mathbb{Z}$, for some $\rho > 1$.

If $|\mathscr{S}^+| = 1$ and in the lattice (non-lattice) case, the extremal process associated to *MM* distributions are semi-selfsimilar (selfsimilar). Their 1-dimensional *df*s are the ones of *MSS*(*MS*) random variables. If $|\mathscr{S}^+| = 2$, although *MM* distributions take the form of the product of two max-semistable (max-stable) *df*s they are no longer max-semistable (max-stable) by themselves. Their associated extremal processes are not semi-selfsimilar (selfsimilar). Rather, it is the maximum of two semi-selfsimilar (selfsimilar) extremal processes.

4. MULTISCALING MODELS AS LIMIT LAWS UNDER GEOMETRIC GROWTH

In this section, we show that max-multiscaling distributions may be seen as limit laws for some renormalized maximum of independent but not necessarily identically distributed random variables. Some work in this direction may be found in Refs.[7,10,14–16,21,22].

It should be emphasized that we have no pretention here to fully characterize the domain of attraction of max-multiscaling laws. In the case of *MSS* distributions, final characterization results of their max-domain of attraction can be found in Theorems 1–3 in Ref.[16].

4.1. The Max-Semistable Case: Preliminaries

Consider a particular lattice *MM* model with $\gamma_i = 1, c_i = c^{r_i}, c \in (0,1), r_i \in \mathbb{N}$, $i = 1, \dots, m$, $\gcd(r_i) = 1$. This guarantees that we are in a max-semistable case with a unique $\alpha > 0$ defined by $\tau(\alpha) = 1$. Define now the integer-valued function $n(r)$ of $r \in \mathbb{N}$, recursively by

$$n(r) = 1(r^* > r) + \sum_{i=1}^{m} n(r - r_i)1(r_i \leq r), \quad r \in \mathbb{N}\backslash\{0\}, \;\; n(0) = 1, \tag{24}$$

with $r^* := \max_{i=1,\dots,m} r_i$. This sequence is a deterministic multitype branching process for which the number of individuals at discrete time r is obtained as follows: at time $r = 0$, a single ancestor is available; this ancestor gives birth to m first generation sons as a whole, a type $-i$ son coming to life at time $r_i > 0$. The ancestor dies at time r^* when it gives birth to its last son. Each first generation son repeats the same splitting program, starting from its birth time, and so forth for the subsequent generations. This construction simply is a deterministic version of the age-dependent Crump-Mode branching process. Using the renewal structure of $n(r)$ and singularity analysis of the generating function of $\{n(r), r \geq 0\}$, we could check that $\lim_{r\uparrow\infty} n(r)^{1/r} = \rho$, with $\rho > 1$ defined by $\sum_{i=1}^{m} \rho^{-r_i} = 1$, and even that $n(r) \sim_{r\uparrow\infty} K\rho^r$, for some $K > 0$. Recalling that the condition $\tau(\alpha) = 1$ reads

$$\sum_{i=1}^{m} c_i^{\alpha} = \sum_{i=1}^{m} c^{\alpha r_i} = 1,$$

we conclude that $\rho c^{\alpha} = 1$.

Let $(\chi_j, j \geq 1)$ be a sequence of *iid* random variables, distributed like a random variable χ, with *df* F_χ. Let $Z(r) := \max_{j=1,\dots,n(r)} \chi_j$. In the following proposition, we will say that χ is in the *MDA* of such type I (respectively II and III) *MM* models (all in the *MSS* class) if there exists $\sigma(r) > 0$ and $x(r)$ such that $\widetilde{Z}(r) := [Z(r) - x(r)]/\sigma(r)$, $r \in \mathbb{N}$, converges in law to any of the type I (respectively II and III) models.

Let $b := \sup\{v : F_\chi(v) < 1\}$. We have

Proposition 7. *Sufficient conditions for χ to be in the MDA of the three MSS types are*

Type I: *$b = \infty$ and for some scale function $s \in \mathscr{E}_\alpha$, some slowly varying function L at ∞*

$$\overline{F}_\chi(v) \sim \frac{1}{K} \cdot (v/L(v))^{-\alpha} s(\log(v/L(v))) \quad \textit{for large } v \tag{25}$$

Type II: *$b < \infty$ and the df of $-1/(\chi - b)$ satisfies (25).*

Type III: *$b \leq \infty$ and χ has no atom at right-endpoint b; there exists $x_0 < b$ such that for some Von Mises function $\exp\left\{-\int_{x_0}^{x} \frac{\alpha}{a(u)} du\right\}$ with positive and absolutely continuous auxiliary function $a(u)$ satisfying $a'(u) \to_{u\uparrow b} 0$, for some scale function $s \in \mathscr{E}_\alpha$ and some function $c(x) \to_{x\uparrow b} 1$*

$$\overline{F}_\chi(x) = \frac{c(x)}{K} \cdot s\left(\int_{x_0}^{x} \frac{1}{a(u)} du\right) \cdot \exp\left\{-\int_{x_0}^{x} \frac{\alpha}{a(u)} du\right\}, \quad x_0 < x < b. \tag{26}$$

Proof. Type I: First, define $\sigma(r)$ by $[\sigma(r)/L(\sigma(r))]^{-\alpha} = 1/\rho^r$ (assuming without loss of generality that $L \in C^\infty$, see Ref.[3], Proposition $A3.5$, p. 566) and let $x(r) = 0$. As $\rho c^\alpha = 1$, we note that $\sigma(r)/L(\sigma(r)) = c^{-r}$. Second, with $\varepsilon(r) \to_{r\uparrow\infty} 0$, we have

$$\mathbb{P}(\widetilde{Z}(r) \leq v) = (1 - \overline{F}_\chi(v\sigma(r)))^{n(r)} = \exp\{-n(r)\overline{F}_\chi(v\sigma(r))(1 + \varepsilon(r))\}.$$

From (25) and the definition of $\sigma(r)$, we get: $n(r)\overline{F}_\chi(v\sigma(r)) \sim_{r\uparrow\infty} v^{-\alpha}s(\log(v\sigma(r)/L(v\sigma(r))))$. As $r \in \mathbb{N}$ and as $s(\log v)$ has period $-\log c$, it follows from the definition of $\sigma(r)$ and the slowly varying character of L, that, at each point of continuity of function s, it holds

$$\mathbb{P}(\widetilde{Z}(r) \leq v) \underset{r\uparrow\infty}{\to} \exp\{-v^{-\alpha}s(\log v)\}$$

which is the df of some strictly *MSS* random variable of type I.

Type II: Exploiting the connection between distributions of *MSS* models of type I and II this result can easily be obtained.

Type III: First, let $x(r) \in (x_0, b)$ and $\sigma(r)$ be defined by

$$\exp\left\{-\int_{x_0}^{x(r)} \frac{\alpha}{a(u)} du\right\} = 1/\rho^r \quad \text{and} \quad \sigma(r) = a(x(r)). \tag{27}$$

As $c(x) \to_{x\uparrow b} 1$ and $s(\cdot)$ is positive and bounded, it holds that $x(r) \to_{r\uparrow\infty} b$. When $b < \infty$, we have $\sigma(r) = o(b - x(r)) \to_{r\uparrow\infty} 0$ (see Ref.[3], Proposition 3.3.24 and the remark that follows on p. 141). It holds $x(r) + x\sigma(r) \to b$ for all x. From (26) we have

$$\begin{aligned}\mathbb{P}(\widetilde{Z}(r) \le x) &= (1 - \overline{F}_{\chi}(x(r) + x\sigma(r)))^{n(r)} \\ &= \left(1 - \frac{c(x(r) + x\sigma(r))}{K\rho^r} s\left(\int_{x_0}^{x(r)+x\sigma(r)} \frac{1}{a(u)} du\right)\right. \\ &\quad \left. \times \exp\left\{-\int_{x(r)}^{x(r)+x\sigma(r)} \frac{\alpha}{a(u)} du\right\}\right)^{n(r)}.\end{aligned}$$

Now, from the choice of $x(r)$ and $\sigma(r)$ in (27), and following Ref.[16], Lemma 2,

$$\int_{x_0}^{x(r)+x\sigma(r)} \frac{1}{a(u)} du = \int_{x_0}^{x(r)} \frac{1}{a(u)} du + \int_{x(r)}^{x(r)+x\sigma(r)} \frac{1}{a(u)} du = -r \log c + x + \varepsilon(r)$$

where $\epsilon(r) \to_{r\uparrow\infty} 0$. As a result, under our assumptions

$$\mathbb{P}(\widetilde{Z}(r) \le x) = \left(1 - \frac{c(x(r) + x\sigma(r))}{K\rho^r} s(x - r\log c + \varepsilon(r)) \exp\{-\alpha(x + \varepsilon(r))\}\right)^{n(r)}$$

Finally, since $s(x)$ has period $-\log c$, with $c \in (0, 1)$ and as $r \in \mathbb{N}$, at each point of continuity of s, we get

$$\mathbb{P}\big(\widetilde{Z}(r) \le x\big) \underset{r\uparrow\infty}{\to} \exp\{-s(x)e^{-\alpha x}\}$$

which is the df of some *MSS* random variable of type III, as required (with $\beta_i = -\log c_i = -r_i \log c$).

Remark 2. This construction is a simple illustrative example of results obtained by Ref.[16]. It provides an important building block of the maximum domain of attraction of *MSS* laws. However, as $n(r)$ is given here in advance, with a natural connection with parameters ($\gamma_i = 1, c_i, i = 1, \ldots, m$) of this particular *MM* model, the full domain of attraction, as described by Ref.[16], cannot be obtained in this way (see the comment on p. 978 of Ref.[16]).

4.2. The Full Max-Multiscaling Case

In the *MM* model satisfying (1), let us now simply assume that $c_i, \gamma_i > 0$, $i = 1, \ldots, m$ are such that $\mathcal{S}^+ = \{\alpha > 0 : \tau(\alpha) = 1\}$ is not empty. We focus on the lattice case when there exists $c \in (0, 1)$, $r_i \in \mathbb{Z}$ such that $c_i = c^{r_i}$, $i = 1, \ldots, m$, with $\gcd(|r_i|) = 1$.

Let $N(r)$ be a Poisson process with intensity $\lambda(r)$ satisfying the functional equation

$$\lambda(r) = \sum_{i=1}^{m} \gamma_i \lambda(r - r_i), \quad \forall r \in \mathbb{Z}. \tag{28}$$

Solutions of the type $\lambda(r) = K\rho^r$ with $k > 0$ and $\rho > 0$, exist if there exists $\rho > 0$ satisfying $\sum_{i=1}^{m} \gamma_i \rho^{-r_i} = 1$. Recalling that condition $\tau(\alpha) = 1$ reads

$$\sum_{i=1}^{m} \gamma_i c_i^{\alpha} = \sum_{i=1}^{m} \gamma_i c^{\alpha r_i} = 1, \quad \text{for each } \alpha \in \mathscr{S}^+,$$

we can take $\rho = \rho_\alpha$ where, for each $\alpha \in \mathscr{S}^+$, $\rho_\alpha c^\alpha = 1$ and $\rho_\alpha > 1$.

For each $\alpha \in \mathscr{S}^+$, we will denote by $N_\alpha(r)$ the Poisson process with intensity $\lambda_\alpha(r) = K_\alpha \rho_\alpha^r$, $K_\alpha > 0$. Let $\lambda(r) := \sum_{\alpha \in \mathscr{S}^+} \lambda_\alpha(r)$. If $|\mathscr{S}^+| = 2$, we assume these two Poisson processes to be independent so that $N(r) = \sum_{\alpha \in \mathscr{S}^+} N_\alpha(r)$ is a Poisson process with intensity $\lambda(r)$. For each $\alpha \in \mathscr{S}^+$, let $\left(\mathscr{X}_\alpha \stackrel{d}{=} \mathscr{X}_{\alpha,j}, j \geq 1\right)$ be an *iid* sequence, independent of $N_\alpha(r)$ and let $F_\alpha(v)$ be the *df* of $\mathscr{X}_\alpha$. If $|\mathscr{S}^+| = 2$, we assume that $\left(\mathscr{X}_\alpha \stackrel{d}{=} \mathscr{X}_{\alpha,j}, j \geq 1, N_\alpha(r)\right), \alpha \in \mathscr{S}^+$, are mutually independent. Consider now the max processes

$$Z_\alpha(r) := \max_{j=1,\ldots,N_\alpha(\tau)} \mathscr{X}_{\alpha,j} \quad \text{and} \quad Z(r) := \max_{\alpha \in S^+} Z_\alpha(r). \tag{29}$$

$Z(r)$ can be interpreted as the maximum of independent random variables having one of two specified distributions. We observe that $Z(r) \stackrel{d}{=} \max_{j=1,\ldots,N(r)} \mathscr{X}_j(r)$, where $\left(\mathscr{X}(r) \stackrel{d}{=} \mathscr{X}_j(r), j \geq 1\right)$ is an *iid* sequence for each r. Here, $\mathscr{X}(r)$ is defined as the Bernoulli mixture: $\mathscr{X}(r) \stackrel{d}{=} B_r \mathscr{X}_{\alpha_1} + (1 - B_r)\mathscr{X}_{\alpha_2}$, where $\mathscr{S}^+ = \{\alpha_1, \alpha_2\}$ and $B_r \in \{0, 1\}$ is a Bernoulli random variable with $\mathbb{P}(B_r = 1) = \lambda_{\alpha_1}(r) / \sum_{\alpha \in \mathscr{S}^+} \lambda_\alpha(r)$, independent of $(\mathscr{X}_{\alpha_1}, \mathscr{X}_{\alpha_2})$. Note that the sample $(\mathscr{X}_j(r); j = 1, \ldots, N(r); r \geq 1)$ constitutes a triangular array. In the following theorem, we will show that there exist $\sigma(r) > 0$ and $x(r)$ such that

$$\widetilde{Z}(r) := [Z(r) - x(r)]/\sigma(r), \quad r \in \mathbb{N}$$

converges in law (as $r \uparrow \infty$) to any of the type I, II or III strictly *MM* models. We have

Theorem 8. *Let $b_\alpha := \sup\{v : F_\alpha(v) < 1\}$. Sufficient conditions on $\mathscr{X}_\alpha$s laws which guarantee that $\widetilde{Z}(r)$ converges in distribution to an MM law of the three types are*

Type I: $b_\alpha = \infty$ *and for some scale function $s_\alpha \in \varepsilon_\alpha$ and some slowly varying function L at ∞*

$$\overline{F}_\alpha(v) \sim \frac{1}{K_\alpha} \cdot [v/L(v)]^{-\alpha} s_\alpha(\log(v/L(v))) \text{ for large } v \text{ and for each } \alpha \in \mathscr{S}^+. \tag{30}$$

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Type II: *For each* $\alpha, b_\alpha = b < \infty$ *and the df of* $-1/(\mathscr{X}_\alpha - b)$ *satisfies* (30).

Type III: *For each* $\alpha, b_\alpha = b \leq \infty$ *and* $\mathscr{X}_\alpha$ *has no atom at* b; *there exists* $x_0 < b$ *such that for some Von Mises function* $\exp\{-\int_{x_o}^{x} \frac{\alpha}{a(u)} du\}$ *with positive absolutely continuous auxiliary function* $a(u)$ *satisfying* $a'(u) \rightarrow_{u\uparrow b} 0$, *for some scale function* $s_\alpha \in \mathscr{E}_\alpha$ *and some function* $c_\alpha(x) \rightarrow_{x\uparrow b} 1$

$$\overline{F}_\alpha(x) = \frac{c_\alpha(x)}{K_\alpha} \cdot s_\alpha\left(\int_{x_0}^{x} \frac{1}{a(u)} du\right) \cdot \exp\left\{-\int_{x_0}^{x} \frac{\alpha}{a(u)} du\right\}, \quad x_0 < x < b. \tag{31}$$

Proof. Type I: First, let $\sigma(r)/L(\sigma(r)) = c^{-r}$ and $x(r) = 0$. From the definition of $\widetilde{Z}(r)$ and the independence assumptions, we get

$$\mathbb{P}(\widetilde{Z}(r) \leq v) = \exp\left\{-\sum_{\alpha\in\mathscr{S}^+} \lambda_\alpha(r)\overline{F}_\alpha(\sigma(r)v)\right\}.$$

We note that, under our assumptions, $[\sigma(r)/L(\sigma(r))]^{-\alpha} = 1/\rho_\alpha^r$ for each $\alpha \in \mathscr{S}^+$. Now,

$$\lambda_\alpha(r)\overline{F}_\alpha(\sigma(r)v) \sim v^{-\alpha}s_\alpha(\log(v\sigma(r)/L(v\sigma(r)))), \quad \text{for each } \alpha \in \mathscr{S}^+.$$

As $r \in \mathbb{N}$ and as $s_\alpha(\log v)$ has period $-\log c$, then $s_\alpha(\log(v\sigma(r)/L(v\sigma(r)))) \rightarrow_{r\uparrow\infty} s_\alpha(\log v)$ for each $\alpha \in \mathscr{S}^+$ and at each point of continuity of s_α. It follows that, at each such point

$$\mathbb{P}(\widetilde{Z}(r) \leq v) \underset{r\uparrow\infty}{\rightarrow} \exp\left\{-\sum_{a\in\mathscr{S}^+} v^{-\alpha}s_\alpha(\log v)\right\}$$

which is the *df* of some *MM* random variable of type I, in its full generality ($|\mathscr{S}^+| \in \{1, 2\}$).

Type II: Exploiting the connection between distributions of *MM* models of type I and II this result can easily be obtained.

Type III: Let $x(r)$ and $\sigma(r)$ be defined by

$$\exp\left\{-\int_{x_0}^{x(r)} \frac{1}{a(u)} du\right\} = c^r \quad \text{and} \quad \sigma(r) = a(x(r)).$$

Note that, for each $\alpha \in \mathscr{S}^+$, it holds: $\exp\left\{-\int_{x_0}^{x(\tau)} \frac{\alpha}{\alpha(u)} du\right\} = 1/\rho_\alpha^\tau$. As it was done for type I, adapting the proof of Propostion 7 concerning type III to the Poisson structure of this model, our claim follows.

Remark 3. Since $N_\alpha(r)/K_\alpha\rho_\alpha^r \rightarrow 1$ in probability (*as* $r \uparrow \infty$), deterministic sample sizes of the form $n_\alpha(r) \sim K_\alpha\rho_\alpha^r$ would work as well. The advantage of considering Poisson sample sizes is the natural connection with the original parameters $(\gamma_i, c_i; i = 1, \ldots, m)$ suplied in (28) which parallels (24).

5. DISCUSSION OF MAIN RESULTS AND PERSPECTIVES

Let us summarize the main results. The telling features of the class of *MSS* distributions (as a first-step extension of *MS* ones) have been briefly introduced. We then focus on the wider class of *MM* distributions. These can be defined as the fixed point of some functional equations for their dfs. We identify the distribution functions being solutions of these functional equations. *MM* laws are then shown to be limiting laws for maxima of specific independent random variables when the sample size grows geometrically, after some appropriate affine normalization. Further extensions of *MM* distributions are presently under study, namely the ones obtained when $m = \infty$ and when randomizing the integer m, and/or the location-scale and intensity parameters $(\beta_i, c_i, \gamma_i), i = 1, \dots, m$ in (10), following the works of Refs.[4,11,13], in the context for sums.

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