Noname manuscript No.
(will be inserted by the editor)

Weighted L^p-theory for Poisson, biharmonic and Stokes problems on periodic unbounded strips of $\mathbb{R}^n$

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the date of receipt and acceptance should be inserted later

Abstract This paper establishes isomorphisms for Laplace, biharmonic and Stokes operators in weighted Sobolev spaces. The $W^{m,p}_{\alpha}(\mathbb{R}^n)$-spaces are similar to standard Sobolev spaces $W^{m,p}(\mathbb{R}^n)$, but they are endowed with weights $(1+|x|^2)^{\alpha/2}$ prescribing functions' growth or decay at infinity. Although well established in $\mathbb{R}^n$ [3], these weighted results do not apply in the specific hypothesis of periodicity. This kind of problem appears when studying singularly perturbed domains (roughness, sieves, porous media, etc) : when zooming on a single perturbation pattern, one often ends with a periodic problem set on an infinite strip. We present a unified framework that enables a systematic treatment of such problems in the context of periodic strips. We provide existence and uniqueness of solutions in our weighted Sobolev spaces. This gives a refined description of solution's behavior at infinity which is of importance in the multi-scale context. The isomorphisms are valid for any relative integer m, any p in $(1,\infty)$, and any real α out of a countable set of critical values for the Stokes, the bi-harmonic and the Laplace operators.

Keywords periodic infinite strip; weighted Sobolev spaces; Hardy inequality; isomorphisms; Laplace operator; Stokes equations; Green function; boundary layers;

1 Introduction

In [11, 24, 25], the first author studied blood-flow in stented arteries using homogenization techniques. Blood flows were modeled through Laplace or Stokes equations (which is plausible since the Reynolds number in arteries is relatively small) and the stent device was introduced as a thin singular perturbation between separated

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domains. Asymptotic expansions separate scales and lead to microscopic boundary layer problems. These can be seen as a zoom on minimal periodic patches of the stent device, where the macroscopic effects are experienced as boundary condition at infinity. Thus the boundary layer problems are defined on infinite periodic strips with obstacles. In a great number of articles [21,20,1,23,27], the analysis of such problems is done in various ways, mostly in the Hilbertian context, out of a generic frame. The aim of this paper is to provide an adequate framework for a systematic analysis of these problems. To that end, we follow the ideas developed in [17,3,4] : since the operators we are investigating are linear, we first solve them in periodic strips without obstacles in order to focus only on the solutions' behavior at infinity. Then using techniques from [4], we combine the latter results and solve the associated exterior problems. This will be done in a forthcoming paper.

Thus in this paper we consider the Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } \mathbb{R}^n, \\ \operatorname{div} \boldsymbol{u} = g & \text{in } \mathbb{R}^n, \end{cases} \tag{1}$$

where the velocity field $\boldsymbol{u} : \mathbb{R}^n \mapsto \mathbb{R}^n$, the pressure $\pi : \mathbb{R}^n \mapsto \mathbb{R}$ and the data $\boldsymbol{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}$ are $\boldsymbol{L}$-periodic with respect to the first $n-1$ directions, that is, for any $\boldsymbol{y} = (\boldsymbol{y}', y_n) \in \mathbb{R}^n$,

$$\boldsymbol{u}(\boldsymbol{y}' + \boldsymbol{L}, y_n) = \boldsymbol{u}(\boldsymbol{y}), \quad \pi(\boldsymbol{y}' + \boldsymbol{L}, y_n) = \pi(\boldsymbol{y}), \tag{2}$$

and

$$\boldsymbol{f}(\boldsymbol{y}' + \boldsymbol{L}, y_n) = \boldsymbol{f}(\boldsymbol{y}), \quad g(\boldsymbol{y}' + \boldsymbol{L}, y_n) = g(\boldsymbol{y}), \tag{3}$$

where $\boldsymbol{L} = (L_j)_{j\in\{1,\dots,n-1\}}$ is a vector of positive real numbers.

We shall also consider the biharmonic problem:

$$-\Delta^2 u = f \quad \text{in } \mathbb{R}^n, \tag{4}$$

where, for any $\boldsymbol{y} \in \mathbb{R}^n$,

$$u(\boldsymbol{y}' + \boldsymbol{L}, y_n) = u(\boldsymbol{y}) \quad \text{and} \quad f(\boldsymbol{y}' + \boldsymbol{L}, y_n) = f(\boldsymbol{y}). \tag{5}$$

This problem is closely related to the Stokes problem.

In order to be able to define functional spaces on the strip $\Pi_{k=1}^{k=n-1}[0, L_k) \times \mathbb{R}$, we follow the ideas developed in [22,33]. We identify the strip $\Pi_{k=1}^{k=n-1}[0, L_k) \times \mathbb{R}$ with the set $G := \Pi_{k=0}^{k=n-1}(\mathbb{R}/L_k\mathbb{Z}) \times \mathbb{R}$, which together with addition as group operation and the canonical quotient topology inherited from $\mathbb{R}^n$ yields a locally compact abelian group. The Haar measure associated to G is, modulo some normalization, the product of Lebesgue's measures associated to each direction. The Stokes problem (1)–(3) can be rephrased in G as:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } G, \\ \operatorname{div} \boldsymbol{u} = g & \text{in } G. \end{cases} \tag{6}$$

Similarly the biharmonic problem (4)–(5) can be rewritten as:

$$-\Delta^2 u = f \quad \text{in } G. \tag{7}$$

Since the solutions of (6) and (7) correspond to boundary layer, their behavior at infinity in the y_n-direction are of importance: rescaled properly, they provide averaged feed-backs on the macroscopic scale (see [11] and references therein). Therefore we choose to solve (6) and (7) in weighted Sobolev spaces in order to describe behaviors of the solutions and of the data (polynomial growth or decay). To this aim, the weights, when adapted to our problem, are polynomial functions at infinity and regular bounded functions in the neighborhood of the origin : they are powers of $\rho(\boldsymbol{y}) := (1 + y_n^2)^{1/2}$. The literature on weighted Sobolev spaces is wide [18,4,16,15,10,9,7,30,5,6] and deals with various types of domains. To our knowledge, this type of weights has not been applied to problems (6), (7) and (8) in the context of periodic strips.

This paper combines various tools from several fields mixing Fourier analysis on locally compact abelian groups [22,33], functional analysis with Muckenhoupt weights [13,34], weighted Sobolev spaces techniques for the whole space [3,2,29]. We prove these results in the framework of $W^{m,p}_\alpha(G)$ spaces as defined in [3], for $p \in (1, \infty)$ and α in $\mathbb{R}$ out of a discrete set of critical values. The core of the paper are the isomorphisms of the Laplace operator for any relative integer m, any $p \in (1, \infty)$ any weight α : for a given data f we look for u solving

$$-\Delta u = f \quad \text{in } G. \tag{8}$$

Firstly, these results are obtained on a starting interval of weights $\alpha \in (-1/p, 1/p')$ for which we prove Theorem 3. Then, using induction arguments, higher regularity is obtained combined with a natural weight shift. Calderón-Zygmund-type results from [34] provide another set of regularity results, the weight index α being kept in the initial interval. The combination of these latter steps extends then the first isomorphisms to the general non critical case as stated in Theorem 5.

Composing isomorphisms of the Laplace operator, we provide similar results for the poly-harmonic operator Δ^m, m being any given positive integer. For sake of simplicity these results are shown in detail for the bi-harmonic operator.

At this stage, one solves the Stokes problem (6) : eliminating the divergence equation, the pressure is computed using isomorphisms of the Laplace operator, then the velocity is obtained through another use of these isomorphism, the gradient of the pressure becoming part of the right hand side in the first equation of (6). The biharmonic operator is used in order to characterize the kernel of the Stokes operator.

The structure of the paper follows lines explained above. First in Section 2, we define the differentiable structure on our locally compact abelian group. Then in Section 3, we set up the weighted function spaces and establish their basic properties. In Section 4, we construct isomorphisms of the Laplace operator, in Sections 5 and 6 we then use the latter results in order to provide isomorphisms for the biharmonic and the Stokes operators. Finally, in the appendix, we provide some results on distributions with compact support defined on the Pontryagin's dual group $\hat{G}$.

2 Fourier analysis on the LCA-group G

As in [34], we briefly define the structure inherent to periodic strips. We denote Ξ_G the canonical quotient mapping

$$\Xi_G : \mathbb{R}^n \to G, \quad \Xi_G(\boldsymbol{y}) := ([y_1], \dots, [y_{n-1}], y_n), \tag{9}$$

where $[y_j] \in \mathbb{R}/L_j\mathbb{Z}$ is the equivalence class of $y_j \in \mathbb{R}$, for $j \in \{1, \dots, n-1\}$.

Let $\boldsymbol{\lambda} \in \mathbb{N}^n$, be a multi-index an n-uple of nonnegative integers λ_j, we denote $D_{y_j} = \partial_j$ the partial derivative with respect to the j-th coordinate, then $D^{\boldsymbol{\lambda}} := \Pi_{j=1}^n D_{y_j^{\lambda_j}}^{\lambda_j}$ denotes the differential operator of order $|\boldsymbol{\lambda}| = \sum_{j=1}^n \lambda_j$. For $m \in \mathbb{N} \cup \{\infty\}$ let us define

$$C^m(G) := \{u : G \to \mathbb{C}, \text{ such that } \exists \tilde{u} \in C^m(\mathbb{R}^n) \text{ and } \tilde{u} = u \circ \Xi_G\}$$

the space of m-times differentiable functions on G. We define the derivatives via

$$D^{\boldsymbol{\lambda}} u = D^{\boldsymbol{\lambda}} \tilde{u}|_{\Pi_{k=1}^{n-1}[0,L_k)\times\mathbb{R}},$$

where $\boldsymbol{\lambda} \in \mathbb{N}^n$ with $|\boldsymbol{\lambda}| \leq m$ and we identified G with $\Pi_{k=1}^{n-1}[0, L_k) \times \mathbb{R}$. We define $C(G)$ the space of $C^0(G)$ functions that vanish when $|y_n| \to \infty$ and $\mathcal{D}(G) := C_0^\infty(G)$, the set of infinitely many differentiable functions with compact support. The Schwartz-Bruhat space associated to G reads

$$\mathcal{S}(G) := \{u \in C^\infty(G) \text{ such that } \mathfrak{n}_j(u) < \infty, \quad \forall j \in \mathbb{N}^*\}$$

where we set the semi-norm $\mathfrak{n}_j(u) := \sup_{\boldsymbol{y} \in G}(1 + |y_n|^2)^{j/2}|\mathbf{D}^j(u)|$ and $\mathbf{D}^j u := (D^{\boldsymbol{\lambda}} u)_{|\boldsymbol{\lambda}|\leq j}$ (see [36]). The corresponding dual is well defined with respect to the weak-* topology and denotes $\mathcal{S}'(G)$, the space of tempered distributions on G. This allows to define the derivatives of elements of $\mathcal{S}'(G)$ via the primal space :

$$< D^{\boldsymbol{\lambda}} T, u >= (-1)^{|\boldsymbol{\lambda}|} < T, D^{\boldsymbol{\lambda}} u >, \quad u \in \mathcal{S}(G), \ \boldsymbol{\lambda} \in \mathbb{N}^n$$

and proves that they are also tempered distributions. The Pontryagin dual of G is identified with $\hat{G} := \Pi_{k=1}^{k=n-1}(2\pi/L_k\mathbb{Z}) \times \mathbb{R}$. It is also a locally compact abelian group see [32, Theorem 1.2.6]. By default $\hat{G}$ is equipped with the compact-open topology which in this case coincides with the product of discrete topologies on each $2\pi/L_k\mathbb{Z}$ and the Euclidian topology on $\mathbb{R}$. The Haar measure on $\hat{G}$ is simply the product of counting measures on $2\pi/L_k\mathbb{Z}$ for each $k \in \{1, \dots, n-1\}$ and Lebesgue's measure on $\mathbb{R}$. The associated differentiable structure similar to the group G is then

$$C^\infty(\hat{G}) := \left\{ u \in \hat{G} \to \mathbb{C} \text{ such that } u(\mathbf{k}, \cdot) \in C^\infty(\mathbb{R}), \ \forall \mathbf{k} \in \Pi_{\ell=1}^{\ell=n-1}(2\pi\mathbb{Z}/L_\ell) \right\}$$

where we define the derivatives $D^\lambda u(\mathbf{k}, \zeta) = D^\lambda_{\zeta^\lambda} u(\mathbf{k}, \zeta)$ together with the norms $\hat{\mathfrak{n}}_j(u) := \sup_{\boldsymbol{\eta}\in\hat{G}}(1 + |\boldsymbol{\eta}|)^j |\mathbf{D}^j u(\boldsymbol{\eta})|$ where $\mathbf{D}^j u := (D^\lambda_{\xi^\lambda} u)_{\lambda\leq j}$. The Schwartz space on $\hat{G}$ then reads

$$\mathcal{S}(\hat{G}) := \{u \in C^\infty(\hat{G}) \text{ s.t. } \hat{\mathfrak{n}}_j(u) < \infty, \quad \forall j \in \mathbb{N}^*\}$$

and again this provides the correct setting in order to define the associated dual space. For more details see [22,34] and references therein. By the same arguments as in [19, Theorem 1.5.2], one can prove that $\mathcal{E}'(\hat{G})$, the set of tempered distributions with compact support in $\hat{G}$ is identical to the dual space of $C^\infty(\hat{G})$ with the topology defined by the semi norms $\hat{\mathfrak{m}}_{j,K}(u) := \sup_{\boldsymbol{\eta}\in K} |\mathbf{D}^j u(\boldsymbol{\eta})|$ where K ranges over all compact sets of $\hat{G}$, and j over all non-negative integers.

We are in the position to define the Fourier transform $\mathcal{F}_G : L^1(G) \to C(\hat{G})$ as

$$\mathcal{F}_G(u)(\mathbf{k}, \xi) := \frac{1}{|\Sigma|} \int_\Sigma \int_{\mathbb{R}} u(\boldsymbol{y}', y_n) \exp(-i\boldsymbol{y}' \cdot \mathbf{k} - iy_n\xi) dy_n d\boldsymbol{y}'$$

where we denote $\Sigma := \Pi_{\ell=1}^{n-1}(0, L_\ell)$ and $|\Sigma|$ its Lebesgue measure. Several remarks are to be made, $\mathcal{F}_G$ maps $L^1(G)$ into $C(\hat{G})$ by the Riemann-Lebesgue lemma. One proves [12] that $\mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(\hat{G})$ is a homeomorphism. The corresponding inverse Fourier transform reads $\mathcal{F}_G^{-1} : L^1(\hat{G}) \to C(G)$ where

$$\mathcal{F}_G^{-1}(u)(\boldsymbol{y}', y_n) := \sum_{\mathbf{k}\in \Pi_{k=1}^{k=n-1}(2\pi/L_k\mathbb{Z})} \int_{\mathbb{R}} u(\mathbf{k}, \xi) \exp(i\boldsymbol{y}' \cdot \mathbf{k} + iy_n\xi) dy_n d\boldsymbol{y}'.$$

By the Pontryagin duality theorem, there exists also a Fourier transform $\mathcal{F}_{\hat{G}} : \mathcal{S}(\hat{G}) \to \mathcal{S}(G)$, which again has an inverse denoted $\mathcal{F}_{\hat{G}}^{-1} : \mathcal{S}(G) \to \mathcal{S}(\hat{G})$. We have the correspondence $\mathcal{F}_G^{-1}(\mathcal{F}_G(f))(\boldsymbol{y}) = f(\boldsymbol{y}) = \mathcal{F}_{\hat{G}}(\mathcal{F}_G(f))(-\boldsymbol{y})$ for all $\boldsymbol{y} \in G$ and every $f \in \mathcal{S}(G)$. One is able to define the Fourier and the inverse Fourier transforms of a tempered distribution by duality $\mathcal{F}_G : \mathcal{S}'(G) \to \mathcal{S}'(\hat{G})$ and $\mathcal{F}_G^{-1} : \mathcal{S}'(\hat{G}) \to \mathcal{S}'(G)$ as

$$< \mathcal{F}_G(T), \varphi > = < T, \mathcal{F}_{\hat{G}}(\varphi) >, \quad \forall \varphi \in \mathcal{S}(\hat{G})$$

and

$$< \mathcal{F}_G^{-1}(T), \varphi > = < T, \mathcal{F}_{\hat{G}}^{-1}(\varphi) >, \quad \forall \varphi \in \mathcal{S}(G).$$

For all $T \in \mathcal{S}'(G)$ there is a correspondence between derivation and multiplication in the frequency domain : $\mathcal{F}_G(D^{\boldsymbol{\lambda}} T) = i^{|\boldsymbol{\lambda}|} \boldsymbol{\eta}^{\boldsymbol{\lambda}} \mathcal{F}_G(T)$ for any $\boldsymbol{\lambda} \in \mathbb{N}^n$ and $\boldsymbol{\eta} = (\mathbf{k}, \xi) \in \hat{G}$.

3 Properties of the weighted Sobolev spaces

Definition 1 *We define the weight function with respect to the normal coordinate* $\rho(\boldsymbol{y}) := \sqrt{1 + y_n^2}$.

Proposition 2 *The sequence* $(U_k)_{k\in\mathbb{Z}}$ *defined as*

$$U_k := \Pi_{j=1}^{j=n-1} \left([0, r_{k,j}) \cup (L_j - r_{k,j}, L_j) \right) \times \left(-2^k, 2^k \right), \quad r_{k,j} := \min(2^k, L_j),$$

together with $A = 2$ *and* $\theta(k) = k + 1$, *forms a local base of* $0 \in G$ i.e.

i) $\cup_{k\in\mathbb{Z}} U_k = G$
ii) $U_k \subset U_{k'}$ *if* $k < k'$

iii) there exist a positive constant A and a mapping $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $\boldsymbol{x} \in G$, $k < \theta(k)$, $U_k - U_k \in U_{\theta(k)}$, and $\mu(\boldsymbol{x} + U_{\theta(k)}) < A\mu(\boldsymbol{x} + U_k)$. The set difference stems for $U - V := \{z \in G\,;\, \exists u \in U \text{and } v \in V,\, z = u - v\}$.

This proposition shows that the structure of G is compatible with Assumptions 1.1 in [33].

Definition 3 *Let $p \in (1, \infty)$. A weight $w \in L^1_{\text{loc}}(G)$ belongs to the Muckenhoupt class $A_p(G)$ iff*

$$\mathcal{A}_p(w) := \sup_{U \in G} \left(\frac{1}{\mu(U)} \int_U w d\mu \right) \left(\frac{1}{\mu(U)} \int_U w^{-\frac{p'}{p}} d\mu \right)^{\frac{p}{p'}} < \infty$$

where the supremum runs over all base sets $U \in G$.

Proposition 4 *For any $p \in (1, \infty)$ and any real number $\alpha \in (-1/p, 1/p')$, the weight $\rho^{\alpha p}(\boldsymbol{y})$ is of Muckenhoupt type $A_p(G)$.*

Proof : As the weight ρ is constant with respect to $\boldsymbol{y}'$, the Muckenhoupt criterion reduces to check whether $\rho^{\alpha p}(y_n)$ is in $A_p(\mathbb{R})$. This is true under the assumption that $\alpha \in (-1/p, 1/p')$ by using Lemma 2.3 (v) p. 258 [13].

For $\alpha \in \mathbb{R}$, we introduce the weighted space :

$$L^p_\alpha(G) \equiv W^{0,p}_\alpha(G) := \{u \in L^p_{\text{loc}}(G) \text{ s.t. } \rho^\alpha u \in L^p(G)\},$$

which is a Banach space equipped with its natural norm

$$\|u\|_{L^p_\alpha(G)} = \|\rho^\alpha u\|_{L^p(G)}.$$

Proposition 4 allows to use [34, Lemma 3.1] and claim that $\mathcal{D}(G)$ is dense in $L^p_\alpha(G)$ for $\alpha \in (-1/p, 1/p')$. This yields the general density result stated below.

Lemma 5 *For any real number α, $\mathcal{D}(G)$ is dense in $L^p_\alpha(G)$.*

Proof : The proof is obvious considering that for any α, the mapping $u \in L^p_\alpha(G) \mapsto \rho^\gamma u \in L^p_{\alpha-\gamma}(G)$ is an isomorphism for any real γ. Then using the previous density result established in the range $\alpha \in (-1/p, 1/p')$, the claim is proved.

For any integer $m > 0$, any $p \in (1, \infty)$ and any real number α, we now define the weighted Sobolev space

$$W^{m,p}_\alpha(G) := \left\{ u \in \mathcal{S}'(G);\, \forall \boldsymbol{\lambda} \in \mathbb{N}^n \,:\, 0 \leq |\boldsymbol{\lambda}| \leq m,\, \rho^{\alpha - m + |\boldsymbol{\lambda}|} D^{\boldsymbol{\lambda}} u \in L^p(G) \right\},$$

which is a Banach space endowed with its natural norm

$$\|u\|_{W^{m,p}_\alpha(G)} = \left(\sum_{0 \leq |\boldsymbol{\lambda}| \leq m} \|\rho^{\alpha - m + |\boldsymbol{\lambda}|} D^{\boldsymbol{\lambda}} u\|^p_{L^p(G)} \right)^{1/p}.$$

We define the semi-norm

$$|u|_{W^{m,p}_\alpha(G)} = \left(\sum_{|\boldsymbol{\lambda}| = m} \|\rho^\alpha D^{\boldsymbol{\lambda}} u\|^p_{L^p(G)} \right)^{1/p}.$$

For detailed studies on this functional space defined in the whole $\mathbb{R}^n$, the reader can refer to [3,17,18]. Among several properties, we have the following algebraic and topological inclusions :

$$W^{m,p}_{\alpha}(G) \subset W^{m-1,p}_{\alpha-1}(G) \cdots \subset L^p_{\alpha-m}(G). \tag{10}$$

The mapping

$$u \in W^{m,p}_{\alpha}(G) \mapsto D^{\boldsymbol{\lambda}} u \in W^{m-|\boldsymbol{\lambda}|,p}_{\alpha}(G) \tag{11}$$

is continuous for $\boldsymbol{\lambda} \in \mathbb{N}^n$. Using the fact that for any $\lambda, \gamma \in \mathbb{R}$,

$$|D^{\lambda}_{y_n^{\lambda}} \rho^{\gamma}| \leq C\rho^{\gamma-\lambda}, \tag{12}$$

the mapping

$$u \in W^{m,p}_{\alpha}(G) \mapsto \rho^{\gamma} u \in W^{m,p}_{\alpha-\gamma}(G) \tag{13}$$

is an isomorphism for any $\alpha, \gamma \in \mathbb{R}$.

Lemma 6 *For any $(m,p,\alpha) \in \mathbb{N} \times (1,\infty) \times \mathbb{R}$, $\mathcal{D}(G)$ is dense in $W^{m,p}_{\alpha}(G)$.*

Proof : The ideas of the proof come from [3] and [28]. Let u be in $W^{m,p}_{\alpha}(G)$.

(i) We first approximate u by functions with compact support in the y_n-direction. As this is a standard procedure, we refer to [18] p. 230–231 for further details.

(ii) We define ω, the $\boldsymbol{y}'$-periodic transform of a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\omega\varphi(\boldsymbol{y}) := \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} L_\ell \mathbb{Z}} \varphi(y_1 - k_1, \ldots, y_{n-1} - k_{n-1}, y_n), \quad \forall \boldsymbol{y} \in G.$$

A simple computation shows that for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\|\omega\varphi\|_{L^p_{\alpha}(G)} \leq c(|\text{supp } \varphi|)\|\varphi\|_{L^p_{\alpha}(\mathbb{R}^n)}. \tag{14}$$

There exists a function $\theta \in \mathcal{D}(\mathbb{R}^{n-1})$ such that $\omega\theta = 1$. Indeed, for any non-negative function $\varphi \in \mathcal{D}(\mathbb{R}^{n-1})$ whose support contains at least one period Σ, one can choose $\theta := \varphi/(\omega\varphi)$. Such a function is called a $\boldsymbol{y}'$-periodic partition of unity. Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence such that $\alpha_j \in \mathcal{D}(\mathbb{R}^n)$, $\alpha_j \geq 0$, $\int_{\mathbb{R}^n} \alpha_j(\boldsymbol{x})d\boldsymbol{x} = 1$ and the support of α_j is included in the closed ball of radius $r_j > 0$ and centered at 0 where $r_j \to 0$ as $j \to \infty$. It is well known that as $j \to \infty$, α_j converges in the distributional sense to the Dirac measure. We set $w(\boldsymbol{y}) = \theta(\boldsymbol{y}')\tilde{u}(\boldsymbol{y})$, where $\tilde{u}$ denotes the periodic extension of u in $\mathbb{R}^n$. Then w belongs to $W^{m,p}_{\alpha}(\mathbb{R}^n)$ and has a compact support. Moreover, since $\omega\theta = 1$, we have $\omega w = \omega(\theta\tilde{u}) = (\omega\theta)u = u$. We define $\varphi_j = w * \alpha_j$. Then φ_j belongs to $\mathcal{D}(\mathbb{R}^n)$ and converges to w in $W^{m,p}_{\alpha}(\mathbb{R}^n)$ as j tends to ∞. Let $\psi_j := \omega\varphi_j$, then ψ_j belongs to $\mathcal{D}(G)$ and thanks to (14), ψ_j converges to $\omega\, w = u$ in $W^{m,p}_{\alpha}(G)$ as j tends to ∞.

Definition 7 *If $j \in \mathbb{Z}$ then we denote by $\mathcal{P}_j$ the set of polynomials of degree less or equal to j that only depend on the y_n-direction, with the convention that if $j < 0$ then $\mathcal{P}_j = \{0\}$. For any integer $m \geq 1$, we denote by $\mathcal{P}^{\Delta^m}_j$ the set of polyharmonic polynomials of order m of $\mathcal{P}_j$. In particular, $\mathcal{P}^{\Delta}_j$ is the space of harmonic polynomials of $\mathcal{P}_j$.*

Remark 8 *Observe that since polynomials of $\mathcal{P}_j$ only depend on the y_n direction, their degree cannot exceed $2m-1$. In other words, for any integer $m \geq 1$, we have*

$$\mathcal{P}_j^{\Delta^m} = \mathcal{P}_{\min(j,2m-1)}.$$

Proposition 9 *Let $m \geq 0$ be an integer, $p \in (1,\infty)$ and α be real numbers. We introduce the integer*

$$q(m,p,\alpha) = \begin{cases} \lfloor m - 1/p - \alpha \rfloor & \text{if } 1/p + \alpha \notin \mathbb{Z}, \\ m - 1/p - \alpha - 1 & \text{otherwise}, \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor part of its argument. Then $\mathcal{P}_{q(m,p,\alpha)}$ is the biggest set of polynomials included in $W^{m,p}_\alpha(G)$,

Proposition 10 *For any $\alpha \in \mathbb{R}$ and $p \in (1,\infty)$, one has the continuous embedding*

$$\mathcal{S}(G) \hookrightarrow L^p_\alpha(G) \hookrightarrow \mathcal{S}'(G).$$

where we identify $u \in L^p_\alpha(G)$ with $T_u \in \mathcal{S}'(G)$ via $< T_u, \psi >:= \int_G u\psi d\boldsymbol{y}$ for all $\psi \in \mathcal{S}(G)$.

Proof : If $\alpha < -1/p$ then a simple application of Hölder's inequality leads to

$$\|\varphi\|_{L^p_\alpha(G)} \leq c \sup_{\boldsymbol{y} \in G} |\varphi(\boldsymbol{y})| \leq c\, \mathfrak{n}_0(\varphi)$$

which proves the continuous injection in this case. Otherwise, if $\alpha \geq -1/p$, then for any $\varphi \in \mathcal{S}(G)$, we can write

$$\begin{aligned} \|\varphi\|^p_{L^p_\alpha(G)} &= \int_G |\varphi(\boldsymbol{y})|^p \rho^{\alpha p}(\boldsymbol{y}) d\boldsymbol{y} \leq \sup_{\boldsymbol{y} \in G} (1+\rho(\boldsymbol{y}))^{jp} |\varphi(\boldsymbol{y})|^p \int_G \rho^{\alpha p - jp}(\boldsymbol{y}) d\boldsymbol{y} \\ &\leq c\, \mathfrak{n}_j(\varphi)^p \int_G \rho^{\alpha p - jp}(\boldsymbol{y}) d\boldsymbol{y}, \end{aligned}$$

the latter integral is finite as soon as $j := \lfloor \alpha + 1/p \rfloor + 1 > \alpha + 1/p$, which ends the proof of the first continuous injections for all $\alpha \in \mathbb{R}$. Using these injections one has directly that

$$\left| \int_G u(\boldsymbol{y}) \varphi(\boldsymbol{y}) d\boldsymbol{y} \right| \leq \|u\|_{L^p_\alpha(G)} \|\varphi\|_{L^{p'}_{-\alpha}(G)} \leq c \|u\|_{L^p_\alpha(G)} \mathfrak{n}_j(\varphi)$$

for a certain j to be chosen accordingly. Thus u is a linear continuous form on $\mathcal{S}(G)$ and the latter continuous injection is proved.

Corollary 1 *Let $(m,p,\alpha) \in \mathbb{N} \times (1,\infty) \times \mathbb{R}$ then*

$$\mathcal{S}(G) \hookrightarrow W^{m,p}_\alpha(G) \hookrightarrow \mathcal{S}'(G).$$

Definition 11 *Let $\alpha \in \mathbb{R}$, m be a negative integer, and $p \in (1,\infty)$, then the Sobolev space $W^{m,p}_\alpha(G)$ is defined as the dual space of $W^{-m,p'}_{-\alpha}(G)$, where p' is the conjugate of p.*

Corollary 2 *Let $\alpha \in \mathbb{R}$, m be a negative integer, and $p \in (1,\infty)$, then elements of Sobolev spaces $W^{m,p}_\alpha(G)$ are tempered distributions.*

Theorem 1 *Let $m \geq 1$ be an integer, $p \in (1, \infty)$ and α be real numbers such that $1/p + \alpha \notin \{1, \dots, m\}$. Then there exists a constant $C > 0$, depending only on m, p and n such that*

$$\forall u \in W^{m,p}_\alpha(G), \quad \|u\|_{W^{m,p}_\alpha(G)/\mathcal{P}_{q'(m,p,\alpha)}} \leq C|u|_{W^{m,p}_\alpha(G)}$$

where $q'(m,p,\alpha) = \min(q(m,p,\alpha), m-1)$. In other words, the semi-norm $|\cdot|_{W^{m,p}_\alpha(G)}$ defines a norm on $W^{m,p}_\alpha(G)/\mathcal{P}_{q'(m,p,\alpha)}$ that is equivalent to the quotient norm.

The proof follows the same lines as in [3, Theorem 8.3 p 598]. As a direct consequence of Theorem 1, we can prove isomorphism results on the gradient and the divergence operator. To that end, let us define the spaces

$$\mathcal{V} := \{\boldsymbol{\varphi} \in \mathcal{D}(G)^n, \ \operatorname{div} \boldsymbol{\varphi} = 0\} \quad \text{and} \quad H_{p,\alpha} := \{\mathbf{v} \in L^p_\alpha(G)^n, \ \operatorname{div} \mathbf{v} = 0\}.$$

Given a Banach space B and a closed subspace X of B, we denote by $B' \perp X$ (or more simply $X^\perp$, if there is no ambiguity as to the duality product), the subspace of B' orthogonal to X, *i.e.*

$$B' \perp X = X^\perp = \{f \in B' : \forall v \in X, < f, v >= 0\} = (B/X)',$$

the space $X^\perp$ being also called the polar space of X in B' and possibly denoted X^0.

Proposition 12 *Let $p \in (1, \infty)$ and α be real numbers such that $1/p + \alpha \neq 1$. Then the operators defined by*

$$\nabla : W^{1,p}_\alpha(G)/\mathcal{P}_{q'(1,p,\alpha)} \to L^p_\alpha(G)^n \perp H_{p',-\alpha} \tag{15}$$

and

$$\operatorname{div} : L^p_\alpha(G)^n / H_{p,\alpha} \to W^{-1,p}_\alpha(G) \perp \mathcal{P}_{q'(1,p',-\alpha)} \tag{16}$$

are isomorphisms.

Proof : The gradient operator defined by (15) is linear, continuous and its kernel is reduced to $\mathcal{P}_{q'(1,p,\alpha)}$. Actually, thanks to Proposition 9, this means that if $\alpha < 1/p'$ then $q'(1,p,\alpha) = 0$ otherwise $q(1,p,\alpha) < 0$ which implies that $\mathcal{P}_{q'(1,p,\alpha)} = \{0\}$. Since $1/p + \alpha \neq 1$, then thanks to Theorem 1, the semi-norm $|\cdot|_{W^{1,p}_\alpha(G)}$ is a norm on the quotient space $W^{1,p}_\alpha(G)/\mathcal{P}_{q'(1,p,\alpha)}$. Thus the gradient is an isomorphism from $W^{1,p}_\alpha(G)/\mathcal{P}_{q'(1,p,\alpha)}$ onto its range R_g and R_g is a closed subspace of $L^p_\alpha(G)$. Hence by the Closed Range Theorem $R_g = (\ker(\operatorname{div}))^0$, where div is the operator defined by

$$\operatorname{div} : L^{p'}(G) \to (W^{1,p}_\alpha(G)/\mathcal{P}_{q'(1,p,\alpha)})'.$$

We deduce that $R_g = (H_{p',-\alpha})^0$, so that the gradient operator defined by (15) is an isomorphism and through duality and transposition the divergence operator defined by (16) is also an isomorphism.

In the remaining part of this section, our aim is to characterize tempered distributions by means of their gradients.

Lemma 13 *Let $p \in [1,\infty)$ and $\alpha \in \mathbb{R}$. If $\mathbf{f} \in L^p_{\mathrm{loc}}(G)^n$ such that*

$$\langle \mathbf{f}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \mathcal{V},$$

then there exists $p \in L^p_{\mathrm{loc}}(G)$ such that $\mathbf{f} = \nabla p$.

Proof : For every $\boldsymbol{y}$ in $\mathbb{R}^n$ we associate the periodic extension of $\mathbf{f}$ reading :

$$\tilde{\mathbf{f}}(\boldsymbol{y}) := \mathbf{f}(\Xi_G(\boldsymbol{y})), \quad \text{a.e. } \boldsymbol{y} \in \mathbb{R}^n.$$

where the restriction operator Ξ_G was introduced in (9). As $\mathbf{f} \in L^p_{\mathrm{loc}}(G)^n$, then for any $K \subset \mathbb{R}^n$ compact set, one can write:

$$\int_K |\tilde{\mathbf{f}}(\boldsymbol{y})|^p d\boldsymbol{y} = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} L_\ell \mathbb{Z}} \int_{(G+\sum_{\ell=1}^{n-1} k_\ell \mathbf{e}_\ell)\cap K} |\tilde{\mathbf{f}}(\boldsymbol{y})|^p d\boldsymbol{y} \leq C(|K|) \int_{K'} |\mathbf{f}(\boldsymbol{y})|^p d\boldsymbol{y}$$

where $K' \subset G$ is a compact set. Then $\tilde{\mathbf{f}}$ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^n)^n$ and as a consequence $\tilde{\mathbf{f}}$ also belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)^n$. Let now $\tilde{\boldsymbol{\varphi}}$ be in $\mathcal{D}(\mathbb{R}^n)^n$ and satisfy $\operatorname{div} \tilde{\boldsymbol{\varphi}} = 0$ in $\mathbb{R}^n$. Recalling that ω is the $\boldsymbol{y}'$-periodic transform defined in Lemma 6, then $\omega\tilde{\boldsymbol{\varphi}} \in \mathcal{V}$ and is a $\boldsymbol{y}'$-periodic function. Moreover one has

$$\int_{\mathbb{R}^n} \tilde{\mathbf{f}}(\boldsymbol{y})\tilde{\varphi}(\boldsymbol{y}) d\boldsymbol{y} = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} L_\ell \mathbb{Z}} \int_{G+\sum_{\ell=1}^{n-1} k_\ell \mathbf{e}_\ell} \tilde{\mathbf{f}}(\boldsymbol{y})\tilde{\varphi}(\boldsymbol{y}) d\boldsymbol{y} = \int_G \mathbf{f}(\boldsymbol{y})\omega\tilde{\varphi}(\boldsymbol{y}) d\boldsymbol{y} = 0.$$

Thanks to [14, Lemma III.1.1. p. 144], there exists $\tilde{p} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ s.t. $\nabla \tilde{p} = \tilde{\mathbf{f}}$ almost everywhere in $\mathbb{R}^n$. Thanks to [14, Lemma II.6.1. p. 81], $\tilde{p} \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$. Then defining $p := \tilde{p}|_{\Pi_{\ell=1}^{n-1}(0,L_\ell)\times\mathbb{R}}$ proves that p belongs to $L^p_{\mathrm{loc}}(G)$.

Hereafter we adapt to our case arguments from [26]. We define $X_{p,\alpha}$ the closure of $\mathcal{V}$ for the $L^p_\alpha(G)$ norm. We also define $A_{p,\alpha} := X^\perp_{p',-\alpha}$, the annihilator of $X_{p',-\alpha}$.

Lemma 14 *For any $p \in [1,\infty)$, and $\alpha \in \mathbb{R}$, $A_{p,\alpha} \equiv \{\nabla p \in L^p_\alpha(G)^n,\ p \in L^p_{\mathrm{loc}}(G)\}$*

Proof : Since $\mathbf{f} \in L^p_\alpha(G)^n$, integrations by part are well defined so that if $\mathbf{f} = \nabla p$, then we can write

$$\langle \mathbf{f}, \mathbf{u} \rangle = \langle \nabla p, \mathbf{u} \rangle = -\langle p, \operatorname{div} \mathbf{u} \rangle = 0, \quad \forall \mathbf{u} \in \mathcal{V}.$$

this proves the inclusion of the gradients in the annihilator set. Thanks to the previous Lemma one has the converse claim.

Proposition 15 *The tempered distribution $u \in \mathcal{S}'(G)$ satisfies $\Delta u = 0$ in G if and only if $u \in \mathcal{P}^\Delta_1$.*

Proof : It is clear that if $u \in \mathcal{P}^\Delta_1$, then $\Delta u = 0$. Conversely, let $u \in \mathcal{S}'(G)$ satisfy $\Delta u = 0$. Applying the Fourier transform, one has

$$|\boldsymbol{\eta}|^2 \, \mathcal{F}_G(u)(\boldsymbol{\eta}) = 0, \quad \text{a.e. } \boldsymbol{\eta} \in \hat{G},$$

which implies that the transform is compactly supported in $\{0\} \subset \hat{G}$. Then by Theorem 11 (see Appendix) there exists $q \in \mathbb{N}$ such that

$$< \mathcal{F}_G(u), \psi >_{\mathcal{E}'(\hat{G})\times\mathcal{E}(\hat{G})} = \sum_{r=0}^{q} c_r D^r_{\xi^r} \psi(0), \quad \forall \psi \in \mathcal{E}(\hat{G}).$$

Now we prove that $\mathcal{F}_G^{-1}(\mathcal{F}_G(u))$ is a polynomial. Indeed,

$$\begin{aligned}
&\left\langle \mathcal{F}_G^{-1}\left(\mathcal{F}_G(u)\right), \varphi \right\rangle_{\mathcal{S}'(G)\times\mathcal{S}(G)} = \left\langle \mathcal{F}_G(u), \mathcal{F}_{\hat{G}}^{-1}(\varphi) \right\rangle_{\mathcal{S}'(\hat{G})\times\mathcal{S}(\hat{G})} \\
&= \sum_{r=0}^{p} c_r \left(D^r_{\xi^r} \left(\mathcal{F}_G(\varphi)(-\boldsymbol{\eta})\right)\right)\Big|_{\boldsymbol{\eta}=0} = \frac{1}{|\Sigma|}\sum_{r=0}^{p} c_r \left(D^r_{\xi^r} \int_G \varphi(\boldsymbol{y}) \exp(i\boldsymbol{\eta}\cdot\boldsymbol{y}) d\boldsymbol{y}\right)\Big|_{\boldsymbol{\eta}=0} \\
&= \frac{1}{|\Sigma|}\sum_{r=0}^{p} c_r \int_G \varphi(\boldsymbol{y})(iy_n)^r \exp(i\boldsymbol{\eta}\cdot\boldsymbol{y}) d\boldsymbol{y}\Big|_{\boldsymbol{\eta}=0} = \int_G \left\{ \frac{1}{|\Sigma|}\sum_{r=0}^{p} c_r (iy_n)^r \right\} \varphi(\boldsymbol{y}) d\boldsymbol{y}.
\end{aligned}$$

Since the polynomial should be harmonic and in $\mathcal{S}'(G)$ it shall belong by definition to $\mathcal{P}_1^{\Delta}$.

Lemma 16 *Let $p \in (1,\infty)$ and $\alpha \in (-1/p, 1/p')$ be real numbers. Moreover, assume that $q \in L^p_{\mathrm{loc}}(G)$ is such that $\Delta q = 0$ and $\nabla q \in L^p_\alpha(G)^n$. Then $\nabla q = 0$.*

Proof : Since $\nabla q \in L^p_\alpha(G)$ then $\nabla q \in \mathcal{S}'(G)$. As q is harmonic, ∇q is harmonic thus by Proposition 15, ∇q is either constant or proportional to y_n. Since neither y_n nor constants belong to $L^p_\alpha(G)^n$ for $\alpha \in (-1/p, 1/p')$, the gradient is zero.

Corollary 3 *If $p \in (1,\infty)$ and $\alpha \in (-1/p, 1/p')$, then $H_{p,\alpha} \cap A_{p,\alpha} = \{0\}$.*

Proposition 17 *If $p \in (1,\infty)$ and $\alpha \in (-1/p, 1/p')$, there exists a bounded operator $\Theta_{p,\alpha}$ from $L^p_\alpha(G)^n$ onto $H_{p,\alpha}$ such that $\Theta_{p,\alpha}\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in H_{p,\alpha}$.*

Proof : In a first step we give a formal picture of the rigorous proof below : inspired by [26, Proposition 1.5, p. 119], we aim at solving, in the sense of distributions $-\Delta q = \operatorname{div}\mathbf{u}$, so that we use the Fourier transform and write that $\mathcal{F}_G(q) = -i\boldsymbol{\eta}\cdot\mathcal{F}_G(\mathbf{u})/|\boldsymbol{\eta}|^2$. Formally, the Fourier transform of ∇q can be computed as :

$$\mathcal{F}_G(\nabla q) = \underline{m}(\boldsymbol{\eta})\mathcal{F}_G(\mathbf{u}), \quad \text{with } \underline{m}(\boldsymbol{\eta}) := \frac{\boldsymbol{\eta}\otimes\boldsymbol{\eta}}{|\boldsymbol{\eta}|^2}$$

where $\otimes$ denotes the tensorial product in $\mathbb{R}^n$. But as we want to avoid circular arguments about the resolution of the Laplace problem and since we are interested, for Helmholtz decomposition issues, into the gradient of q, in what follows, $\mathbf{g}$ will denote a rigorous computation of the formal quantity ∇q. As $\underline{m}$ is a bounded function but is not continuous near the origin, the transference principle applies [34,8], but after a suitable regularization.

We define the matrix :

$$\underline{m}_\dagger : \hat{G} \to \mathcal{M}_n(\mathbb{C}), \quad \underline{m}_\dagger := (1 - \underline{m}_P(\boldsymbol{\eta}))(\boldsymbol{\eta}\otimes\boldsymbol{\eta})/|\boldsymbol{\eta}|^2,$$

where $\underline{m}_P(\boldsymbol{\eta}) := \mathbb{1}_{\{0\}}(\mathbf{k})$ for all $\boldsymbol{\eta} := (\mathbf{k}, \xi) \in \hat{G}$ and we define the corresponding continuous equivalent to $\underline{m}_\dagger$:

$$\underline{M}_\dagger : \mathbb{R}^{n-1}\times\mathbb{R} \to \mathcal{M}_n(\mathbb{C}), \quad \underline{M}_\dagger(\boldsymbol{\zeta}) = \underline{M}_\dagger(\boldsymbol{\kappa}, \xi) := (1 - \varphi(|\boldsymbol{\kappa}|))(\boldsymbol{\zeta}\otimes\boldsymbol{\zeta})/|\boldsymbol{\zeta}|^2$$

where $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi(0) = 1$ and $\operatorname{supp}\varphi \in (-\frac{1}{2}, \frac{1}{2})$. The symbol $\underline{M}_\dagger$ vanishes in a neighborhood of $\boldsymbol{\kappa} = 0$, and since it is bounded and continuous, it is a

Mikhlin multiplier. We define P, the mean value operator with respect to $\boldsymbol{y}'$ and its complement :

$$(P\psi)(y_n) := \frac{1}{|\Sigma|}\int_{\Sigma} \psi(\boldsymbol{y}', y_n) d\boldsymbol{y}', \quad P_\dagger \psi = \psi - P\psi, \quad \forall \psi \in \mathcal{D}(G).$$

Then we set

$$\mathbf{g} = P u_n \mathbf{e}_n + \mathcal{F}_G^{-1}\left(\underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u})\right) =: \overline{\mathbf{g}} + \mathbf{g}_\dagger$$

one has trivially that $\|\overline{\mathbf{g}}\|_{L^p_\alpha(G)} \leq \|\mathbf{u}\|_{L^p_\alpha(G)}$, and since $\underline{m}_\dagger$ and $\underline{M}_\dagger$ are well defined, Proposition 5.3 [34] shall be used giving :

$$\left\|\mathbf{g}_\dagger\right\|_{L^p_\alpha(G)} \leq c\left\|P_\dagger \mathbf{u}\right\|_{L^p_\alpha(G)} \leq c'\|\mathbf{u}\|_{L^p_\alpha(G)},$$

where we used the decomposition of $L^p_\alpha(G)$ provided in Lemma 5.6 [34]. At this stage, using the Fourier transform, it is easy to check that $\operatorname{div} \mathbf{g} = \operatorname{div} \mathbf{u}$ in $\mathcal{S}'(G)$. Indeed, one has :

$$\begin{aligned} < \operatorname{div} \mathbf{g}, \psi > &= - < \mathbf{g}, \nabla\psi >= - < \overline{\mathbf{g}} + \mathbf{g}_\dagger, \nabla\psi > \\ &= - < P u_n, \partial_n \psi > - < \underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u}), \mathcal{F}_{\hat{G}}^{-1}(\nabla\psi) > \\ &=< \partial_n P u_n, \psi > + < \underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u}), i\,\boldsymbol{\eta}\, \mathcal{F}_G^{-1}(\psi) > \end{aligned}$$

Now since $\underline{m}_\dagger \in C^\infty(\hat{G})$, it is trivial to prove that $< \underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u}), i\boldsymbol{\eta}\mathcal{F}_G^{-1}(\psi) >=< \mathcal{F}_G(P_\dagger \mathbf{u}), i\underline{m}_\dagger \boldsymbol{\eta} \mathcal{F}_G^{-1}(\psi)) >$ and whence as $\underline{m}_\dagger \boldsymbol{\eta} = (1 - \underline{m}_P)\boldsymbol{\eta}$, this gives in turn

$$\begin{aligned} &< \operatorname{div} \mathbf{g}, \psi >=< \partial_n P u_n, \psi > + < (1 - \underline{m}_P)\mathcal{F}_G(P_\dagger \mathbf{u}), i\boldsymbol{\eta}\mathcal{F}_G^{-1}(\psi) > \\ &=< \partial_n P u_n, \psi > - < (1 - \underline{m}_P)\mathcal{F}_G(P_\dagger \mathbf{u}), \mathcal{F}_{\hat{G}}^{-1}(\nabla\psi) > \\ &=< \partial_n P u_n, \psi > - < \mathcal{F}_G(P_\dagger \mathbf{u}), \mathcal{F}_{\hat{G}}^{-1}(\nabla\psi) >=< \partial_n P u_n, \psi > - < P_\dagger \mathbf{u}, \nabla\psi >, \end{aligned}$$

where the brackets are intended in the $\mathcal{S}'(G)$ or the $\mathcal{S}'(\hat{G})$ senses, and we have used that $(1 - \underline{m}_P)\mathcal{F}_G(P_\dagger \mathbf{u}) = \mathcal{F}_G(P_\dagger \mathbf{u})$ since $\mathcal{F}_G(P_\dagger \mathbf{u})(\mathbf{k} = 0, \xi) = 0$, which is indeed the case since $\mathcal{F}_G(P_\dagger \mathbf{u}) = \mathcal{F}_G(\mathbf{u}) - \mathcal{F}_G(P\mathbf{u}) = (1 - \underline{m}_P)\mathcal{F}_G(\mathbf{u})$ (see (19) p.14 [34]). Now one defines $\Theta_{p,\alpha}\mathbf{u} = \mathbf{u} - \mathbf{g}$ and the claim follows.

Theorem 2 *If $p \in (1,\infty)$ and $\alpha \in (-1/p, 1/p')$ then*

i) there is a Helmholtz decomposition of $L^p_\alpha(G)^n = H_{p,\alpha} \oplus A_{p,\alpha}$
ii) the space $\mathcal{V}$ is dense in $H_{p,\alpha}$.

Proof : i) By our construction of $\Theta_{p,\alpha}$, one has that $\mathcal{D}(G)^n \subset H_{p,\alpha} \oplus A_{p,\alpha} \subset L^p_\alpha(G)^n$, which proves that $H_{p,\alpha} \oplus A_{p,\alpha}$ is dense in $L^p_\alpha(G)^n$. As the projector $\Theta_{p,\alpha} : L^p_\alpha(G)^n \to H_{p,\alpha}(G)$ is a bounded linear operator and $H_{p,\alpha}$, $A_{p,\alpha}$ are closed, $H_{p,\alpha} \oplus A_{p,\alpha}$ is closed. Thus *i)* holds.

ii) By i), $H_{p,\alpha} = L^p_\alpha(G)^n / A_{p,\alpha}$. Hence

$$H'_{p,\alpha} = (L^p_\alpha(G)^n / A_{p,\alpha})' = A^\perp_{p,\alpha} \subset H_{p',-\alpha}.$$

The latter inclusion holds. Indeed, suppose that $\mathbf{u} \in A^\perp_{p,\alpha}$ then it belongs to $L^{p'}_{-\alpha}(G)$ and one writes :

$$< \operatorname{div} \mathbf{u}, \varphi >_{\mathcal{S}'(G)\times\mathcal{S}(G)} = - < \mathbf{u}, \nabla\varphi >_{L^{p'}_{-\alpha}(G)\times L^p_\alpha(G)} = 0, \quad \forall \varphi \in \mathcal{S}(G),$$

since $\varphi \in \mathcal{S}(G)$, $\nabla\varphi \in L^p_\alpha(G)$ and thus $\nabla\varphi \in A_{p,\alpha}$ by Lemma 14. Thus $\operatorname{div} \mathbf{u} = 0$ and since $\mathbf{u} \in L^{p'}_{-\alpha}(G)$ one concludes that $\mathbf{u} \in H_{p',-\alpha}$.
The subset $X_{p',-\alpha}$ is total in $H_{p',-\alpha}$ if and only if the only continuous linear functional that vanishes on $X_{p',-\alpha}$ is the 0 functional. Thus suppose that there exists $\mathbf{w} \in (H_{p',-\alpha})'$ s.t. for any $\mathbf{v} \in X_{p',-\alpha}$, $< \mathbf{w}, \mathbf{v} >= 0$ (where the brackets are understood in the $L^p_\alpha(G), L^{p'}_{-\alpha}(G)$ duality), this means that $\mathbf{w} \in X^\perp_{p',-\alpha} = A_{p,\alpha}$, but since $(H_{p',-\alpha})' \subset H_{p,\alpha}$ then $\mathbf{w} \in H_{p,\alpha} \cap A_{p,\alpha} = \{0\}$, which ends the proof.

Proposition 18 *Assume that $p \in (1,\infty)$ and $\alpha \in (-1/p, 1/p')$ are real numbers. Let $u \in \mathcal{S}'(G)$ be such that $\nabla u \in L^p_\alpha(G)^n$. Then $u \in W^{1,p}_\alpha(G)$ and there exists $c > 0$ independent on u such that*

$$\|u\|_{W^{1,p}_\alpha(G)/\mathbb{R}} \leq c\|\nabla u\|_{L^p_\alpha(G)^n}.$$

Proof : If $u \in \mathcal{S}'(G)$ and $\nabla u \in L^p_\alpha(G)^n$, then for all $\varphi \in \mathcal{V}$ one has

$$< \nabla u, \varphi >= - < u, \operatorname{div} \varphi >= 0$$

where the brackets are to be understood in the $\mathcal{S}'(G), \mathcal{S}(G)$-duality sense. The left hand side of the previous equality is nevertheless an integral as well since ∇u is in $L^p_\alpha(G)$. Since $\alpha \in (-1/p, 1/p')$, and thanks to Theorem 2, $\mathcal{V}$ is dense in $H_{p',-\alpha}$ and we see that $\nabla u \in L^p_\alpha(G) \perp H_{p',-\alpha}$. Moreover, under the assumptions on α, we have $q'(1,p,\alpha) = 0$. Thus by Proposition 12, there exists $w \in W^{1,p}_\alpha(G)/\mathbb{R}$ such that $\nabla w = \nabla u$. Since these functions differ at most by a constant ([35], Theorem VI, Chap. 2) and, under the assumptions on α, constants are in $W^{1,p}_\alpha(G)$, u belongs to the latter space and the quotient norm provides the estimates.

4 The Laplace operator

In this section, we solve the Laplace problem

$$-\Delta u = f \quad \text{in } G.$$

We begin by characterizing the kernel.

Proposition 19 *Let $m \geq 0$ be an integer, α be a real number. A function $u \in W^{m,p}_\alpha(G)$ satisfies $\Delta u = 0$ in G if and only if $u \in \mathcal{P}^\Delta_{q(m,p,\alpha)}$.*

Proof : In view of Remark 8, it is clear that if $u \in \mathcal{P}^\Delta_{q(m,p,\alpha)}$, then $\Delta u = 0$. Conversely, if $u \in W^{m,p}_\alpha(G)$, by Corollary 1, $u \in \mathcal{S}'(G)$. Proposition 15 implies then that u is a polynomial in $\mathcal{P}^\Delta_1$, as it is also in $W^{m,p}_\alpha(G)$, it is necessarily in $\mathcal{P}^\Delta_{q(m,p,\alpha)}$.

In order to establish the first isomorphism result for the Laplace operator, we compute the fundamental solution of the Laplace equation in G.

Proposition 20 *The fundamental solution of the Laplace equation in G reads:*

$$\Phi(\boldsymbol{y}) = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell}\mathbb{Z}} \frac{\exp(-|\mathbf{k}||y_n| + i(\mathbf{k}, \boldsymbol{y}'))}{2|\mathbf{k}|} - \frac{1}{2}|y_n|,$$

Proof : The Fourier transform of the fundamental solution is: $\mathcal{F}_{\hat{G}}(\Phi)(\boldsymbol{\eta}) = 1/|\boldsymbol{\eta}|^2$ for all $\boldsymbol{\eta} \in \hat{G}$. Inverting this expression yields formally:

$$\Phi(\boldsymbol{y}) = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell}\mathbb{Z},} \int_{\mathbb{R}} \frac{\exp(i\xi y_n)}{(|\mathbf{k}|^2 + \xi^2)} d\xi \exp(i(\mathbf{k}, \boldsymbol{y}')).$$

Then separating the case $\mathbf{k} \equiv 0$ from the rest provides the result using residue calculus as in [31, example 1, Chap. IX, p. 58].

Lemma 21 *The fundamental solution Φ is in $\mathcal{S}'(G)$.*

Proof : A simple computation gives that $|y_n| \in L^1_{\text{loc}}(G)$ is a tempered distribution since for any $\varphi \in \mathcal{S}(G)$, one has

$$|< |y_n|, \varphi >| \leq c_1 \int_G \frac{|y_n|}{(1 + |y_n|^2)^{j/2}} d\boldsymbol{y}\, \mathfrak{n}_j(\varphi)$$

the integral being bounded for $j \geq 3$. We define $\Phi_{\tilde{\mathbf{k}}}(\boldsymbol{y}) := \exp(-|\tilde{\mathbf{k}}|y_n + i(\tilde{\mathbf{k}}, \boldsymbol{y}')))/(2|\tilde{\mathbf{k}}|)$ for any $\tilde{\mathbf{k}} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell}\mathbb{Z}$: it is a tempered distribution. Moreover, one has that

$$\widehat{\Phi_{\tilde{\mathbf{k}}}}(\boldsymbol{\eta}) := \mathcal{F}_G(\Phi_{\tilde{\mathbf{k}}})(\mathbf{k}, \xi) = \frac{\mathbb{1}_{\{\mathbf{k}=\tilde{\mathbf{k}}\}}(\mathbf{k})}{|\mathbf{k}|^2 + \xi^2}.$$

A simple computation shows that $\widehat{\Phi_{\tilde{\mathbf{k}}}}$ is indeed in $L^1(\hat{G})$, and thus, below, the duality bracket in $(\mathcal{S}'(\hat{G}), \mathcal{S}(\hat{G}))$ becomes an integral with respect to the Haar measure associated to $\hat{G}$. Let now M be a positive real number large enough, define $Z_M := \{\tilde{\mathbf{k}} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell}\mathbb{Z} \text{ s.t. } \tilde{\mathbf{k}} \neq 0 \text{ and } |\tilde{\mathbf{k}}| < M\}$ and

$$\Phi^M(\boldsymbol{y}) := \sum_{\tilde{\mathbf{k}} \in Z_M} \Phi_{\tilde{\mathbf{k}}}(\boldsymbol{y}).$$

Then applying the Fourier transform,

$$\widehat{\Phi_M}(\boldsymbol{\eta}) := \mathcal{F}_G(\Phi^M)(\boldsymbol{\eta}) = \sum_{\tilde{\mathbf{k}} \in Z_M} \frac{\mathbb{1}_{\{\mathbf{k}=\tilde{\mathbf{k}}\}}(\mathbf{k})}{|\mathbf{k}|^2 + \xi^2}.$$

The previous arguments show that $\widehat{\Phi_M} \in L^1(\hat{G})$, but the bound is not uniform with respect to M, since it behaves as $\sum_{\mathbf{k} \in Z_M} |\mathbf{k}|^{-1}$. Nevertheless for any $\tilde{\varphi} \in \mathcal{S}(\hat{G})$, we have

$$\begin{aligned} \left\langle \widehat{\Phi_M}, \tilde{\varphi} \right\rangle_{\mathcal{S}'(\hat{G}), \mathcal{S}(\hat{G})} &= \sum_{\mathbf{k} \in Z_M} \int_{\mathbb{R}_+} \frac{\tilde{\varphi}(\mathbf{k}, \xi)}{|\mathbf{k}|^2 + \xi^2} d\xi \\ &\leq \hat{\mathfrak{n}}_j(\tilde{\varphi}) \sum_{\mathbf{k} \in Z_M} (1 + |\mathbf{k}|)^{-j} \int_{\mathbb{R}_+} \frac{1}{|\mathbf{k}|^2 + \xi^2} d\xi \leq c_2 \hat{\mathfrak{n}}_j(\tilde{\varphi}). \end{aligned}$$

For j sufficiently large with respect to n, the constant c_2 does not depend on M. This ends the proof since the Fourier transform is homeomorphic between $\mathcal{S}'(G)$ and $\mathcal{S}'(\hat{G})$.

Lemma 22 *If $f \in \mathcal{S}(G)$, then $\Phi * f \in \mathcal{S}'(G)$. Moreover if $f \in \mathcal{D}(G)$ then*

$$< \Phi * f, \varphi >_{\mathcal{S}'(G)\times\mathcal{S}(G)} = < f, \Phi * \varphi >_{\mathcal{E}'(G)\times\mathcal{E}(G)}, \quad \forall \varphi \in \mathcal{S}(G).$$

Proof : The proof of the first part is obvious and relies on the Fourier transform: indeed $\widehat{\Phi * f} = \hat{\Phi}\hat{f}$ and as the product $\hat{\Phi}\hat{f}$ is again in $\mathcal{S}'(\hat{G})$, then so it is for the inverse transform giving the result. Moreover, one has :

$$< \Phi * f, \varphi > = < \Phi * \varphi, f >, \quad \forall \varphi \in \mathcal{S}(G), \quad \forall f \in \mathcal{S}(G).$$

Indeed, since Φ^M belongs to $L^1(G)$, it allows to convert the brackets into sums and integrals and use Lebesgues : one obtains the second result for any fixed M. Then passing to the limit necessitates the same arguments as in the previous Lemma. Since f and φ are both in $\mathcal{E}(G)$ so are the respective convolutions with Φ. Moreover, if we suppose that $f \in \mathcal{D}(G)$ then it is also in $\mathcal{E}'(G)$ and thanks to these arguments, the result follows, and the bracket can be transformed :

$$< \Phi * \varphi, f >_{\mathcal{S}'(G)\times\mathcal{S}(G)} = < f, \Phi * \varphi >_{\mathcal{E}'(G)\times\mathcal{E}(G)} .$$

We can now state the first isomorphism result.

Theorem 3 *Let $p \in (1,\infty)$ and $\alpha \in \left(-1/p, 1/p'\right)$ be real numbers. Then the Laplace operator defined by*

$$W^{1,p}_\alpha(G)/\mathbb{R} \overset{\Delta}{\mapsto} W^{-1,p}_\alpha(G)\perp\mathbb{R}$$

is an isomorphism.

Proof : We adapt ideas from [3, Theorem 5.1, p. 586–587]. The operator is clearly linear and continuous, thanks to Proposition 19, it is also injective. Hereafter we prove surjectivity. Given $f \in W^{-1,p}_\alpha(G)\perp\mathbb{R}$ an obvious candidate could be $\Phi * f$, the problem is that neither Φ nor f are not compactly supported : the latter convolution does not make sense *a priori.*

i) Thanks to Proposition 12, there exists $\mathbf{v} \in L^p_\alpha(G)$ s.t. $\operatorname{div} \mathbf{v} = f$ and

$$\|\mathbf{v}\|_{L^p_\alpha(G)} \leq c\|f\|_{W^{-1,p}_\alpha(Z)}$$

where the constant is independent on $\mathbf{v}$. Now since $\mathcal{D}(G)$ is dense in $L^p_\alpha(G)$, there exists a sequence $\mathbf{v}_j \in \mathcal{D}(G)$ s.t. $\mathbf{v}_j \to \mathbf{v}$ in $L^p_\alpha(G)$.

ii) Set $f_j = \operatorname{div} \mathbf{v}_j$, and $\psi_j = \Phi * f_j$, for all $i = 1, \dots, n$ and $\varphi \in \mathcal{D}(G)$ we have :

$$\begin{aligned}&< \partial_i \psi_j, \varphi >_{\mathcal{S}'(G)\times\mathcal{S}(G)} = - < \Phi * f_j, \partial_i \varphi >_{\mathcal{S}'(G)\times\mathcal{S}(G)} \\ &= - < f_j, \partial_i(\Phi * \varphi) >_{\mathcal{E}'(G)\times\mathcal{E}(G)} = < \mathbf{v}_j, \nabla\partial_i(\Phi * \varphi) >_{\mathcal{E}'(G)\times\mathcal{E}(G)},\end{aligned}$$

the change in the sense of duality brackets is justified by Lemma 22. Then thanks to Proposition 4, $\rho^{-p'\alpha}(y_2)$ is a Muckenhoupt weight, we apply the weighted Calderon-Zygmund inequality from the proof of Theorem 1.1 p. 15-16 [34], and one has

$$\begin{aligned}\left|< \partial_i \psi_j, \varphi >\right| &\leq \left\|\mathbf{v}_j\right\|_{L^p_\alpha(G)} \|\nabla\partial_i(\Phi * \varphi)\|_{L^{p'}_{-\alpha}(G)} \\ &\leq c\left\|\mathbf{v}_j\right\|_{L^p_\alpha(G)} \|\Delta(\Phi * \varphi)\|_{L^{p'}_{-\alpha}(G)} \leq c'\|f\|_{W^{-1,p}_\alpha(G)} \|\varphi\|_{L^{p'}_{-\alpha}(G)}\end{aligned}$$

So that $\nabla\psi_j$ is uniformly bounded with respect to j in $L^p_\alpha(G)$.

iii) We can apply Proposition 18, so there exists constants $(c_j(\psi_j))_{j\in\mathbb{N}}$ and $c>0$ s.t.

$$\psi_j + c_j \in W^{1,p}_{\alpha}(G), \quad \left\|\psi_j + p_j\right\|_{W^{1,p}_{\alpha}(G)} \leq c\|f\|_{W^{-1,p}_{\alpha}(G)}.$$

From this it follows that $\psi_j + p_j$ converges weakly to some function u in $W^{1,p}_{\alpha}(G)$ and it solves (8) so that the mapping is indeed surjective.

In order to extend Theorem 3 to other values of α, we first prove a regularity result.

Proposition 23 *Let ℓ be a non-negative integer, $p\in(1,\infty)$ and $\alpha\in(-1/p,1/p')$ be real numbers. Then the Laplace operator defined by*

$$W^{1+\ell,p}_{\alpha+\ell}(G)/\mathbb{R} \overset{\Delta}{\mapsto} W^{-1+\ell,p}_{\alpha+\ell}(G)\perp\mathbb{R} \tag{17}$$

is an isomorphism.

Proof : Owing to Theorem 3, the claim is true for $\ell=\ 0$. Assume that it is true for $\ell=k$ and let us prove that it is still true for $\ell=k+1$. The Laplace operator defined by (17) is clearly linear and continuous. It is also injective : if $u\in W^{2+k,p}_{\alpha+k+1}(G)$ and $\Delta u=0$ then $u\in\mathcal{P}^{\Delta}_{\lfloor 1-1/p-\alpha\rfloor}\equiv\mathbb{R}$ since $\alpha\in(-1/p,1/p')$. To prove that it is onto, let f be given in $W^{k,p}_{\alpha+k+1}(G)\perp\mathbb{R}$. According to (10), f belongs to $W^{-1+k,p}_{\alpha+k}(G)\perp\mathbb{R}$. Then the induction assumption implies that there exists $u\in W^{1+k,p}_{\alpha+k}(G)$ such that $\Delta u=f$. Next, we have

$$\Delta(\rho\partial_i u)=\rho\partial_i f+\partial_i u\Delta\rho+2\nabla\rho\cdot\nabla(\partial_i u). \tag{18}$$

Using , (10), (11), (12), and (13), all the terms of the right-hand side belong to $W^{-1+k,p}_{\alpha+k}(G)$. This implies that $\Delta(\rho\partial_i u)$ also belongs to $W^{-1+k,p}_{\alpha+k}(G)$. In order to apply again the induction argument, it remains to prove that $\Delta(\rho\partial_i u)$ is orthogonal to constants. Let us first note that $\Delta(\rho\partial_i u)$ also belongs to $W^{-2+k,p}_{\alpha+k-1}(G)$. Moreover, since u belongs to $W^{1+k,p}_{\alpha+k}(G)$ implies that $\rho\partial_i u$ belongs $W^{k,p}_{\alpha+k-1}(G)$, then, for any $\varphi\in W^{2-k,p'}_{-\alpha-k+1}(G)$, we can write

$$\langle\Delta(\rho\partial_i u),\varphi\rangle_{W^{-2+k,p}_{\alpha+k-1}(G)\times W^{2-k,p'}_{-\alpha-k+1}(G)}=\langle\rho\partial_i u,\Delta\varphi\rangle_{W^{k,p}_{\alpha+k-1}(G)\times W^{-k,p'}_{-\alpha-k+1}(G)}.$$

Since $\mathbb{R}\subset W^{2-k,p'}_{-\alpha-k+1}(G)$, we can take $\varphi\in\mathbb{R}$ which implies that $\Delta\varphi=0$. It follows that $\Delta(\rho\partial_i u)$ is indeed orthogonal to constants. Thanks to the induction assumption, there exists v in $W^{1+k,p}_{\alpha+k}(G)$ such that

$$\Delta v=\Delta(\rho\partial_i u).$$

Hence, the difference $v-\rho\partial_i u$ is a constant. Since the constants are in $W^{1+k,p}_{\alpha+k}(G)$, we deduce that $\rho\partial_i u$ belongs to $W^{1+k,p}_{\alpha+k}(G)$ which in turn implies that u belongs to $W^{2+k,p}_{\alpha+k+1}(G)$.

We continue by establishing a new family of isomorphism using Calderón-Zygmund's inequality and Theorem 1.

Proposition 24 *Let $p \in (1,\infty)$, $\alpha \in (-1/p, 1/p')$ be real numbers and $m \geq 2$ be an integer. Then the Laplace operator defined by*

$$W^{m,p}_{\alpha}(G)/\mathcal{P}_{m-1} \overset{\Delta}{\mapsto} W^{m-2,p}_{\alpha}(G)/\mathcal{P}_{m-3}$$

is an isomorphism.

Proof : The mapping defined in the claim is bounded linear and injective. Again as in the proof of Theorem 3, as $\alpha \in (-1/p, 1/p')$, the weights are of Muckenhoupt type and one can apply the weighted Calderón-Zygmund estimates giving :

$$\begin{aligned}&\forall u \in W^{m,p}_{\alpha}(G), \quad \forall \lambda \in \mathbb{N}^n \ : \ |\lambda| = m-2, \ \forall i,j = 1,\dots,n\\ &\left\|\partial_i\partial_j D^{\lambda}u\right\|_{L^p_{\alpha}(G)} \leq c\left\|\Delta D^{\lambda}u\right\|_{L^p_{\alpha}(G)} \leq c\|\Delta u\|_{W^{m-2,p}_{\alpha}(G)/\mathcal{P}_{m-3}}.\end{aligned}$$

Moreover, using again the assumption on α, we have $m-1 < m-1/p-\alpha < m$. Then thanks to Theorem 1, the semi-norm $|\cdot|_{W^{m,p}_{\alpha}(G)}$ is a norm on the quotient space $W^{m,p}_{\alpha}(G)/\mathcal{P}_{m-1}$. This yields

$$\|u\|_{W^{m,p}_{\alpha}(G)/\mathcal{P}_{m-1}} \leq c\|\Delta u\|_{W^{m-2,p}_{\alpha}(G)/\mathcal{P}_{m-3}}$$

which shows that the range of the operator is a closed subspace of the target quotient space. The Closed Range Theorem and the injectivity of the adjoint operator imply that the range is the whole target space.

A direct consequence of this result is the

Proposition 25 *Let $p \in (1,\infty)$, $\alpha \in (-1/p, 1/p')$ be real numbers and $m \geq 2$ be an integer. Then the Laplace operator defined by*

$$W^{m,p}_{\alpha}(G)/\mathcal{P}_{1} \overset{\Delta}{\mapsto} W^{m-2,p}_{\alpha}(G)$$

is an isomorphism.

We can now extend Theorem 3 to other values of α.

Theorem 4 *Let $p \in (1,\infty)$, $\alpha \in (-1/p, 1/p')$ be real numbers and $\ell \geq 1$ be an integer. Then the Laplace operators defined by*

$$W^{1,p}_{\alpha-\ell}(G)/\mathcal{P}_{1} \overset{\Delta}{\mapsto} W^{-1,p}_{\alpha-\ell}(G) \tag{19}$$

and

$$W^{1,p'}_{-\alpha+\ell}(G) \overset{\Delta}{\mapsto} W^{-1,p'}_{-\alpha+\ell}(G)\perp\mathcal{P}_{1} \tag{20}$$

are isomorphisms.

Proof : Let us recall that due to Proposition 25, the mapping

$$W^{m,p}_{\alpha}(G)/\mathcal{P}_{1} \overset{\Delta}{\mapsto} W^{m-2,p}_{\alpha}(G)$$

is an isomorphism. Now through duality and transposition, the mapping

$$W^{-m+2,p'}_{-\alpha}(G) \overset{\Delta}{\mapsto} W^{-m,p'}_{-\alpha}(G)\perp\mathcal{P}_{1}$$

is an isomorphism. Next, using the same arguments as in the proof of Proposition 23, we are able to show that for any integer $\ell \geq 1$, the mapping

$$W^{-m+2+\ell,p'}_{-\alpha+\ell}(G) \overset{\Delta}{\mapsto} W^{-m+\ell,p'}_{-\alpha+\ell}(G)\perp\mathcal{P}_1$$

is an isomorphism. Choosing $\ell = m-1$ yields the isomorphism result defined by (20) which through duality and transposition also enables to obtain the one defined by (19).

Summarizing Theorems 3 and 4, we deduce

Theorem 5 *For $p \in (1,\infty)$ and any $\alpha \in \mathbb{R}$ satisfying*

$$\alpha + 1/p \notin \mathbb{Z} \quad \textit{and} \quad \alpha - 1/p' \notin \mathbb{Z}, \tag{21}$$

the mapping

$$W^{1,p}_{\alpha}(G)/\mathcal{P}^{\Delta}_{\lfloor 1-1/p-\alpha \rfloor} \overset{\Delta}{\mapsto} W^{-1,p}_{\alpha}(G)\perp\mathcal{P}^{\Delta}_{\lfloor 1-1/p'+\alpha \rfloor} \tag{22}$$

is an isomorphism. As a consequence, for any $m \in \mathbb{Z}$ and for any $\alpha \in \mathbb{R}$ satisfying (21)*, the mapping*

$$W^{m+2,p}_{\alpha}(G)/\mathcal{P}^{\Delta}_{\lfloor m+2-1/p-\alpha \rfloor} \overset{\Delta}{\mapsto} W^{m,p}_{\alpha}(G)\perp\mathcal{P}^{\Delta}_{\lfloor -m-1/p'+\alpha \rfloor} \tag{23}$$

is an isomorphism.

5 The biharmonic operator

Here we consider the polyharmonic problem : given f, we look for u solution of

$$\Delta^2 u = f \quad \text{in } G.$$

Proceeding as in Proposition 19, we can prove the following characterization of the kernel of the biharmonic operator.

Proposition 26 *Let α a real number. A function $u \in W^{m,p}_{\alpha}(G)$ satisfies $\Delta^2 u = 0$ in G if and only if $u \in \mathcal{P}^{\Delta^2}_{\lfloor m-1/p-\alpha \rfloor}$.*

We first have the following isomorphism result.

Theorem 6 *Let $p \in (1,\infty)$ and $\alpha \in (-1/p, 1/p')$ be real numbers. Then the biharmonic operator defined by*

$$\Delta^2 : W^{2,p}_{\alpha}(G)/\mathcal{P}_1 \mapsto W^{-2,p}_{\alpha}(G)\perp\mathcal{P}_1$$

is an isomorphism.

Proof : Since $\alpha \in (-1/p, 1/p')$, applying Proposition 24 for $m = 2$, we see that the Laplace operator defined by

$$W_{\alpha}^{2,p}(G)/\mathcal{P}_1 \to L_{\alpha}^{p}(G) \tag{24}$$

is an isomorphism. Then using duality and transposition we also see that

$$L_{-\alpha}^{p'}(G) \to W_{-\alpha}^{-2,p'}(G) \perp \mathcal{P}_1$$

is an isomorphism. Since $\alpha \in (-1/p, 1/p')$ implies that $-\alpha \in (-1/p', 1/p)$, it follows that the Laplace operator defined by

$$L_{\alpha}^{p}(G) \to W_{\alpha}^{-2,p}(G) \perp \mathcal{P}_1 \tag{25}$$

is also an isomorphism. Composing (24) and (25) proves the statement.

Proposition 27 *Let $p \in (1, \infty)$ and α be real numbers satisfying* (21). *Then the biharmonic operator defined by*

$$\Delta^2 : W_{\alpha}^{2,p}(G)/\mathcal{P}_{\lfloor 2-1/p-\alpha \rfloor}^{\Delta^2} \mapsto W_{\alpha}^{-2,p}(G) \perp \mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}^{\Delta^2} \tag{26}$$

is an isomorphim.

Proof : Thanks to Theorem 6, the statement is already proved for $\alpha \in (-1/p, 1/p')$.

Assume now that $\alpha < -1/p$.

The biharmonic operator defined above is clearly linear and continuous. Its injectivity follows from Proposition 26.

Let us prove that the compatibility condition is necessary. For any $u \in W_{\alpha}^{2,p}(G)$ and $\varphi \in \mathcal{S}(G)$, we can write

$$\begin{aligned} |\langle \Delta^2 u, \varphi \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)}| &\leq \int_G |\Delta u| |\Delta \varphi| \, d\boldsymbol{y} \\ &\leq \|\Delta u\|_{L_{\alpha}^{p}(G)} \|\Delta \varphi\|_{L_{-\alpha}^{p'}(G)} \\ &\leq \|\Delta u\|_{L_{\alpha}^{p}(G)} |\varphi|_{W_{-\alpha}^{2,p'}(G)}. \end{aligned}$$

But if $\alpha < -1/p$, then the semi-norm $|\cdot|_{W_{-\alpha}^{2,p'}(G)}$ is a norm on $W_{-\alpha}^{2,p'}(G)/\mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}$. Moreover if $\alpha < -1/p$ the spaces $\mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}$, $\mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}^{\Delta}$ and $\mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}^{\Delta^2}$ coincide. Hence Δ^2 belongs to $W_{\alpha}^{-2,p}(G) \perp \mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}^{\Delta^2}$.

Let now prove that the operator is onto. Take f in $W_{\alpha}^{-2,p}(G) \perp \mathcal{P}_{[2-1/p'+\alpha]}$. Then thanks to the isomorphism result of the Laplace operator defined by (23), there exists $v \in L_{\alpha}^{p}(G)$ such that $\Delta v = f$ in G. Since $\alpha < -1/p$, there are no polynomials in the space $\mathcal{P}_{[-1/p'+\alpha]}$, then using again (23), there exists $u \in W_{\alpha}^{2,p}(G)$ such that $\Delta u = v$ in G which implies that $\Delta^2 u = f$ in G. As a consequence the operator defined by (26) is an isomorphism if $\alpha < -1/p$. Then using duality and transposition it is also an ismorphism if $\alpha > 1/p'$ which ends the proof.

As a consequence we have the following general result.

Theorem 7 *Let $p \in (1, \infty)$ and α be real numbers satisfying* (21) *and $\ell \in \mathbb{Z}$. Then the biharmonic operator defined by*

$$\Delta^2 : W_{\alpha}^{2+\ell,p}(G)/\mathcal{P}_{\lfloor 2+\ell-1/p-\alpha \rfloor}^{\Delta^2} \mapsto W_{\alpha}^{-2+\ell,p}(G) \perp \mathcal{P}_{\lfloor 2-\ell-1/p'+\alpha \rfloor}^{\Delta^2}$$

is an isomorphim.

6 The Stokes equation

This section is devoted to the Stokes problem: given f and g, we look for a pair $(\boldsymbol{u}, \pi)$ satisfying

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} & \text{in } G, \\ \operatorname{div} \boldsymbol{u} = g & \text{in } G. \end{cases} \tag{27}$$

For any integer k, we introduce the space of polynomials

$$\mathcal{N}_k = \{(\boldsymbol{\theta}, \lambda) \in \mathcal{P}_k^{\Delta^2} \times \mathcal{P}_{k-1}^{\Delta},\ \operatorname{div} \boldsymbol{\theta} = 0,\ -\Delta \boldsymbol{\theta} + \nabla \lambda = 0\}.$$

Proposition 28 *Let $m \geq 0$ be an integer and α a real number. A pair $(\boldsymbol{u}, \pi) \in W_\alpha^{m+1,p}(G) \times W_\alpha^{m,p}(G)$ satisfies*

$$-\Delta \boldsymbol{u} + \nabla \pi = 0 \quad \textit{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \textit{in } G, \tag{28}$$

if and only if $(\boldsymbol{u}, \pi) \in \mathcal{N}_{\lfloor m+1-1/p-\alpha \rfloor}$.

Proof : Let the pair $(\boldsymbol{u}, \pi)$ belong to $W_\alpha^{m+1,p}(G) \times W_\alpha^{m,p}(G)$ and suppose it satisfies (28). Taking the divergence of the first equation of (28) implies that $\Delta \pi = 0$ and therefore π is a polynomial of $\mathcal{P}^{\Delta}_{\lfloor m-1/p-\alpha \rfloor}$. As a consequence, we have $\Delta^2 \boldsymbol{u} = \Delta(\nabla \pi) = 0$ which implies that $\boldsymbol{u}$ belongs to $\mathcal{P}^{\Delta^2}_{\lfloor m+1-1/p-\alpha \rfloor}$.

We first look for solutions in $W_\alpha^{1,p}(G) \times L_\alpha^p(G)$.

Theorem 8 *Let α be a real number satisfying (21). Let $\boldsymbol{f} \in W_\alpha^{-1,p}(G)$, $g \in L_\alpha^p(G)$ satisfy the compatibility condition*

$$\forall (\boldsymbol{\theta}, \lambda) \in \mathcal{N}_{\lfloor 1-1/p'+\alpha \rfloor}, \quad \langle \boldsymbol{f}, \boldsymbol{\theta} \rangle_{W_\alpha^{-1,p}(G) \times W_{-\alpha}^{1,p'}(G)} + \langle g, \lambda \rangle_{L_\alpha^p(G) \times L_{-\alpha}^{p'}(G)} = 0. \tag{29}$$

Then the Stokes equations (27) have a unique solution $(\boldsymbol{u}, \pi) \in (W_\alpha^{1,p}(G) \times L_\alpha^p(G))/\mathcal{N}_{\lfloor 1-1/p-\alpha \rfloor}$.

Proof : Let us introduce the Stokes operator

$$T : \quad (\boldsymbol{u}, \pi) \mapsto (-\Delta \boldsymbol{u} + \nabla \pi, -\operatorname{div} \boldsymbol{u}).$$

The statement amounts to prove that the operator

$$T : (W_\alpha^{1,p}(G) \times L_\alpha^p(G))/\mathcal{N}_{\lfloor 1-1/p-\alpha \rfloor} \mapsto (W_\alpha^{-1,p}(G) \times L_\alpha^p(G)) \perp \mathcal{N}_{\lfloor 1-1/p'+\alpha \rfloor}.$$

is an isomorphism.
We assume first $\alpha < 1/p'$.

The operator T defined above is clearly linear and continuous. Its injectivity follows from Proposition 28.

Let us show that the compatibility condition is necessary. Note that under the assumtion $\alpha < 1/p'$, there are no polynomials in $L_{-\alpha}^{p'}(G)$ and the polynomials of $W_{-\alpha}^{1,p'}(G)$ are at most constants. Then (29) is reduced to show that

$\boldsymbol{f} \in W_\alpha^{-1,p}(G) \perp \mathcal{P}_{\lfloor 1-1/p'+\alpha \rfloor}$. Consider now $(\boldsymbol{u}, \pi) \in W_\alpha^{1,p}(G) \times L_\alpha^p(G)$. Then $-\Delta \boldsymbol{u} + \nabla \pi \in W_\alpha^{-1,p}(G)$ and for any $\boldsymbol{\theta} \in \mathcal{P}_{\lfloor 1-1/p'+\alpha \rfloor}$, we have

$$\begin{aligned}
&\langle -\Delta \boldsymbol{u} + \nabla \pi, \boldsymbol{\theta} \rangle_{W_\alpha^{-1,p}(G) \times W_{-\alpha}^{1,p'}(G)} \\
&= \langle \boldsymbol{u}, -\Delta \boldsymbol{\theta} \rangle_{W_\alpha^{1,p}(G) \times W_{-\alpha}^{-1,p'}(G)} - \langle \pi, \operatorname{div} \boldsymbol{\theta} \rangle_{L_\alpha^p(G) \times L_{-\alpha}^{p'}(G)} \\
&= 0
\end{aligned}$$

since $\boldsymbol{\theta}$ is at most a constant vector. This shows that the compatibility condition is necessary.

Let us now show that the operator T is onto. Let $\boldsymbol{f} \in W_\alpha^{-1,p}(G)$ and $g \in L_\alpha^p(G)$. Then this implies that $\operatorname{div} \boldsymbol{f} + \Delta g$ belongs to $W_\alpha^{-2,p}(G)$. Since $\alpha < 1/p'$, the degree of polynomials in $\mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}$ is at most one. Hence Δg belongs to $W_\alpha^{-2,p}(G) \perp \mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}$. Then since $\boldsymbol{f} \in W_\alpha^{-1,p}(G) \perp \mathcal{P}_{\lfloor 1-1/p'+\alpha \rfloor}$, it follows that $\operatorname{div} \boldsymbol{f} + \Delta g \in W_\alpha^{-2,p}(G) \perp \mathcal{P}_{\lfloor 2-1/p'+\alpha \rfloor}$. Thanks to the isomorphism result (23) of the Laplace operator, there exists $\pi \in L_\alpha^p(G)$ such that

$$\Delta \pi = \operatorname{div} \boldsymbol{f} + \Delta g \quad \text{in } G.$$

It follows that $\boldsymbol{f} - \nabla \pi \in W_\alpha^{-1,p}(G) \perp \mathcal{P}_{\lfloor 1-1/p'+\alpha \rfloor}$. Thanks again to (23), there exists $\boldsymbol{u} \in W_\alpha^{1,p}(G)$ such that

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \quad \text{in } G.$$

As a consequence $\operatorname{div} \boldsymbol{u} - g \in L_\alpha^p(G)$ and satisfies

$$\Delta(\operatorname{div} \boldsymbol{u} - g) = 0 \quad \text{in } G.$$

Therefore $\operatorname{div} \boldsymbol{u} - g = \lambda \in \mathcal{P}^{\Delta}_{\lfloor -1/p-\alpha \rfloor}$. Let $\gamma(y_n) = \int_0^{y_n} \lambda(s) ds$, then γ belongs to $\mathcal{P}_{\min(2, \lfloor 1-1/p-\alpha \rfloor)} \subset \mathcal{P}^{\Delta^2}_{\lfloor 1-1/p-\alpha \rfloor}$. Let now $\boldsymbol{\theta}$ be a polynomial such that $\theta_j = 0$ for any $j = 1, \ldots, n-1$ and $\theta_n = \gamma$. Then $\boldsymbol{\theta}$ belongs to $\mathcal{P}^{\Delta^2}_{\lfloor 1-1/p-\alpha \rfloor}$ and satisfies $\operatorname{div} \boldsymbol{\theta} = \lambda$. Thus $(\boldsymbol{u} - \boldsymbol{\theta}, \pi)$ satisfies (27) and as a consequence, the operator T is onto.

Through duality the operaor T is also an isomorphism if $\alpha > -1/p$ and this ends the proof.

Using again regularity and duality arguments, we have the following statement.

Theorem 9 *Let α be a real number satisfying (21) and m be an integer. Let $\boldsymbol{f} \in W_\alpha^{m,p}(G)$, $g \in W_\alpha^{m+1,p}(G)$ satisfy the compatibility condition*

$$\begin{aligned}
&\forall (\boldsymbol{\theta}, \lambda) \in \mathcal{N}_{\lfloor -m-1/p'+\alpha \rfloor}, \\
&\qquad \langle \boldsymbol{f}, \boldsymbol{\theta} \rangle_{W_\alpha^{m,p}(G) \times W_{-\alpha}^{-m,p'}(G)} + \langle g, \lambda \rangle_{W_\alpha^{m+1,p}(G)) \times W_{-\alpha}^{-m-1,p'}(G)} = 0.
\end{aligned}$$

Then the Stokes equations (27) has a unique solution $(\boldsymbol{u}, \pi) \in (W_\alpha^{m+2,p}(G) \times W_\alpha^{m+1,p}(G)) / \mathcal{N}_{\lfloor m+2-1/p-\alpha \rfloor}$.

A Compactly supported distributions, $\mathcal{E}'(\hat{G})$

Theorem 10 *If $u \in \mathcal{E}'(\hat{G})$ of order p, then for any test function $\psi \in \mathcal{E}(\hat{G})$ s.t. $D^j_{\xi^j}\psi(\eta) = 0$ for all points η in the support of u and all $j \in \{0, \dots, p\}$, one has $< u, \psi >= 0$, where the brackets denote the duality in $\mathcal{E}'(\hat{G}), \mathcal{E}(\hat{G})$.*

Proof : As the support of u is compact in $\hat{G}$ there exists a finite collection of integers $J \subset \mathbb{N}$ and finite sequence of compact intervals $(K(j))_{j\in J}$ s.t. supp$u \in \{(\mathbf{k}_j, K(j))_{j\in J}\}$, we denote the set of all $S := \{(\mathbf{k}_j)_{j\in J}\}$, a finite subset of $\Pi_{\ell=1}^{n-1}(2\pi\,\mathbb{Z}/L_\ell)$. Let $K_\epsilon(j) := \{\zeta \in \mathbb{R} \text{ s.t. } d(\zeta, K(j)) \leq \epsilon\}$. We define the smooth cut-off function

$$\chi_\epsilon(\mathbf{k}, \xi) := \frac{1}{\epsilon}\begin{cases} \int_{K_{2\epsilon}(j)} \varphi((\xi-\zeta)/\epsilon)d\zeta & \text{if } \mathbf{k} \in S \\ 0 & \text{otherwise} \end{cases}$$

where $\varphi \in C^\infty_0(\mathbb{R})$ s.t. $\varphi \geq 0$, $\int_{\mathbb{R}} \varphi(\xi)d\xi = 1$ and supp$\varphi \subset B(0,1)$. Then it is clear that $\chi_\epsilon(\mathbf{k}_j, \cdot) = 1$ in $K_\epsilon(j)$ and that supp$\chi_\epsilon(\mathbf{k}_j, \cdot) \subset K_{3\epsilon}(j)$. One has then $< u, \psi >=< u, \psi\chi_\epsilon >$ since $\psi(1-\chi_\epsilon) = 0$ in a neighborhood of supp$u$. Hence

$$| < u, \psi > | \leq C \sup_{\eta\in\hat{G}, j\leq p} \left| D^j_{\xi^j}(\psi\chi_\epsilon) \right|$$

which by arguments similar to Theorem 1.5.4 [19] is bounded uniformly by ϵ, and thus tends to 0, as $\epsilon \to 0$. Indeed, for any fixed $\mathbf{k}_j$ using

$$\left|\psi(\mathbf{k}_j, \xi')\right| \leq \frac{1}{(p+1)!} \sup_{t\in(0,1)} \left| D^{p+1}_{\xi^{p+1}} \psi(\mathbf{k}_j, \xi + t(\xi' - \xi)) \right| |\xi' - \xi|^{(p+1)}$$

for all $\xi \in \mathbb{R}$ and similar estimates for all derivatives of higher order, it implies that $\sup_{\xi'\in K_{3\epsilon}}$ $|D^q\psi(\mathbf{k}_j, \xi')| \leq c_{j,q}\epsilon^{p+1-q}$. On the other hand $\sup_{\xi\in K_{3\epsilon}(j)} |D^{q'}\chi_\epsilon(\mathbf{k}_j, \xi)| \leq c'_{j,q'}\epsilon^{-q'}$, which together with the Leibnitz formula gives that

$$| < u, \psi > | \leq c \sup_{j\in J} \sup_{\xi\in K_{3\epsilon}(j)} \binom{p}{q} c_{j,q}\epsilon^{p+1-q} c'_{j,p-q}\epsilon^{-(p-q)} \leq c\epsilon.$$

Theorem 11 *If $u \in \mathcal{E}'(\hat{G})$ s.t. supp$u \subset \{0\}$ then for any $\varphi \in \mathcal{E}(\hat{G})$, there exists a $p \in \mathbb{N}$ s.t. $< u, \varphi >= \sum_{r=0}^p c_r \partial^r_{\xi^r}\varphi(0)$.*

Proof : Let p be the order of u. Set $\psi(\mathbf{k}, \xi) = \varphi(\mathbf{k}, \xi) - \sum_{r=0}^p \partial^r_{\xi^r}\varphi(\mathbf{k}, 0)\frac{\xi^r}{r!}$ it is a smooth function s.t. any derivative in direction ξ vanishes up to order p. Since u is compactly supported, one defines $\chi(\mathbf{k}, \xi) = \mathbb{1}_{\mathbf{k}=0}$ which is in $C^\infty(\hat{G})$. Applying the previous result it comes $< u, \psi >=< u, \chi\psi >= 0$. Then using that u is linear, one recovers :

$$< u, \varphi >=< u, \chi_{\mathbf{k}=0} \sum_{r=0}^p \partial^r_{\xi^r}\varphi(\mathbf{k}, 0)\frac{\xi^r}{r!} >= \sum_{r=0}^p < u, \chi_{\mathbf{k}=0}\frac{\xi^r}{r!} > \partial^r_{\xi^r}\varphi(0),$$

which gives the definition of the constants $(c_r)_{r\in\{0,\dots,p\}}$.

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