ON A STRUCTURED MODEL FOR THE LOAD DEPENDENT REACTION KINETICS OF TRANSIENT ELASTIC LINKAGES MEDIATING NONLINEAR FRICTION. *

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Abstract. We consider a microscopic model for friction mediated by transient elastic linkages introduced in [3, 5]. In this study we extend results and the general approach employed in [2]. We introduce a new unknown and reformulate the model. Based on this framework, we derive new a-priori estimates. In a first step this approach allows us to reproduce results of [2] concerning the convergence of the system to a macroscopic friction law in the semi-coupled case, but under weaker assumptions. Furthermore we consider the fully coupled case and prove existence and uniqueness of the solution.

Key words. friction coefficient, protein linkages, cell adhesion, renewal equation, effect of chemical bonds, integral equation, Volterra kernel.

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1. Introduction. In this study we consider the following system of equations which describes the evolution of the time-dependent position of a single binding site denoted by \( z_\varepsilon(t) \in \mathbb{R} \),

\[
\begin{cases}
\frac{1}{\varepsilon} \int_0^\infty (z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)) \rho_\varepsilon(t,a) \, da = f(t) , & t \geq 0 , \\
z_\varepsilon(t) = z_p(t) , & t < 0 ,
\end{cases}
\]

where the function \( f = f(t) \in \mathbb{R} \) represents a given exterior force. The known past positions are given by the Lipschitz function \( z_p(t) \in \mathbb{R} \) for \( t < 0 \). The dimensionless parameter \( \varepsilon \) denotes the typical age of linkages as compared to the timescale of the problem and hence represents the rate of linkage turnover. The time-dependent density function \( \rho_\varepsilon(t,a) \) represents the age-distribution of the linkages and solves itself an aged structured problem with a non-local boundary term,

\[
\begin{cases}
\varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta_\varepsilon \rho_\varepsilon = 0 , & t > 0 , a > 0 , \\
\rho_\varepsilon(a = 0, t) = \beta_\varepsilon(t) \left( 1 - \int_0^\infty \rho_\varepsilon(t, a) \, da \right) , & t > 0 , \\
\rho_\varepsilon(a = 0, t = 0) = \rho_{I,\varepsilon}(a) , & a \geq 0 ,
\end{cases}
\]

with the kinetic rate functions \( \beta_\varepsilon = \beta_\varepsilon(t) \in \mathbb{R}_+ \) and \( \zeta_\varepsilon \in \mathbb{R}_+ \), both possibly depending on the dimensionless parameter \( \varepsilon > 0 \). If the off-rate

\[
\zeta_\varepsilon := \zeta_\varepsilon(a, t)
\]

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is a given function, then we call the system semi-coupled: one can solve (1.2) to obtain \( \rho_\varepsilon \) in a first step and then use it in order to solve for \( z_\varepsilon \). From a modeling point of view the more interesting case consists in defining \( \zeta_\varepsilon \) such that it depends on \( z_\varepsilon \), after scaling, as

\[
\zeta_\varepsilon(t,a) := \zeta \left( \frac{|z_\varepsilon(t) - z_\varepsilon(t-a)|}{\varepsilon} \right), \quad t \geq 0, \quad a \geq 0,
\]

for a given monotonically increasing function \( \zeta = \zeta(s) > 0 \). In this case we call the system fully coupled since \( \rho_\varepsilon \) and \( z_\varepsilon \) are now interdependent. In fact by (1.4) the linkages’ off-rate depends on their extension i.e on their mechanical load. A typical situation is, for example, an exponential increase of the off-rate as the elastic linker is extended, \( \zeta_\varepsilon = \zeta_0 \exp \left( \frac{|z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)|}{\varepsilon} \right) \) (cf. [7, 1]).

We are especially interested in the behavior of solutions as \( \varepsilon \) tends to zero. The tuple \((\rho_0, z_0)\) satisfies the formal limit of the system above

\[
\begin{align*}
\mu_{1,0} \partial_t z_0 &= f \quad \text{with} \quad \mu_{1,0}(t) := \int_0^\infty a \rho_0(t,a) \, da, \quad t > 0, \\
z_0(t=0) &= z_p(0),
\end{align*}
\]

where the limit distribution \( \rho_0 \) is explicitly given by

\[
\rho_0(t,a) = \frac{1}{\beta_0(t)} + \int_0^\infty \exp \left( - \int_0^a \zeta_0(\tilde{a}) \, d\tilde{a} \right) \, da \left( 1 - \int_0^a \rho_0(t,\tilde{a}) \, d\tilde{a} \right),
\]

being the solution of

\[
\begin{align*}
\partial_t \rho_0 + \zeta_0 \rho_0 &= 0, \quad t > 0, \quad a > 0, \\
\rho_0(t,a = 0) &= \beta_0(t) \left( 1 - \int_0^\infty \rho_0(t,\tilde{a}) \, d\tilde{a} \right), \quad t > 0.
\end{align*}
\]

In the fully coupled case (1.4) the limit off-rate is given by

\[
\zeta_0 = \zeta(a |\partial_t z_0|),
\]

whereas in the semi-coupled case \( \zeta_0 \) is defined as the limit of the family \( \zeta_\varepsilon \) as \( \varepsilon \) tends to zero.

Combining (1.5) and (1.6) we are able to give an explicit expression for the viscosity constant \( \mu_{1,0} \), which represents the macroscopic friction effect, in terms of the microscopic rate constants. In the general fully coupled case (1.4) the viscosity \( \mu_{1,0} \) depends on the velocity of the binding site \( \partial_t z_0 \) and it is given by

\[
\mu_{1,0}(t) = \frac{\int_0^\infty a \exp \left( - \int_0^a \zeta_0(\tilde{a} |\partial_t z_0|) \, d\tilde{a} \right) \, da}{\beta_0(t)} + \frac{\int_0^\infty \exp \left( - \int_0^a \zeta_0(\tilde{a} |\partial_t z_0|) \, d\tilde{a} \right) \, da}{\beta_0(t)}.
\]

For the semi-coupled system (1.1), (1.2), (1.3) the viscosity \( \mu_{1,0} \), however, does not depend on the velocity of the binding site, and, in the special case where the limit off-rate does also not depend on age, \( \zeta_0 = \zeta_0(t) \), the viscosity constant is given by

\[
\mu_{1,0}(t) = \frac{1}{\zeta_0(t)(1 + \zeta_0(t)/\beta(t))}.
\]
Friction effects mediated by protein linkages play an important role in the modeling of intracellular processes related to cell movement, vesicle transport, cell shape changes, mitosis and many more. In [4, 5] a microscopic mathematical model for the effect of elastic protein bonds attaching transiently to moving binding sites has been introduced. The model was formulated having cytoskeletal cross-linker proteins in mind as well as integrins, yet its range of possible applications reaches as far as to the description of rubber friction (cf. [6]).

In the semi-coupled case (1.3), the authors gave a first series of results concerning existence and uniqueness for fixed \( \varepsilon \) in [2]. They proved as well the convergence of \((\rho_\varepsilon, z_\varepsilon)\) to \((\rho_0, z_0)\) when \( \varepsilon \) goes to 0. In this study it is our aim to introduce a new analytic method to deal with the system (1.1), (1.2) and either (1.3) or (1.4) and to obtain new results based on it. To this end we introduce the new variable \( u_\varepsilon \) as the solution of the system

\[
\begin{cases}
\varepsilon \partial_t u_\varepsilon + \partial_a u_\varepsilon = \frac{1}{\mu_{0,\varepsilon}} \left( \varepsilon \partial_t f + \int_0^\infty \zeta_\varepsilon u_\varepsilon \rho_\varepsilon \, da \right), & t > 0, \ a > 0, \\
u_\varepsilon(t, 0) = 0, & t > 0, \\
u_\varepsilon(0, a) = u_{I,\varepsilon}(a), & a \geq 0,
\end{cases}
\]

where \( \mu_{0,\varepsilon}(t) := \int_0^\infty \rho_\varepsilon(\tilde{a}, t) \, d\tilde{a} \) and

\[
u_{I,\varepsilon}(a) := \frac{z_\varepsilon(0) - z_p(-\varepsilon a)}{\varepsilon},
\]

and where according to (1.1) it holds that

\[
z_\varepsilon(0) = \frac{1}{\mu_{0,\varepsilon}(0)} \left( \int_0^\infty z_p(-\varepsilon a) \rho_{I,\varepsilon}(a) \, da + \varepsilon f(0) \right).
\]

In fact \( z_\varepsilon \) being the solution of (1.1) and \( u_\varepsilon \) being the solution of (1.8) contain the same information. We state, on the one hand, that, given \( z_\varepsilon \), one actually obtains the function \( u_\varepsilon \) according to

\[
u_\varepsilon(t, a) = \begin{cases}
\frac{z_\varepsilon(t) - z_\varepsilon(t-\varepsilon a)}{\varepsilon}, & t > \varepsilon a, \\
\frac{z_\varepsilon(t)}{\varepsilon} - \frac{z_\varepsilon(t - \varepsilon a)}{\varepsilon}, & t \leq \varepsilon a.
\end{cases}
\]

On the other hand, given \( u_\varepsilon \) one obtains \( z_\varepsilon \) evaluating

\[
z_\varepsilon(t) = z_\varepsilon(0) + \int_0^t \frac{1}{\mu_{0,\varepsilon}(\tilde{t})} \left( \varepsilon \partial_{\tilde{t}} f(\tilde{t}) + \int_0^\infty \zeta_\varepsilon(\tilde{t}, a) u_\varepsilon(\tilde{t}, a) \, d\tilde{t} \right) \, d\tilde{t}.
\]

Finally the original integral equation (1.1) may be recasted as

\[
\int_0^\infty \rho_\varepsilon(t, a) \, u_\varepsilon(t, a) \, da = f(t), \quad t \geq 0,
\]

and in the fully coupled case (1.4) is replaced by

\[
\zeta_\varepsilon = \zeta(|u_\varepsilon|),
\]

which defines the coupling with (1.2) in a straightforward way.
In our analysis system (1.8) replaces the original integral equation (1.1) and allows to derive a priori bounds for \( u_\varepsilon \). First we obtain the previously unknown a priori estimate

\[
\int_0^\infty \rho_\varepsilon(t,a) |u_\varepsilon(t,a)| \, da \leq \int_0^\infty \rho_{I,\varepsilon}(a) |u_{I,\varepsilon}(a)| \, da + \int_0^t |\partial_t f| \, dt,
\]

which holds provided \( \zeta_\varepsilon \geq 0 \) only. Observe that the estimate formulated in (1.13) includes a \( \varepsilon \)-dependent weight-function. Then under the supplementary hypothesis that \( \zeta_\varepsilon \) is bounded from above, we get a pointwise bound on \( u_\varepsilon \)

\[
|u_\varepsilon(t,a)| \leq \alpha_0 + \alpha_1a, \quad \forall (t,a) \in (0,T) \times \mathbb{R}_+,
\]

where the coefficients \( \alpha_0 \) and \( \alpha_1 \) depend on the data, on \( T \) but not on \( \varepsilon \).

In the semi-coupled case (1.3) we obtain weak convergence results for \( u_\varepsilon \) converging towards the formal limit of the model (1.8). Finally this argument ensures also the strong convergence of \( z_\varepsilon \) to \( z_0 \), thus reproducing the results obtained in [2]. The main advantage of this approach is that no hypotheses are required on the monotonicity of \( \zeta_\varepsilon \).

In the fully coupled case (1.4), we prove existence and uniqueness of a solution to (1.2), (1.8), (1.12) for a fixed \( \varepsilon \).

The analysis in this paper relies on the following set of assumptions. The initial data for the density model (1.2) satisfies the following hypotheses.

**Assumption 1.1.** The initial condition \( \rho_{I,\varepsilon} \in L^\infty_a(\mathbb{R}_+) \) is

(i) nonnegative, i.e. \( \rho_{I,\varepsilon}(a) \geq 0 \), a.e. in \( \mathbb{R}_+ \).

(ii) Moreover, the total initial population satisfies

\[
0 < \int_0^\infty \rho_{I,\varepsilon}(a) \, da < 1,
\]

(iii) and higher moments are bounded,

\[
0 < \int_0^\infty a^p \rho_{I,\varepsilon}(a) \, da \leq c_p, \quad \text{for } p = 1, 2,
\]

where \( c_p \) are positive constants depending only on \( p \).

Concerning the integral equation (1.1) and its new analogue (1.8) we assume

**Assumption 1.2.** The time dependent exterior force in (1.1) is represented by a uniform Lipschitz function \( f = f(t) \) on \([0,T]\) for any \( T > 0 \). The past condition \( z_p \) belongs to \( \text{Lip}((\infty,0)) \), the set of uniform Lipschitz functions on \( \mathbb{R}_- \).

For the chemical reaction rates we assume:

**Assumption 1.3.** The dimensionless parameter \( \varepsilon > 0 \) is assumed to induce a family of on-rates for the protein linkages that satisfy

(i) For any \( T > 0 \) the function \( \beta_\varepsilon(t) \) is a uniform Lipschitz function in \([0,T]\).

(ii) For the limit function \( \beta_0 \in L^\infty_t \) it holds that \( \|\beta_\varepsilon - \beta_0\|_{L^\infty_t} \to 0 \) as \( \varepsilon \to 0 \).

(iii) We also assume that there are upper and lower bounds such that

\[
0 < \beta_{\min} \leq \beta_\varepsilon(t) \leq \beta_{\max},
\]

for all \( \varepsilon > 0 \) and \( t > 0 \).
For the off-rates we distinguish between the fully coupled and the semi-coupled cases. In the semi-coupled case we make assumptions for the off-rates which are analogues of Assumptions 1.3.

**Assumption 1.4.** There is a family $\zeta_{\varepsilon}$ of functions that satisfy

(i) for any $T > 0$, it holds that $\zeta_{\varepsilon}(t, a)$ is in $L^\infty((0, T) \times \mathbb{R}_+)$.

(ii) the limit function $\zeta_0 \in W^{1, \infty}(\mathbb{R}_+; L^\infty(\mathbb{R}_+))$, and $\|\zeta_{\varepsilon} - \zeta_0\|_{L^\infty L^\infty} \to 0$ as $\varepsilon \to 0$.

(iii) We also assume that there are upper and lower bounds such that

$$0 < \zeta_{\min} \leq \zeta_{\varepsilon}(t, a) \leq \zeta_{\max}$$

for all $\varepsilon > 0$, $a \geq 0$ and $t > 0$.

In the fully coupled case we assume instead

**Assumption 1.5.** The function $\zeta = \zeta(s)$, $s \in \mathbb{R}$ is Lipschitz-continuous with Lipschitz-constant $\zeta_{lip} := \|\zeta(\cdot)\|_{L^\infty(\mathbb{R})}$ and there are upper and lower bounds such that

$$0 < \zeta_{\min} \leq \zeta_{\varepsilon}(s) \leq \zeta_{\max}$$

for all $s \in \mathbb{R}$.

In order to set up the analytic framework to deal with the function $u_\varepsilon$, we introduce the weight function

$$\omega(a) := \frac{1}{1 + a}$$

and we define the functional space

$$X_T := \left\{ g \in L^\infty_{loc}((0, T) \times \mathbb{R}_+) \text{ s.t. } \sup_{t \in (0, T)} \|g(t, a)\omega(a)\|_{L^\infty} < \infty \right\}$$

and the corresponding norm is denoted $\|\cdot\|_{X_T}$. Given two real times $T_{n+1} > T_n$, one denotes as well $X_{(T_n, T_{n+1})}$ the space where the time interval in the (1.16) is replaced by $(T_n, T_{n+1})$.

Our first result applies to the semi-coupled system of equations (1.1), (1.2) and (1.3) and relaxes the technical assumptions used in [2].

**Theorem 1.1.** Let assumptions 1.1, 1.2, 1.3 and 1.4 hold, then for every fixed $\varepsilon > 0$ there exists a unique solution of the coupled system (1.1), (1.2), (1.3) denoted by $(z_\varepsilon, \rho_\varepsilon) \in \text{Lip}([0, T]) \times (C^0([0, T]; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^+))$. Let $(z_0, \rho_0)$ be the unique solution to the formal limit system (1.3) then for every $T > 0$ it holds that

$$\|z_\varepsilon - z_0\|_{C^0([0, T])} + \|\rho_\varepsilon - \rho_0\|_{C^0([0, T]; L^1(\mathbb{R}_+))} \to 0$$

as $\varepsilon \to 0$.

We consider in a second step the fully coupled system described by (1.2) coupled to (1.8), (1.12) or, alternatively, (1.2), (1.1), (1.4). We prove existence and uniqueness of the solution $(u_\varepsilon, \rho_\varepsilon)$ for a fixed $\varepsilon > 0$.

**Theorem 1.2.** Let assumptions 1.1, 1.2, 1.3 and 1.4 hold, then there exists a unique solution $(\rho_\varepsilon, u_\varepsilon) \in C([0, T]; L^1(\mathbb{R}_+)) \times X_T$ solving the coupled system (1.2) - (1.8) - (1.12) for any positive time $T$. The maximal time of existence is infinite and the stability results Lemma 4.1 (i.e. (1.13)) and Lemma 4.2 (i.e. (1.14)) hold. This results provide existence and uniqueness of the couple $(\rho_\varepsilon, z_\varepsilon) \in C([0, T]; L^1(\mathbb{R}_+)) \times \text{Lip}([0, T])$ solving (1.1) - (1.2) - (1.4).
We underline that such an existence and uniqueness result is completely new. An open problem is still the convergence of such a non-linear model when $\varepsilon$ goes to zero.

The outline of the paper is as follows. First in Section 2 we recall results concerning the $\rho_\varepsilon$ model (1.2) already established in [2]. In Section 3 we introduce then the variable $u_\varepsilon$ and eliminate $z_\varepsilon$ from the system so to express the problem in terms of $(\rho_\varepsilon, u_\varepsilon)$ exclusively. Furthermore, in Section 4 we prove a priori estimates for the reformulated system. These results apply to the semi-coupled case as well as to the fully coupled case. Hence in Section 5 we treat the semi-coupled case and prove the existence of a unique solution of the fully coupled system (1.1), (1.2), (1.4) for fixed $\varepsilon$.

2. Preliminary results. The following preliminary results on the solution $\rho_\varepsilon$ of (1.2) have been obtained in [2].

Theorem 2.1. Let assumptions 1.1, 1.3 and 1.4 hold, then for every fixed $\varepsilon$ there exists a unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of the problem (1.2). It satisfies (1.2) in the sense of characteristics, namely

$$
\rho_\varepsilon(t,a) = \begin{cases} 
\beta_\varepsilon(t-a) (1 - \int_0^\infty \rho_\varepsilon(\tilde{a}, t-a) \, d\tilde{a}) \times \\
\times \exp \left( - \int_0^a \zeta(\tilde{a}, t-a) \, d\tilde{a} \right), & \text{if } a < t/\varepsilon, \\
\rho_I(\varepsilon a) \exp \left( - \frac{1}{\varepsilon} \int_0^a \zeta((t-t)/\varepsilon + a, t) \, dt \right), & \text{if } a \geq t/\varepsilon.
\end{cases}
$$

Lemma 2.1. Let $\rho_\varepsilon$ be the unique solution of problem (1.2) according to Theorem 2.1 then it satisfies a weak formulation of this problem, namely

$$
\int_0^\infty \int_0^T \rho_\varepsilon(t,a) (\varepsilon \partial_t \varphi + \partial_a \varphi + \zeta_\varepsilon \varphi) \, dt \, da - \varepsilon \int_0^\infty \rho_\varepsilon(t,a) \varphi(a, t=T) \, da + \\
+ \int_0^T \rho_\varepsilon(a=0,t) \varphi(0,t) \, dt + \varepsilon \int_0^\infty \rho_I(\varepsilon a) \varphi(a, t=0) \, da = 0,
$$

for every $T > 0$ and every test function $\varphi \in C^\infty(\mathbb{R}_+^2) \cap L^\infty(\mathbb{R}_+^2)$.

The following two Lemmas formulate bounds on the moments of $\rho_\varepsilon$ which we denote by

$$
\mu_{p,\varepsilon}(t) := \int_0^\infty a^p \rho_\varepsilon(t,a) \, da, \quad \text{where } p = 1, 2.
$$

Lemma 2.2. Let assumptions 1.1, 1.3 and 1.4 hold, then the unique solution $\rho_\varepsilon \in C^0(\mathbb{R}_+; L^1(\mathbb{R}_+)) \cap L^\infty(\mathbb{R}_+^2)$ of the problem (1.2) from Theorem 2.1 satisfies

$$
\rho_\varepsilon(t,a) \geq 0 \quad \text{a.e. in } \mathbb{R}_+^2 \quad \text{and}
$$

$$
\mu_{0,\min} \leq \mu_{0,\varepsilon}(t) < 1, \quad \forall t \in \mathbb{R}_+ \quad \text{where } \mu_{0,\min} := \min \left( \mu_{0,\varepsilon}(0), \frac{\hat{\beta}_{\min}}{\hat{\beta}_{\min} + \zeta_{\max}} \right).
$$
In a more straightforward manner one gets for higher moments as well

**Lemma 2.3.** Let assumption [1.1] hold, then

\[ \mu_{p,\min} < \mu_{p,\varepsilon}(t) \leq k \quad \text{for} \quad p = 1, 2, \quad \text{where} \quad \mu_{p,\min} := \min \left( \mu_{p,\varepsilon}(0), \frac{\mu_{p-1,\min}}{\zeta_{\max}} \right), \]

and the generic constant \( k \) is independent of both time and \( \varepsilon \).

Furthermore the following results on the convergence of \( \rho_{\varepsilon} \) as \( \varepsilon \) tends to 0 have been obtained. We define the Lyapunov functional

\[ \mathcal{H}[u] := \int_0^\infty |u(a)| \, da + \int_0^\infty |u(a)| \, da, \]

and we obtain

**Lemma 2.4.** Let \( \zeta_{\min} > 0 \) be the lower bound to \( \zeta_{\varepsilon}(t, a) \) according to assumption [1.4] then it holds for all \( t \geq 0 \) that

\[ \mathcal{H}[\rho_{\varepsilon}(t, \cdot) - \rho_0(t, \cdot)] \leq \mathcal{H}[\rho_{\varepsilon, I} - \rho_0(0, \cdot)] e^{-\zeta_{\min} t} + \frac{2}{\zeta_{\min}} \left\| \mathcal{R}_{\varepsilon} \right\|_{L^1_a(R^+)} + \left\| M_\varepsilon \right\|_{L^\infty_a(R^+)} \]

with \( \mathcal{R}_{\varepsilon} := -\varepsilon \partial_t \rho_0 - \rho_0(\zeta_{\varepsilon} - \zeta_0) \) and \( M_\varepsilon := (\beta_\varepsilon - \beta_0) \left( 1 - \int_0^\infty \rho_0 \, da \right) \). As a consequence we conclude

**Theorem 2.2.** Let \( \rho_{\varepsilon} \) be the solution to the system [1.2] according to Theorem 2.1 and let the \( \rho_0 \) be as defined in [1.6], then it holds that

\[ \rho_{\varepsilon} \to \rho_0 \quad \text{in} \quad C^0([0, \infty); L^1_a(R^+)) \quad \text{as} \quad \varepsilon \to 0, \]

where the convergence with respect to time is in the sense of uniform convergence on compact subintervals.

**Remark 2.1.** Note that in general \( \rho_{\varepsilon, I} \) does not converge to \( \rho_0(0, \cdot) \) in \( L^1_a \) as \( \varepsilon \to 0 \). A boundary layer will be observable if their difference does not oscillate and its profile will be shaped like a multiple of \( e^{-\zeta_{\min} t} \), which is again a consequence of Lemma 2.4.

In the opposite case we obtain

**Corollary 2.1.** Considering the asymptotic behavior as \( \varepsilon \to 0 \) Under the additional assumption that \( \rho_{\varepsilon, I} \to \rho_0(0, \cdot) \) in \( L^1_a(R^+) \) it holds by coercivity that \( \mathcal{H}[\rho_{\varepsilon, I} - \rho_0(0, \cdot)] \to 0 \) and therefore the convergence \( \rho_{\varepsilon} \to \rho_0 \) in \( L^1_a \) is uniform with respect to \( t \in R^+ \). In fact it holds that

\[ \left\| \rho_{\varepsilon} - \rho_0 \right\|_{L^\infty a L^1_a} \leq \sup_{t \geq 0} \mathcal{H}[\rho_{\varepsilon}(t, \cdot) - \rho_0(t, \cdot)] \leq \mathcal{H}[\rho_{\varepsilon, I} - \rho_0(0, \cdot)] + \frac{2}{\zeta_{\min}} \left\| \mathcal{R}_{\varepsilon} \right\|_{L^1_a(R^+)} + \left\| M_\varepsilon \right\|_{L^\infty_a(R^+)}. \]

We provide estimates on the convergence of the first moment as well.

**Lemma 2.5.** Let \( \rho_{\varepsilon} \) be the solution to the system [1.2] according to Theorem 2.1 and let \( \rho_0 \) be defined as in [1.6], then it holds for \( t > 0 \) that

\[ \int_0^t a|\rho_{\varepsilon} - \rho_0| \, da \leq e^{-\zeta_{\min} t} \int_0^\infty a|\rho_{\varepsilon, I}(a) - \rho_0(a, 0)| \, da + \frac{1}{\zeta_{\min}} C_\varepsilon, \]

where the family of constants \( C_\varepsilon \in \mathbb{R} \) is such that \( C_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

**Remark 2.2.** Integrating in time estimates from Lemma 2.4 and above, one concludes that \( (1 + a)\rho_{\varepsilon} \) actually converges to \( (1 + a)\rho_0 \) strongly in \( L^1((0, T) \times \mathbb{R}^+) \) for any finite time \( T \).
3. Reformulating the Volterra integral equation. Given the functions $\rho_\varepsilon$ and $\zeta_\varepsilon$, our goal in this section is to replace $z_\varepsilon = z_\varepsilon(t)$ satisfying (1.1) by a new quantity which we denote by $u_\varepsilon = u_\varepsilon(t,a)$. It is defined as the solution of (1.8). Observe that in Section 5 and Section 6 we will actually prove the existence of a unique solution to (1.8).

In the following two results we state, on the one hand, that, given $z_\varepsilon$, one actually obtains the function $u_\varepsilon$ according to (1.9) and that on the other hand, given $u_\varepsilon$, one recovers $z_\varepsilon$ evaluating (1.10). Finally we also show that as a consequence of (1.8) the original equation (1.1) transforms into (1.11).

In this paper we consider ”mild” solutions to (1.8) which satisfy (1.8) after integration along characteristics, namely

$$u_\varepsilon(t,a) := \begin{cases} \int_0^t \frac{g(t,s-a)}{s-a} ds, & t > \varepsilon a, \\ \int_0^t \frac{g(t,s-a)}{s-a} ds + u_{I,\varepsilon}(a-t/\varepsilon), & t \leq \varepsilon a, \end{cases}$$

where

$$g(t) := \frac{1}{\mu_0(\varepsilon)} \left( \int_0^\infty \rho_\varepsilon(t,a) \zeta_\varepsilon u_\varepsilon(t,a) da \right).$$

In analogy to Lemma 2.1 any such function is also a weak solution of (1.8) in the sense that it satisfies

$$- \int_0^T \int_0^\infty u_\varepsilon(\varepsilon \partial_t \varphi + \partial_a \varphi) \, ds, \, \varepsilon dt + \varepsilon \int_0^\infty u_\varepsilon(s,a) \varphi(s,a) \, ds = \int_0^\infty \frac{1}{\mu_0(\varepsilon)} \left( \varepsilon \partial_t \varphi + \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \, da \right) \left( \int_0^\infty \varphi(t,a) \, da \right) \, dt$$

for any function $\varphi \in C_\varepsilon([0,T];C_\varepsilon(\mathbb{R}^+))$.

**Lemma 3.1.** Let $z_\varepsilon$ be a Lipschitz-continuous solution to (1.1), then the function $u_\varepsilon$ recovered according to (1.9) satisfies the system in the sense of integration along characteristics (3.1).

**Proof.** In fact, the function $u_\varepsilon$ if defined by (1.9) solves the following system

$$u(t,a) := \begin{cases} \int_0^a z'_\varepsilon(t-s-a) ds, & t > \varepsilon a, \\ \int_0^{a/\varepsilon} z'_\varepsilon(t-s-a) ds + u_{I,\varepsilon}(a-t/\varepsilon), & t \leq \varepsilon a. \end{cases}$$

Hence the weak formulation of the following system holds

$$\int_0^\infty \varepsilon \partial_t u_\varepsilon + \partial_a u_\varepsilon = z'_\varepsilon(t), \quad a > 0, \quad t > 0,$$

$$u_\varepsilon(t,0) = 0, \quad t > 0,$$

$$u_\varepsilon(0,a) = u_{I,\varepsilon}, \quad a \geq 0.$$

Testing against $\rho_\varepsilon$ and integrating in age, one gets that

$$- \int_0^\infty \rho_\varepsilon u_\varepsilon da + \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon da = z'_\varepsilon(t) \mu_0(\varepsilon(t),$$

but then one uses again the fact $z_\varepsilon$ solves (1.1), which in term of $u_\varepsilon$ means (1.11). According to Lemma 2.2 it holds that $\mu_0(\varepsilon) > \mu_{0,\min}$, hence we might isolate $z'_\varepsilon(t)$ in (3.4) and combine it with (1.11) and (3.3) to obtain (3.1). □

**Lemma 3.2.** Let $u_\varepsilon$ satisfy (1.8) in the sense of (3.1), then the reformulated version (1.11) of the original equation (1.1) holds and $z_\varepsilon$ recovered from $u_\varepsilon$ according to (1.10) satisfies (1.1).
Proof. Testing (1.8) against $\rho_\varepsilon$ and integrating in age gives

$$\varepsilon \frac{d}{dt} \int_0^\infty u_\varepsilon(t,a)\rho_\varepsilon(t,a) \, da = \varepsilon \partial_t f,$$

which after integration in time gives

$$\int_0^\infty u_\varepsilon(t,a)\rho_\varepsilon(t,a) \, da - f(t) = \int_0^\infty u_{1,\varepsilon}(a)\rho_\varepsilon(0,a) \, da - f(0) = 0, \quad \forall t > 0,$$

implying (1.11).

Now we would like to confirm that $z_\varepsilon$ given by (1.10) satisfies (1.1). Using (3.1) and (1.10), one checks that, if $t \geq \varepsilon a$,

$$\frac{z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)}{\varepsilon} = \frac{1}{\varepsilon} \int_{t-\varepsilon a}^t g(\tilde{t})d\tilde{t} = \int_0^a g(t - \varepsilon \tilde{a})d\tilde{a} = u_\varepsilon(t,a),$$

and if $t \leq \varepsilon a$,

$$\frac{z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)}{\varepsilon} = \frac{1}{\varepsilon} \left( \int_0^t g(\tilde{t})d\tilde{t} + z_\varepsilon(0) - z_\varepsilon(t - \varepsilon a) \right)$$

$$= \int_0^{t/\varepsilon} g(t - \varepsilon \tilde{a})d\tilde{a} + u_{1,\varepsilon}(a - t/\varepsilon) = u_\varepsilon(t,a).$$

This gives when evaluating the left hand side of (1.1) that

$$\int_{\mathbb{R}_+} \left( \frac{z_\varepsilon(t) - z_\varepsilon(t - \varepsilon a)}{\varepsilon} \right) \rho_\varepsilon(t,a) \, da = \int_{\mathbb{R}_+} u_\varepsilon(t,a)\rho_\varepsilon(t,a) \, da,$$

which thanks to (1.11) proves that $z_\varepsilon$ actually solves (1.1). $\square$

In the rest of the paper we solve the coupled model for the tuple $(\rho_\varepsilon, u_\varepsilon)$.

4. A priori estimates. The uniform a priori estimates below will be very useful for the further analysis.

Lemma 4.1. If $\zeta_\varepsilon \geq 0$, then system (1.8) admits the $\varepsilon$-uniform estimate (1.13).

Proof. Multiplying formally the equation by $\text{sign}(u_\varepsilon)$, testing against $\rho_\varepsilon$ and integrating with respect to $a$ gives

$$\varepsilon \partial_t \int_0^\infty \rho_\varepsilon(t,a)|u_\varepsilon(t,a)| \, da + \int_0^\infty \rho_\varepsilon\zeta_\varepsilon|u_\varepsilon| \, da \leq \varepsilon |\partial_t f| + \int_0^\infty \rho_\varepsilon\zeta_\varepsilon|u_\varepsilon| \, da.$$

On both sides the same term $\int_0^\infty \zeta_\varepsilon \rho_\varepsilon |u_\varepsilon| \, da$ appears and thus cancels. One concludes then directly the claim after integration in time. $\square$

The following result refers to the typical profile in age of the function $u_\varepsilon$.

Lemma 4.2. Let $\zeta_\varepsilon$ be such that $0 \leq \zeta_{\min} \leq \zeta_\varepsilon \leq \zeta_{\max}$, then on any fixed time interval $(0,T)$ the profile $a \mapsto u_\varepsilon(t,a)$ is at most linear in age. Indeed, if we suppose that there are two constants $\alpha_0$ and $\alpha_1$ such that

- the growth factor of the rhs is controlled,

$$\alpha_1 \geq \frac{1}{\rho_{0,\min}} \left\{ \| \partial_t f \|_{L^\infty(0,T)} + \zeta_{\max} \left( \int_0^\infty \rho_{1,\varepsilon}|u_{1,\varepsilon}(a)| \, da + \int_0^T |\partial_t f| \, ds \right) \right\},$$
as well as the Lipschitz constant of the past data,
\[ \alpha_1 \geq \|z'\|_{L^\infty(\mathbb{R}^-)} , \quad \alpha_0 \geq \frac{1}{\mu_{0,\min}} \left( \|z'\|_{L^\infty(\mathbb{R}^-)} \mu_{1,\varepsilon}(0) + f(0) \right) . \]

Then one has
\[ |u_\varepsilon(t, a)| \leq \alpha_0 + \alpha_1 a , \quad (t, a) \in (0, T) \times \mathbb{R}_+ . \]

**Proof.** Observe that
\[ \varepsilon \partial_t (\alpha_0 + \alpha_1 a - |u_\varepsilon|) + \partial_a (\alpha_0 + \alpha_1 a - |u_\varepsilon|) \geq \]
\[ \geq \alpha_1 - \frac{1}{\mu_{0,\min}} \left\{ |\partial_t f| + \int_0^\infty \xi_\varepsilon \rho_\varepsilon |u_\varepsilon| da \right\} \]
\[ \geq \alpha_1 - \frac{1}{\mu_{0,\min}} \left\{ \|\partial_t f\|_{L^\infty(0, T)} + \xi_{\max} \int_0^\infty \rho_\varepsilon |u_\varepsilon| da \right\} . \]

Then, using Lemma 4.1 one recovers that under the first assumption on \( \alpha_1 \) one has
\[ \varepsilon \partial_t (\alpha_0 + \alpha_1 a - |u_\varepsilon|) + \partial_a (\alpha_0 + \alpha_1 a - |u_\varepsilon|) \geq 0 , \quad (t, a) \in (0, T) \times \mathbb{R}_+ . \]

Hence it holds that \( \alpha_0 + \alpha_1 a - |u_\varepsilon| \geq 0 \) along the characteristics of the transport operator \( \varepsilon \partial_t + \partial_a \) provided this quantity is nonnegative at the boundaries where \( a = 0 \) or \( t = 0 \) respectively. When \( a = 0 \), it is straightforward to observe that \( \alpha_0 - |u_\varepsilon| = \alpha_0 \geq 0 \). On the other hand, when \( t = 0 \), the boundary term is \( \alpha_0 + \alpha_1 a - |u_\varepsilon| \) and we need to estimate the initial datum,
\[ |u_{1,\varepsilon}(a)| = \left| \frac{z_\varepsilon(0) - z_p(-a)}{\varepsilon} \right| \leq \left| \frac{z_\varepsilon(0) - z_p(0)}{\varepsilon} \right| + \left| \frac{z_p(0) - z_\varepsilon(-a)}{\varepsilon} \right| = \]
\[ = \frac{1}{\mu_{0,\varepsilon}(0)} \int_0^\infty (z_\varepsilon(-a) - z_p(0)) \rho_\varepsilon(a, 0) da + \left| \frac{z_p(0) - z_\varepsilon(-a)}{\varepsilon} \right| \leq \frac{1}{\mu_{0,\min}} \left( \|z'_\varepsilon\|_{L^\infty(\mathbb{R}^-)} \mu_{1,\varepsilon}(0) + f(0) \right) + \|z'_\varepsilon\|_{L^\infty(\mathbb{R}^-)} a \leq \alpha_0 + \alpha_1 a . \]

Due to the assumptions on \( \alpha_0 \) and \( \alpha_1 \) this ends the proof. \( \Box \)

**Lemma 4.3.** Under assumptions \( \text{I.2} \) or \( \text{I.3} \), let \((\rho_\varepsilon, u_\varepsilon)\) be the solution of problem \( \text{I.2} \) or \( \text{I.3} \), then \( z_\varepsilon \) given by formula \( \text{I.10} \) is a Lipschitz continuous function on any finite time interval \((0, T)\).

**Proof.** The proof is a straightforward consequence of Lemmas 4.1 and 4.2. They provide an \( L^\infty \) bound uniform in \( \varepsilon \) on the rhs of \( \text{I.8} \). Thanks to formula \( \text{I.10} \) one gets directly the Lipschitz continuity of \( z_\varepsilon \). \( \Box \)

**Remark 4.1.** Lemma \( \text{I.3} \) completes Lemma 3.2 and proves the equivalence between \((\rho_\varepsilon, u_\varepsilon)\) solutions of system \( \text{I.2} \) and \( \text{I.3} \), \((\rho_\varepsilon, z_\varepsilon)\) solutions of \( \text{I.1} \) and \( \text{I.3} \), \((\rho_\varepsilon, z_\varepsilon)\) solutions of \( \text{I.1} \) and \( \text{I.3} \), solving system \( \text{I.2} \) or \( \text{I.1} \), which are in \( C([0, T]; L^1(\mathbb{R}_+)) \times C([0, T]; L^\infty(\mathbb{R}_+, \omega)) \) and \((\rho_\varepsilon, z_\varepsilon)\), solving system \( \text{I.2} \) or \( \text{I.1} \), which are in \( C([0, T]; L^1(\mathbb{R}_+)) \times \text{Lip}(0, T) \). This results holds in both semi and fully coupled cases.

5. Existence of solutions and convergence in the semi-coupled case. In this section we consider the semi-coupled case which consists of the equations \( \text{I.2} \) and \( \text{I.3} \) coupled to either \( \text{I.8} \) or \( \text{I.1} \) and prove Theorem 1.1. The framework for the analysis is the function space defined in \( \text{I.16} \) which relies on the weight
function $\omega$ defined in (1.15). The following result is a straightforward consequence of the definition.

**Theorem 5.1.** Let the Assumptions 1.1, 1.2, 1.3 and 1.4 hold and let $\rho_\varepsilon$ be the unique solution of (1.2) according to Theorem 2.1, then for any fixed $\varepsilon$ and any $T > 0$ there exists a unique $u_\varepsilon \in X_T$ solving problem (1.8). Moreover the maximal time of existence is infinite and stability results stated in Lemmas 4.1 and 4.2 hold.

**Proof.** The proof follows by a fixed point argument. We define the mapping $\Phi(w) = u$ such that

$$\begin{cases} u(t,a) = \left\{ \begin{array}{ll} \int_0^t h(t - \varepsilon \tilde{a}) \, d\tilde{a}, & t > \varepsilon a, \\ \int_0^{t/\varepsilon} h(t - \varepsilon \tilde{a}) \, d\tilde{a} + u_{I,\varepsilon}(a - t/\varepsilon), & t \leq \varepsilon a, \end{array} \right. \end{cases}$$

where $h(t) = (\varepsilon \partial_t f + \int_0^\infty \rho_\varepsilon(t,a)\zeta_\varepsilon(t,a)w(t,a) \, da)/\mu_0(t)$. A simple computation shows that

$$\|u\|_{X_T} \leq \|h\|_{L^\infty(0,T)} + \|u_{I,\varepsilon}\|_{L^\infty(\mathbb{R},\omega)}$$

which allows then with the specific definition of $h$ to write:

$$\|u\|_{X_T} \leq \frac{1}{\mu_{0,\min}} \varepsilon \|\partial_t f\|_{L^\infty(0,T)} + \zeta_{\max} \left( 1 + \frac{k}{\mu_{0,\min}} \right) \|u\|_{X_T} + \|u_{I,\varepsilon}\|_{L^\infty(\mathbb{R},\omega)},$$

where $k$ is the constant from Lemma 2.3. This proves that $\Phi$ is an endomorphism for any given time $T$. By similar arguments one shows as well that if we set $u_i = \Phi(w_i)$ for $i \in \{1,2\}$, then it holds that

$$\|u_1 - u_2\|_{X_T} \leq C_2 \frac{T}{\varepsilon} \|w_1 - w_2\|_{X_T},$$

for a constant $C_2 > 0$. Choosing then $T < \varepsilon/C_2$ proves local existence in time in the interval $[0,T]$ by the Banach-Picard fixed point theorem. As the contraction time does not depend on the initial data, we can extend the same result by continuation and existence and uniqueness in $X_T$ follow for any $T > 0$.

Next we obtain a weak convergence result for $u_\varepsilon$ which in a second step implies the strong convergence of $z_\varepsilon$.

**Theorem 5.2.** Under the assumptions of Theorem 5.1 one has

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly-* in } X_T$$

as $\varepsilon \to 0$, where $u_0$ satisfies

$$\begin{cases} \partial_a u_0 = \int_0^\infty \zeta_0 \rho_0 \, da, & t \geq 0, \quad a > 0, \\ u_0(t,0) = 0 & t \geq 0, \quad a = 0. \end{cases}$$

and

$$\int_0^\infty u_0(t,a) \rho_0(t,a) \, da = f(t), \quad a.e \ t \in (0,T).$$

Furthermore it also holds that

$$z_\varepsilon \to z_0 \text{ strongly in } L^\infty(0,T) \text{ as } \varepsilon \to 0,$$
where \( z_0 = z_p(0) + \int_0^1 \frac{f(t)}{\mu_{\gamma,0}(t)} \, dt \) is the unique solution of (1.5).

**Proof.** As already noticed in Remark 2.2 Theorem 2.2 and Lemma 2.5 imply that

\[
(1 + \alpha) \rho_\varepsilon \to (1 + \alpha) \rho_0
\]

in \( L^1((0, T) \times \mathbb{R}_+) \) strongly. On the other hand, one has, by Lemma 4.2 that \( \|u_\varepsilon\|_{X_T} \leq \max\{\alpha_0, \alpha_1\} \), which implies (5.2), i.e.

\[
\frac{u_\varepsilon}{1 + \alpha} \to \frac{u_0}{1 + \alpha}
\]

in \( L^\infty((0, T) \times \mathbb{R}_+) \) in the weak-* sense for a limit function \( u_0 \in X_T \). As \( \zeta_\varepsilon \to \zeta_0 \) in \( L^\infty((0, T) \times \mathbb{R}_+) \) by Assumption 1.4, one concludes that for every \( \psi \in L^\infty((0, T) \times \mathbb{R}_+) \) one has

\[
\int_0^T \int_0^\infty (\zeta_\varepsilon u_\varepsilon \rho_\varepsilon \psi) \, da \, dt \to \int_0^T \int_0^\infty (\zeta_0 u_0 \rho_0 \psi) \, da \, dt.
\]

Observe that this implies the weak convergence of \( \int_0^\infty (\zeta_\varepsilon u_\varepsilon \rho_\varepsilon \psi) \, da \) in \( L^1(0, T) \) since we might choose \( \psi = \psi(t) \). Passing hence to the limit \( \varepsilon \to 0 \) in (3.2) we obtain that \( u_0 = w(t) a \) for a function \( w = w(t) \) due to Lemma 3.2.

By analogous arguments one obtains the weak convergence of \( \int_0^\infty u_\varepsilon \rho_\varepsilon \, da \) to \( \int_0^\infty u_0 \rho_0 \, da \) in \( L^1(0, T) \). Hence one concludes that the limit satisfies the identity (5.4) which proves that \( u_0 = w(t) a \) where \( w(t) = f(t)/\mu_{\gamma,0}(t) \).

A triangular inequality gives that

\[
\left| \int_0^T \frac{1}{\mu_{0,\varepsilon}(t)} \int_0^\infty \zeta_\varepsilon \rho_\varepsilon u_\varepsilon \, da \, dt - \int_0^T \frac{1}{\mu_{0,0}(t)} \int_0^\infty \zeta_0 \rho_0 u_0 \, da \, dt \right| \leq
\]

\[
\left( \int_0^T \left| \frac{1}{\mu_{0,\varepsilon}(t)} - \frac{1}{\mu_{0,0}(t)} \right| \, dt \right) \zeta_{\max} \left\| \int_0^\infty \rho_\varepsilon |u_\varepsilon| \, da \right\|_{L^\infty(0, T)} +
\]

\[
\left| \int_0^T \frac{1}{\mu_{0,0}(t)} \int_0^\infty (\zeta_\varepsilon \rho_\varepsilon u_\varepsilon - \zeta_0 \rho_0 u_0) \, da \, dt \right|
\]

where both terms on the right hand side tend to zero as \( \varepsilon \to 0 \) thanks to the weak convergence of \( \int_0^\infty \zeta_\varepsilon u_\varepsilon \rho_\varepsilon \, da \) in \( L^1(0, T) \) combined with the strong convergence of \( 1/\mu_{0,\varepsilon} \) in \( L^1(0, T) \) due to Theorem 2.2 and Lemma 2.2. This allows to pass to the limit in the third term in the right hand side of (1.10). Moreover, as \( z'_{\gamma(t)} \) is uniformly bounded an easy check gives that \( z_\varepsilon(0) \to z_0(0) \) when \( \varepsilon \) goes to zero. These two facts prove that \( z_\varepsilon \to z_0 \) strongly in \( L^\infty(0, T) \), \( z_0 \) solving:

\[
z_0(t) = z_p(0) + \int_0^t \frac{1}{\mu_{0,0}} \left( \int_0^\infty \zeta_0 u_0 \rho_0 \, da \right) \, dt =
\]

\[
= z_p(0) + \int_0^t \partial_x u_0 \, dt = z_p(0) + \int_0^t \frac{f(t)}{\mu_{1,0}(t)} \, dt,
\]

which concludes the proof. \( \square \)

Finally **Theorem 1.1** summarizes the results of the Theorems 2.2 and 5.2.
6. Existence of a unique solution in the fully coupled case. We prove the result stated in Theorem 1.2 using the Banach Fixed Point theorem.

Proof. For an arbitrary time $T_0 > 0$, which we will determine in the end of the proof, let $A_0 := B_{X_{T_0}}(0, C_0)$ be the ball centered at the origin in $X_{T_0}$ with radius $C_0$. We construct a mapping that given $w \in A_0$ defines the function $\rho_\varepsilon \in C([0, T_0]; L^1(\mathbb{R}_+))$ as the solution of

$$
\begin{align*}
\varepsilon \partial_t \rho_\varepsilon + \partial_u \rho_\varepsilon + \zeta(w) \rho_\varepsilon &= 0, \\
\rho_\varepsilon(a = 0, t) &= \beta_\varepsilon(t) \left(1 - \int_0^\infty \rho_\varepsilon(t, \tilde{a}) \, d\tilde{a}\right), \\
\rho_\varepsilon(a, t = 0) &= \rho_{1,\varepsilon}(a),
\end{align*}
(6.1)
$$

Results from Theorem 2.1 and Lemmas 2.1, 2.2 and 2.3 imply existence and uniqueness of a solution as well as uniform bounds for moments of order up to 2. Then $w$ and $\rho_\varepsilon$ are used as the input in order to compute the function $u \in X_{T_0}$ solving the problem:

$$
\begin{align*}
\varepsilon \partial_t u + \partial_u u &= \frac{1}{\mu_0,\varepsilon} \left(\varepsilon \partial_t f + \int_0^\infty \zeta(w) \, w \, \rho_\varepsilon \, da\right), \\
u(t, 0) &= 0, \\
u(0, a) &= u_{I, \varepsilon}(a),
\end{align*}
(6.2)
$$

By the same arguments as in the proof of Theorem 5.1, the solution $u \in X_{T_0}$ exists for any given function $w \in X_{T_0}$ and $u$ can be controlled by the norm in $X_{T_0}$,

$$
\|u\|_{X_{T_0}} \leq \left(\varepsilon \|\partial_t f\|_{\infty} + \zeta_{\text{max}} \left(1 + \frac{k}{\mu_{0,\text{min}}} \right) \|w\|_{X_{T_0}}\right) \frac{T_0}{\varepsilon} + \|u_{I, \varepsilon}\|_{L^\infty}^\omega,
(6.3)
$$

where $c_1, c_2 > 0$ are constants defined by the preceding computation. Choosing the time $T_0$ such that $T_0 \leq t_0$ where $t_0 := \varepsilon(C_0 - \|u_{I, \varepsilon}\|_{L^\infty})/(c_1 + c_2 C_0)$, one then ensures that $u \in A_0$.

Next, we shall prove that this mapping is contractive in $A_0$. To this end, given two elements $(w_1, w_2)$ and their respective images $(u_1, u_2)$, we define $\tilde{u} := u_2 - u_1$ and $\tilde{w} = w_2 - w_1$. We define also the corresponding densities $(\rho_1, \rho_2)$ and the respective zeroth and first order moments $(\mu_{0, 1}, \mu_{0, 2})$, $(\mu_{1, 1}$ and $\mu_{1, 2}$) as well as their differences $\tilde{\mu}_i := \mu_{i, 2} - \mu_{i, 1}$, $i \in \{0, 1\}$. It holds that

$$
\tilde{u}(t, a) = \begin{cases} 
\int_0^a \tilde{g}(t - \varepsilon \tilde{a}) \, d\tilde{a}, & t > \varepsilon a, \\
\int_0^{\varepsilon a} \tilde{g}(t - \varepsilon \tilde{a}) \, d\tilde{a}, & t \leq \varepsilon a,
\end{cases}
(6.4)
$$

where

$$
\tilde{g}(t) = \varepsilon \partial_t f \left(\frac{1}{\mu_{0, 2}} - \frac{1}{\mu_{0, 1}}\right) + \int_0^\infty \left(\zeta(w_2) \frac{\rho_2}{\mu_{0, 2}} - \zeta(w_1) \frac{\rho_1}{\mu_{0, 1}}\right) \, w \, da.
$$

We estimate

$$
|\tilde{g}(t)| \leq \frac{\varepsilon |\partial_t f|}{\mu_{0, 1} \mu_{0, 2}} |\tilde{\mu}_0| + \sum_{i=1}^4 I_i,
$$
and we detail the right hand side as follows,

\[ I_1 := \int_0^\infty \zeta \left| \frac{\rho_2}{\mu_{0,2}} \right| da \leq \zeta \left| \frac{\rho_2}{\mu_{0,2}} \right| \| \hat{w} \| \| w_2 \| \| w_2 \| \| w_2 \| , \]

\[ I_2 := \int_0^\infty |\hat{\rho}| \left| \zeta(w_1) \right| \frac{1}{\mu_{0,2}} \| w_2 \| da \leq \frac{\zeta_{\text{max}}}{\mu_{0,\text{min}}} \| w_2 \| \| w_2 \| \| w_2 \| \int_0^\infty (1 + a) |\hat{\rho}| da , \]

\[ I_3 := \left( \frac{1}{\mu_{0,1}} \right) \| \hat{\rho} \| \| w_2 \| \| w_2 \| \| w_2 \| \leq \frac{\zeta_{\text{max}}}{\mu_{0,\text{min}}} \left( \frac{1}{\mu_{0,\text{min}}} \right) \| w_2 \| \| w_2 \| \| w_2 \| , \]

\[ I_4 := \int_0^\infty |\hat{w}| \left| \zeta(w_1) \right| \frac{1}{\mu_{0,1}} \| w_2 \| da \leq \zeta_{\text{max}} \left( 1 + \frac{k}{\mu_{0,\text{min}}} \right) \| \hat{w} \| \| w_2 \| . \]

Using the same arguments as in the proof of Lemma 2.4 (see the proof of Lemma 3.3, p. 495 in [2]) we show that

\[ \| \hat{\mu}_0 \|_{L^\infty([0,T_0])} \leq \| \hat{\mu} \|_{L^\infty([0,T_0];L^1(\mathbb{R}))} \leq \frac{2}{\zeta_{\text{min}}} \left( \int_0^\infty |\zeta(w_2) - \zeta(w_1)| \| \rho_2 \| da \right)_{L^\infty([0,T_0])} \]

\[ \leq \frac{2\zeta_{\text{tp}}}{\zeta_{\text{min}}} \left( 1 + \frac{k}{\mu_{0,\text{min}}} \right) \| \hat{w} \| \| w_2 \| , \]

and an analogous result to Lemma 2.3,

\[ \| \hat{\mu}_1 \|_{L^\infty([0,T_0])} \leq \| \hat{\mu} \|_{L^\infty([0,T_0];L^1(\mathbb{R}))} \leq C \| \hat{w} \| \| w_2 \| , \]

for a constant \( C > 0 \). Using these results we obtain

\[ \| w_1 - w_2 \| \| w_2 \| \| w_2 \| \leq \frac{T_0}{\varepsilon} (c_3 + c_4C_0) \| w_1 - w_2 \| \]

which is contractive provided that \( T_0 < t_3 \) where \( t_3 := \varepsilon/(c_3 + c_4C_0) \). Choosing for example \( T_0 < \min(t_0, t_1) \) proves local existence of in time in the interval \([0,T_0]\) by the Banach-Picard fixed point theorem.

We extend that result to longer times by induction. We suppose that the solutions \((\rho_\varepsilon, u_\varepsilon)\) solving (1.2)-(1.4) exist until the time \( T_n \) i.e. \((\rho_\varepsilon, u_\varepsilon) \in C([0,T_n]; L^1(\mathbb{R}_+, (1+a)^2)) \times X_{T_n}\) and that one has the bound:

\[ \|(1+a)^2\rho_\varepsilon\|_{L^\infty([0,T_n];L^1(\mathbb{R}_+))} \leq k_1(1 + \mu_{1,\varepsilon}(0) + \mu_{2,\varepsilon}(0)) , \]

where \( k_1 := 2(1 + 1/\zeta_{\text{min}} + 1/\zeta_{\text{min}}^2) \). Then on the next interval \([T_n, T_{n+1}]\), one uses again a fixed point strategy. We set the mapping defined above by solving (6.1)-(6.2) on \((T_n, T_{n+1})\) with initial datum \( u_\varepsilon(T_n, \cdot) \) for \( u_\varepsilon \) and \( \rho_\varepsilon(T_n, \cdot) \) for \( \rho_\varepsilon \). We denote by \((\rho(w), u(w))\) the solutions of (6.1)-(6.2) on \((T_n, T_{n+1})\) for a given function \( w \in X_{(T_n,T_{n+1})}\). Firstly, we prove using similar arguments as in Lemma 2.4 that

\[ \|(1+a)^2\rho(w)\|_{L^\infty((T_n,T_{n+1});L^1(\mathbb{R}_+))} \leq k_1(1 + \mu_{1,\varepsilon}(T_n) + \mu_{2,\varepsilon}(T_n)) , \]

and similarly to Lemma 2.4 one has as well:

\[ \|(1+a)\hat{\rho}\|_{L^\infty((T_n,T_{n+1});L^1(\mathbb{R}_+))} \leq k_2(1 + \mu_{1,\varepsilon}(T_n) + \mu_{2,\varepsilon}(T_n)) \| \hat{w} \| X_{(T_n,T_{n+1})} . \]
where we denote \( k_2 := (1/\zeta_{\min} + 1/\zeta_{\min}^2), \hat{\rho} := \rho_2(w_2) - \rho_1(w_1), \) and \( \hat{w} := w_2 - w_1. \) Thanks to the induction hypothesis (6.5), these bounds can be estimated as

\[
\|(1 + a)^2 \rho(w)\|_{L^\infty((T_n, T_{n+1}); L^1(\mathbb{R}))} \leq k_2'(1 + \mu_1 \varepsilon(0) + \mu_2 \varepsilon(0)) =: k_2',
\]

where \( k_2' := k_1^2 \) and

\[
\|(1 + a)\hat{\rho}\|_{L^\infty((T_n, T_{n+1}); L^1(\mathbb{R}))} \leq k_2'(1 + \mu_1 \varepsilon(0) + \mu_2 \varepsilon(0)) \|\hat{w}\|_{X(T_n, T_{n+1})} =: k_2'^\prime \|\hat{w}\|_{X(T_n, T_{n+1})},
\]

where \( k_2' := k_2 k_2'^\prime. \) The norm of the initial condition for \( u(w) \) might indeed be larger then \( \|u_{\varepsilon, \cdot}\|_{L^\infty(\mathbb{R}_+, \omega)} \), at worst we may have attained the bound \( C_n \) during the previous periods and we might have to choose \( C_{n+1} > C_n. \) Rewriting (6.3) in \([T_n, T_{n+1})\) and denoting \( \Delta T_n := T_{n+1} - T_n, \) one has indeed

\[
\|u\|_{X(T_n, T_{n+1})} \leq \frac{1}{\mu_{0, \min}} \left( \varepsilon \|\partial_t f\|_{L^\infty} + \zeta_{\max} \left( \int_{\mathbb{R}_+} (1 + a)\rho_t(t, a) da \right) \right) \|w\|_{X(T_n, T_{n+1})} \frac{\Delta T_n}{\varepsilon} + C_n \leq (c_1 + c_2 C_{n+1}) \frac{\Delta T_n}{\varepsilon} + C_n,
\]

where we used (6.6). A similar computation gives for the contraction part that:

\[
\|\hat{u}\|_{X(T_n, T_{n+1})} \leq \left( \sum_{i=1}^5 \delta_i \right) \frac{\Delta T_n}{\varepsilon},
\]

where

\[
J_1 := \frac{\varepsilon}{\mu_{0, \min}^2} \|\partial_t f\|_{L^\infty(T_n, T_{n+1})} \|\hat{\rho}_0\|_{L^\infty(T_n, T_{n+1})} \leq \frac{\varepsilon k_2'^\prime}{\mu_{0, \min}^2} \|\partial_t f\|_{L^\infty(T_n, T_{n+1})} \|\hat{w}\|_{X(T_n, T_{n+1})},
\]

\[
J_2 := \frac{\varepsilon}{\mu_{0, \min}^2} \sup_{(T_n, T_{n+1})} \left( \int_{\mathbb{R}_+} \zeta(w) |\rho_t| \rho_t da \right) \leq \frac{\varepsilon k_2'^\prime}{\mu_{0, \min}^2} \|w_2\|_{X(T_n, T_{n+1})} \|\hat{w}\|_{X(T_n, T_{n+1})},
\]

\[
J_3 := \frac{\varepsilon}{\mu_{0, \min}^2} \sup_{(T_n, T_{n+1})} \left( \int_{\mathbb{R}_+} \zeta_1 \hat{w} \rho_t da \right) \leq \frac{\varepsilon k_2'^\prime}{\mu_{0, \min}^2} \|w_1\|_{X(T_n, T_{n+1})} \|\hat{w}\|_{X(T_n, T_{n+1})},
\]

\[
J_4 := \frac{\varepsilon}{\mu_{0, \min}^2} \sup_{(T_n, T_{n+1})} \left( \int_{\mathbb{R}_+} \zeta_1 \hat{w} \rho_t da \right) \leq \frac{\varepsilon \mu_{0, \min}^2}{\mu_{0, \min}^2} \|\hat{w}\|_{X(T_n, T_{n+1})},
\]

\[
J_5 := \frac{\varepsilon}{\mu_{0, \min}^2} \sup_{(T_n, T_{n+1})} \left( \int_{\mathbb{R}_+} \zeta_1 \hat{w} \rho_t da \right) \leq \frac{\varepsilon \mu_{0, \min}^2}{\mu_{0, \min}^2} \|\hat{w}\|_{X(T_n, T_{n+1})},
\]

leading to

\[
\|\hat{u}\|_{X(T_n, T_{n+1})} \leq \left( (c_3 + c_4 C_{n+1}) \frac{\Delta T_n}{\varepsilon} \right) \|\hat{w}\|_{X(T_n, T_{n+1})}.
\]
Existence and uniqueness of the extended solution on the interval $[0, T_{n+1}] = [0, T_n] \cup (T_n, T_{n+1}]$ hold provided that the period $\Delta T_n$ is chosen sufficiently small, i.e.

$$\Delta T_n < \min \left\{ \frac{\varepsilon \Delta C_n}{c_1 + c_2' C_{n+1}}, \frac{\varepsilon}{c_3' + c_4' C_{n+1}} \right\}.$$  

where $\Delta C_n = C_{n+1} - C_n$. The fixed point theorem provides a pair $(\rho_\varepsilon, u_\varepsilon)$ defined on $C([0, T_{n+1}]; L^1(\mathbb{R}^+, (1 + a)^2)) \times X_{T_{n+1}}$, then Lemmas 2.2 and 2.3 establish that $\mu_{0,\varepsilon}(T_{n+1}) \leq 1$ and $\mu_{1,\varepsilon}(T_{n+1}) \leq \mu_{1,\varepsilon}(0) + \varepsilon^{-1}$. For $\mu_{2,\varepsilon}$, the second order moment, a similar estimate holds as well. This proves (6.5) up to $T_{n+1}$. The induction step is complete.

Thus, in an iterative way we are able to extend the solution up to periods $[T_n, T_{n+1})$ for any $n > 0$. We choose $C_n := 2\|u_{I,\varepsilon}\|_{L^{\infty}(\mathbb{R}^+)}(n + 1)$. Since both series, $\frac{\varepsilon \Delta C_n}{c_1 + c_2' C_n}$ and $\frac{\varepsilon}{c_3' + c_4' C_n}$, are scaled versions of the divergent series $\sum_{n=0}^{\infty} \frac{1}{1+n}$, the periods $\Delta T_n$ can be chosen such that $T_n \to \infty$ as $n$ grows large, which finishes the proof. □

REFERENCES