Koszul algebras and MacMahon's Master Theorem

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Overview

- **MacMahon's "Master Theorem"**: statement, some background, references, . . .

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- **Koszul algebras**: a quick introduction

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- **Koszul algebras**: a quick introduction

- Application: a new proof of the **quantum Master Theorem**

Objective

I will explain a **new proof** of the

"quantum MacMahon Master Theorem" (qMMT)

due to Garoufalidis, Lê and Zeilberger

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The manuscript, joint with **Phùng Hô Hai** (University of Duisburg-Essen), has been submitted to the LMS

preprint arXiv: math.QA/0603169

Objective

Doron Zeilberger

Rutgers University

Objective

The precise formulation of qMMT will be given later

– according to [GLZ], qMMT is *"a key ingredient in a finite non-commutative formula for the colored Jones polynomial of a knot"* –

but here is the original MMT . . .

Objective

MacMahon's Master Theorem (original version, 1917):

Given a matrix $A = (a_{ij})_{n\times n}$ *over some commutative ring* R *and commuting indeterminates* $x_1, \ldots, x_n$ *over* R*. For each* $(m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$*, the* R*-coefficient of* $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ *in*

$$\left(\sum_{j=1}^{n} a_{1j}x_j\right)^{m_1} \left(\sum_{j=1}^{n} a_{2j}x_j\right)^{m_2} \cdots \left(\sum_{j=1}^{n} a_{nj}x_j\right)^{m_n}$$

is identical to the corresponding coefficient in

$$\det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right)^{-1}$$

Background

Percy Alexander MacMahon
1854 - 1929

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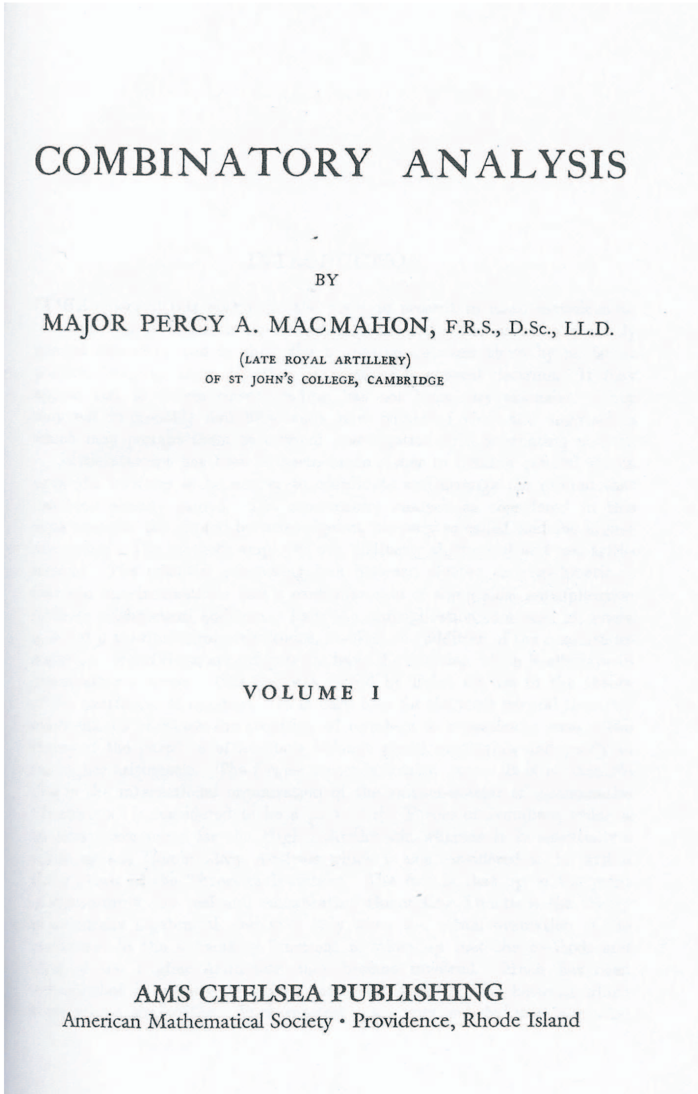
1854 - 1929

- 1882 - 1888: Instructor in Mathematics at the Royal Military Academy
- 1891 - 1898: Instructor/Professor of Physics at the Artillery College
- 1894 - 1896: President of the London Math Society

Background


COMBINATORY ANALYSIS

BY

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VOLUME I

AMS CHELSEA PUBLISHING
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Title page of MacMahon's book containing the "Master Theorem"

(originally published at Cambridge, 1917)

Background

... and here is where the name comes from:

98 APPLICATIONS OF THE THEOREM [SECT. III, CH. II

wherein the denominator is in symbolic form in such wise that on multiplication the factors a_1b_2, $a_1b_2c_3$, ... are to be placed in determinant brackets $|a_1b_2|$, $|a_1b_2c_3|$, ... and denote the co-axial minors of the determinant

$$|a_1b_2 \ldots n_n|,$$

which appertains to the matricular relation."

This is a master theorem in the Theory of Permutations.

We will write

$$V_n$$

for the expression

$$|(1-a_1x_1)(1-b_2x_2)\ldots(1-n_nx_n)|,$$

and the reader will be able to verify that V_n has also the expression

$$(-)^n x_1x_2\ldots x_n \begin{vmatrix} a_1-1/x_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ b_1 & b_2-1/x_2 & \cdot & \cdot & \cdot & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ n_1 & n_2 & \cdot & \cdot & \cdot & n_n-1/x_n \end{vmatrix}$$

Applications of the Theorem.

Background

Some later proofs:

- I. J. Good, *A short proof of MacMahon's 'Master Theorem'*, Math. Proc. Cambridge Philos. Soc. **58** (1960), 160

  $\rightsquigarrow$ **analysis**: contour integration

- Chu Wenchang, *Determinant, permanent, and MacMahon's Master Theorem*, Lin. Alg. and Appl. **255** (1997), 171–183

  $\rightsquigarrow$ **combinatorics**: cycle-generating functions, binomial identities

- I-C. Huang, *Applications of residues to combinatorial identies*, Proc. Amer. Math. Soc. **125** (1997), 1011–1017

  $\rightsquigarrow$ **algebra**: Grothendieck duality

Background

Andrews' Problem:
(from George E. Andrews, *Problems and prospects for basic hypergeometric functions*, In: Theory and application of special functions, Academic Press, New York, 1975, pp. 191–224.)

5. MacMahon's Master Theorem and the Dyson Conjecture.

PROBLEM 5. Are there q-analogs of MacMahon's Master Theorem and the Dyson Conjecture?

First let us recall:

MacMahon's Master Theorem (MacMahon (1894), (1915)). The coefficient of $X_1^{P_1} X_2^{P_2} \ldots X_n^{P_n}$ in

. . . and some motivation:

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

This theorem has remarkable consequences in the theory of permutations and has been used to provide solutions to generalized Rencontre problems (MacMahon (1915)), the Menage problem (Percus (1971)), and numerous other results. Section 3 of MacMahon's book (MacMahon (1915)) as well as pages 18-31 of J. K. Percus (1971) provide ample evidence of this assertion.

I. J. Good (1962) utilized the M

Background

GLZ proved their qMMT in response to this problem.

The GLZ-quantization is **not** the first non-commutative version of MMT – Foata proved one as early as 1965 –

GLZ claim their quantization to be **natural**

Background

This talk is to support this claim

Reformulation of MMT

Recall: $A = (a_{ij}) \in \mathrm{Mat}_{n\times n}(R)$

$x_1, \ldots, x_n$ commuting indeterminates over R

For each $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geq 0}$, let the R-coefficient of $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ in $\prod_{i=1}^n \left(\sum_{j=1}^n a_{ij} x_j\right)^{m_i} \in R[x_1, \ldots, x_n]$ be denoted by $c_A(\mathbf{m})$

MMT: $1 = \det\left(1_{n\times n} - A \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right) \cdot \sum_{\mathbf{m}} c_A(\mathbf{m}) \mathbf{x}^{\mathbf{m}}$

This is an identity in $R[\![x_1, \ldots, x_n]\!]$

Reformulation of MMT

MMT: $$1 = \det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right) \cdot \sum_{\mathbf{m}} c_A(\mathbf{m})\mathbf{x}^{\mathbf{m}}$$

$$\Downarrow \qquad \text{all } x_i \mapsto t$$

MMT': $$1 = \det\left(1_{n\times n} - At\right) \cdot \sum_{d=0}^{\infty}\left(\sum_{|\mathbf{m}|=d} c_A(\mathbf{m})\right) t^d$$

$= \sum_i m_i$

an identity in $R[\![t]\!]$

Reformulation of MMT

MMT: $$1 = \det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right) \cdot \sum_{\mathbf{m}} c_A(\mathbf{m})\mathbf{x}^{\mathbf{m}}$$

$\Uparrow$ choose A "generic"

MMT': $$1 = \det\left(1_{n\times n} - At\right) \cdot \sum_{d=0}^{\infty}\left(\sum_{|\mathbf{m}|=d} c_A(\mathbf{m})\right) t^d$$

Reformulation of MMT

MMT: $$1 = \det\left(1_{n\times n} - A\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right)\cdot \sum_{\mathbf{m}} c_A(\mathbf{m})\mathbf{x}^{\mathbf{m}}$$

$$\Updownarrow$$

MMT': $$1 = \det(1_{n\times n} - At)\cdot \sum_{d=0}^{\infty}\left(\sum_{|\mathbf{m}|=d} c_A(\mathbf{m})\right) t^d$$

$= \sum_{d=0}^{n} \mathrm{trace}(\bigwedge^d A)(-t)^d$

$= \mathrm{trace}(\mathsf{S}^d A)$

Reformulation of MMT

To summarize, we have the following modern interpretation of MMT:

MMT

$$\Updownarrow$$

$$\boxed{1 = \left(\sum_{d=0}^{n} \operatorname{trace}(\textstyle\bigwedge^d A)(-t)^d\right) \cdot \left(\sum_{d=0}^{\infty} \operatorname{trace}(\mathsf{S}^d A)t^d\right)}$$

All this is a well-known

Part II

Next: **Koszul algebras**

my main reference: **Yu. I. Manin**, *Quantum groups and noncommutative geometry*, Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988.

Quadratic algebras

Def: A **quadratic algebra** is a factor of the tensor algebra $\mathsf{T}(V)$ of some finite-dimensional $\Bbbk$-vector space V modulo quadratic relations:

$$A \cong \mathsf{T}(V)/\left(R(A)\right), \quad R(A) \subseteq \mathsf{T}(V)_2 = V^{\otimes 2}$$

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The natural grading of $\mathsf{T}(V)$ descends to a **grading** of A:

$$A = \bigoplus_{d \geq 0} A_d \text{ with } A_0 = \Bbbk,\ A_1 \cong V$$

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In short: $A \leftrightarrow \{A_1, R(A)\}$

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Notation: $\tilde{x}_1, \ldots, \tilde{x}_n$ will be a $\Bbbk$-basis of V
$\rightsquigarrow \quad \mathsf{T}(V) = \Bbbk\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle$, the free algebra
$x_i := \tilde{x}_i \mod R(A)$, algebra generators for A

Quadratic algebras

Example: Quantum affine n-space

For a fixed $0 \neq q \in \mathbb{k}$, define

$$A_q^{n|0} := \mathbb{k}\langle \tilde{x}_1, \ldots, \tilde{x}_n \rangle / \left(\tilde{x}_j \tilde{x}_i - q \tilde{x}_i \tilde{x}_j \mid 1 \leq i < j \leq n \right)$$

Thus, $A_q^{n|0}$ is generated $x_1, \ldots, x_n$ subject to the relations

$$x_j x_i = q x_i x_j \quad \text{for } i < j.$$

Quadratic dual

Def: The **quadratic dual** of $A \leftrightarrow \{V, R(A)\}$ is defined by

$$A^! \leftrightarrow \{V^*, R(A)^\perp\}$$

with $R(A)^\perp = \{f \in \left(V^{\otimes 2}\right)^* \mid f(R(A)) = 0\}$ and $\left(V^{\otimes 2}\right)^* \cong (V^*)^{\otimes 2}$ as usual

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Notation: $\tilde{x}_1, \ldots, \tilde{x}_n$ is a $\Bbbk$-basis of V, as before
Denote the dual basis of V^* by $\tilde{x}^1, \ldots, \tilde{x}^n$
$\rightsquigarrow$ generators $x^i = \tilde{x}^i \mod R(A)^\perp$ for $A^!$

Quadratic dual

Example: Quantum exterior algebra

The quadratic dual of quantum plane $A_q^{n|0}$ is denoted by $A_q^{0|n}$

The procedure described yields algebra generators $x^1, \ldots, x^n$ for $A_q^{0|n}$ satisfying the defining relations

$$x^\ell x^\ell = 0 \quad \text{for all } \ell$$

and

$$x^i x^j + q x^j x^i = 0 \quad \text{for } i < j$$

The category QA

objects: quadratic algebras$/\mathbb{k}$
morphisms: graded algebra maps

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morphisms: graded algebra maps

Some further **operations on** QA:

- ordinary tensor product $A \otimes B$
- Segre product $A \circ B = \bigoplus_n A_n \otimes B_n$
- $A \bullet B \leftrightarrow \{A_1 \otimes B_1, S_{23}(R(A) \otimes R(B))\}$
  where $S_{23} \colon A_1^{\otimes 2} \otimes B_1^{\otimes 2} \to (A_1 \otimes B_1)^{\otimes 2}$ switches the 2^{nd} and 3^{rd} factors.

The category QA

objects: quadratic algebras$/\mathbb{k}$
morphisms: graded algebra maps

$\mathsf{QA}^{\mathrm{op}}$ is the category of "**quantum linear spaces**" over $\mathbb{k}$

Analogies:

- $\bullet \overset{!}{\longleftrightarrow} \circ$ tensor product of quantum spaces
- $\otimes$ direct sum of quantum spaces
- $!$ dualization plus parity change

The bialgebra $\underline{\mathrm{end}}\, A$

Def: For a given quadratic algebra A, one defines

$$\boxed{\underline{\mathrm{end}}\, A = A^! \bullet A}$$

So $\underline{\mathrm{end}}\, A \leftrightarrow \{V^* \otimes V, S_{23}(R(A)^{\perp} \otimes R(A))\}$

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So $\underline{\mathrm{end}}\, A \leftrightarrow \{V^* \otimes V, S_{23}(R(A)^\perp \otimes R(A))\}$

Notation: $\tilde{x}_i, \tilde{x}^j$ dual bases for V and V^* as before
$\rightsquigarrow \tilde{z}_i^j = \tilde{x}^j \otimes \tilde{x}_i$ a basis of $V^* \otimes V$
$\rightsquigarrow z_i^j = \tilde{z}_i^j \mod R(\underline{\mathrm{end}}\, A)$ generate $\underline{\mathrm{end}}\, A$

The bialgebra $\underline{\text{end}}\, A$

Properties:

- $\underline{\text{end}}\, A$ is a **bialgebra** over $\Bbbk$, with comultiplication

$$\Delta\colon \underline{\text{end}}\, A \to \underline{\text{end}}\, A \otimes \underline{\text{end}}\, A\ , \qquad \Delta(z_i^j) = \sum_\ell z_i^\ell \otimes z_\ell^j$$

  and counit

$$\epsilon\colon \underline{\text{end}}\, A \to \Bbbk\ , \qquad \epsilon(z_i^j) = \delta_{i,j}$$

- A is a left $\underline{\text{end}}\, A$-**comodule algebra**; the coaction is

$$\delta_A\colon A \to \underline{\text{end}}\, A \otimes A\ , \qquad \delta_A(x_i) = \sum_j z_i^j \otimes x_j\ ,$$

Example: Right quantum matrices

This is the algebra $\underline{\text{end}}\, A_q^{n|0}$. Defining relations:

column relations: $z_j^\ell z_i^\ell = q\, z_i^\ell z_j^\ell$ (all $\ell, i < j$)

cross relations: $z_i^k z_j^\ell - z_j^\ell z_i^k = q^{-1}\, z_j^k z_i^\ell - q\, z_i^\ell z_j^k$ $(i < j, k < \ell)$

Note the "missing" row relations. The algebra $\underline{\text{end}}\, A_q^{n|0}$ is **non-commutative** even for $q = 1$!

The bialgebra $\underline{\mathrm{end}}\, A$

The relations for the generators z_i^j of $\underline{\mathrm{end}}\, A_q^{n|0}$ are exactly those used by GLZ to define "right quantum matrices".

(GLZ do **not** arrive at these relations via Manin's construction)

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(GLZ do **not** arrive at these relations via Manin's construction)

$\rightsquigarrow$ "**generic right quantum matrix**" $Z = (z_i^j)_{n\times n}$

Any algebra map $\varphi\colon \underline{\mathrm{end}}\, A_q^{n|0} \to R$ ("R-point" of the space defined by $\underline{\mathrm{end}}\, A_q^{n|0}$) yields a right quantum matrix φZ over R

Koszul complexes

For any quadratic algebra A, one has Koszul complexes

$$\mathsf{K}^{\ell,\bullet}(A) : \; 0 \longrightarrow A^{!\,*}_{\ell} \longrightarrow A^{!\,*}_{\ell-1} \otimes A_1 \longrightarrow \cdots \longrightarrow A^{!\,*}_{1} \otimes A_{\ell-1} \longrightarrow A_\ell \longrightarrow 0$$

for all $\ell \geq 0$; for details see [Manin].

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Example: For $A = \mathsf{S}(V) = A^{n|0}_{q=1}$, these are the familiar Koszul complexes

$$\ldots \longrightarrow \wedge^{\ell-i+1}(V) \otimes \mathsf{S}^{i-1}(V) \longrightarrow \wedge^{\ell-i}(V) \otimes \mathsf{S}^{i}(V) \longrightarrow \ldots$$

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for all $\ell \geq 0$; for details see [Manin].

Lemma 1 (PHH & L) *All* $\mathsf{K}^{\ell,\bullet}(A)$ *are complexes of* $\underline{\mathrm{end}}\,A$*-comodules.*

Koszul complexes

For any quadratic algebra A, one has Koszul complexes

$$\mathsf{K}^{\ell,\bullet}(A): \; 0 \longrightarrow A^{!\,*}_{\ell} \longrightarrow A^{!\,*}_{\ell-1} \otimes A_1 \longrightarrow \cdots \longrightarrow A^{!\,*}_{1} \otimes A_{\ell-1} \longrightarrow A_\ell \longrightarrow 0$$

for all $\ell \geq 0$; for details see [Manin].

Def: The quadratic algebra A is said to be **Koszul** iff the complexes $\mathsf{K}^{\ell,\bullet}(A)$ are exact for $\ell > 0$.

Some Koszul facts

- Koszul algebras were introduced by Stewart Priddy in connection with his investigation of Yoneda algebras $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ (Trans AMS, 1970)

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  a graded algebra A is Koszul iff the minimal graded A-resolution of $\Bbbk$ is linear.

- The class of Koszul algebras is quite robust: it is stable under the operations $!$, $\otimes$, $\circ$, $\bullet$, $\underline{\mathrm{end}}$, ...

- A sufficient condition for A to be Koszul is the existence of a **PBW-basis**.

Some Koszul facts

Example: $A_q^{n|0}$ and right quantum matrices

Recall that $A_q^{n|0}$ is generated $x_1, \ldots, x_n$ subject to the relations

$$x_j x_i = q x_i x_j \quad \text{for } i < j.$$

Therefore, A has a $\mathbb{k}$-basis consisting of the ordered monomials $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$; this is a PBW-basis.

$\Longrightarrow$ $A_q^{n|0}$ is Koszul, and hence $\underline{\mathrm{end}}\, A_q^{n|0}$ as well

Characters

Notation: B some bialgebra over $\mathbb{k}$ (later: $B = \underline{\mathrm{end}}\, A$)

$\mathcal{R}_B$ Grothendieck ring of all left B-comodules that are finite-dimensional$/\mathbb{k}$ (or f.g. projective)

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In more detail:

- B-comodule V $\rightsquigarrow$ $[V] \in \mathcal{R}_B$
- $0 \to U \to V \to W \to 0$ exact $\rightsquigarrow$ $[V] = [U] + [W]$ in $\mathcal{R}_B$
- Multiplication in $\mathcal{R}_B$ is given by the tensor product of B-comodules

Characters

Def: Let V be a B-comodule; so have $\delta_V : V \to B \otimes V$
Consider the map

$$\mathrm{Hom}_{\Bbbk}(V, B \otimes V) \cong B \otimes V \otimes V^* \xrightarrow{\mathrm{Id}_B \otimes \langle\, .\, ,\, .\, \rangle} B \otimes \Bbbk \cong B$$

evaluation $V \otimes V^* \to \Bbbk$

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The image of δ_V under this map will be denoted by χ_V and called the **character** of V.

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Explicitly: If $\delta_V(v_j) = \sum_i b_{i,j} \otimes v_i$ for some $\Bbbk$-basis $\{v_i\}$ of V then

$$\chi_V = \sum_i b_{i,i}$$

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Lemma 2 *The map $[V] \mapsto \chi_V$ yields a well-defined ring homomorphism $\chi\colon \mathcal{R}_B \to B$.*

MMT for Koszul algebras

Recall the modern interpretation of the original MMT:

$$1 = \left(\sum_{d=0}^{n} \operatorname{trace}(\bigwedge^{d} A)(-t)^{d} \right) \cdot \left(\sum_{d=0}^{\infty} \operatorname{trace}(\mathsf{S}^{d} A) t^{d} \right)$$

for any $n \times n$-matrix A over some commutative ring

MMT for Koszul algebras

Here is the version for Koszul algebras:

Theorem 1 (PHH & L) *Let A be a Koszul algebra and $B = \underline{\mathrm{end}}\, A$. Then the following identity holds in $B[\![t]\!]$:*

$$1 = \left(\sum_{m \geq 0} \chi_{A^{!*}_m} (-t)^m \right) \cdot \left(\sum_{\ell \geq 0} \chi_{A_\ell} t^\ell \right)$$

MMT for Koszul algebras

Proof: Recall the exact Koszul complex

$$\mathsf{K}^{\ell,\bullet}(A):\; 0 \longrightarrow A_\ell^{!\,*} \longrightarrow A_{\ell-1}^{!\,*} \otimes A_1 \longrightarrow \cdots \longrightarrow A_1^{!\,*} \otimes A_{\ell-1} \longrightarrow A_\ell \longrightarrow 0$$

By Lemma 1, this gives equations in $\mathcal{R}_B$:

$$\sum_i (-1)^i [A_i^{!\,*}][A_{\ell-i}] = 0 \qquad (\ell > 0)$$

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By Lemma 1, this gives equations in $\mathcal{R}_B$:

$$\sum_i (-1)^i [A_i^{!\,*}][A_{\ell-i}] = 0 \qquad (\ell > 0)$$

Defining $P_A(t) = \sum_i [A_i] t^i$, $P_{A^{!*}}(t) = \sum_i [A_i^{!\,*}] t^i \in \mathcal{R}_B[\![t]\!]$, this becomes

$$1 = P_{A^{!*}}(-t) \cdot P_A(t)$$

MMT for Koszul algebras

Proof: Recall the exact Koszul complex

$$\mathsf{K}^{\ell,\bullet}(A) \colon\ 0 \to A^{!}_{\ell}{}^{*} \to A^{!}_{\ell-1}{}^{*} \otimes A_1 \to \cdots \to A^{!}_{1}{}^{*} \otimes A_{\ell-1} \to A_\ell \to 0$$

By Lemma 1, this gives equations in $\mathcal{R}_B$:

$$\sum_i (-1)^i [A^{!}_{i}{}^{*}][A_{\ell-i}] = 0 \qquad (\ell > 0)$$

Defining $P_A(t) = \sum_i [A_i]t^i$, $P_{A^{!*}}(t) = \sum_i [A^{!}_{i}{}^{*}]t^i \in \mathcal{R}_B[\![t]\!]$, this becomes

$$1 = P_{A^{!*}}(-t) \cdot P_A(t)$$

Now apply the ring homomorphism $\chi[\![t]\!] \colon \mathcal{R}_B[\![t]\!] \to B[\![t]\!]$. QED

Garoufalidis, Lê and Zeilberger's qMMT is **exactly** the special case of the Theorem where $A = A_q^{n|0}$

Spelled out in detail . . .

qMMT

Notation: $A = A_q^{n|0} = \Bbbk[x_i \mid i = 1, \ldots, n]$
$B = \underline{\mathrm{end}}\, A_q^{n|0} = \Bbbk[z_i^j \mid i, j = 1, \ldots, n]$
$Z = (z_i^j)_{n \times n}$ are as before

For each $J \subseteq \{1, \ldots, n\}$, I will write $Z_J = (z_i^j)_{i,j \in J}$.

Finally,

$$\det_q(Z_J) = \sum_{\pi \in \mathfrak{S}_m} (-q)^{-l(\pi)} z_{j_{\pi 1}}^{j_1} z_{j_{\pi 2}}^{j_2} \ldots z_{j_{\pi m}}^{j_m}$$

is the **quantum determinant** as defined by [FRT].

qMMT

Theorem 2 (qMMT of GLZ) In $B \otimes A = \bigoplus_{\mathbf{m} \in \mathbb{Z}^n_{\geq 0}} B \otimes \mathbf{x}^{\mathbf{m}}$ put $X_i = \sum_j z_i^j \otimes x_j$ and define $G(\mathbf{m})$ to be the B-coefficient of $\mathbf{x}^{\mathbf{m}}$ in $X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n}$. In $B[\![t]\!]$ put

$$\text{Bos}(Z) := \sum_{\ell \geq 0} \sum_{|\mathbf{m}| = \ell} G(\mathbf{m}) t^{\ell}$$

$$\text{Ferm}(Z) := \sum_{m \geq 0} \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ |J| = m}} \det_q(Z_J)(-t)^m$$

Then:

$$\boxed{\text{Bos}(Z) \cdot \text{Ferm}(Z) = 1}$$

Sketch of proof: In view of Theorem 1 we must show

- $$\chi_{A_\ell} = \sum_{|\mathbf{m}|=\ell} G(\mathbf{m})$$

- $$\chi_{A^{!*}_m} = \sum_{\substack{J\subseteq\{1,\ldots,n\} \\ |J|=m}} \det_q(Z_J)$$

qMMT

For $\chi_{A_\ell} = \sum_{|\mathbf{m}|=\ell} G(\mathbf{m})$, recall:

- the homogeneous component A_ℓ has a $\mathbb{k}$-basis consisting of the ordered monomials $\mathbf{x}^{\mathbf{m}}$ with $|\mathbf{m}| = \ell$;
- the coaction δ_A of $B = \underline{\mathrm{end}}\, A$ on A is given by $\delta_A(x_i) = \sum_j z_i^j \otimes x_j = X_i$.

Therefore,

$$\delta_A(x^{\mathbf{m}}) = X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} = \sum_{|\mathbf{r}|=\ell} b_{\mathbf{r},\mathbf{m}} \otimes \mathbf{x}^{\mathbf{r}}$$

for uniquely determined $b_{\mathbf{r},\mathbf{m}} \in B$. In particular, $b_{\mathbf{m},\mathbf{m}} = G(\mathbf{m})$. The desired equality follows.

qMMT

The proof of $\chi_{A^{!*}_m} = \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ |J| = m}} \det_q(Z_J)$ proceeds similarly using the $\mathbb{k}$-basis of $A^{!*}_m$ consisting of the elements

$$\wedge \tilde{x}_J := \sum_{\pi \in \mathfrak{S}_m} (-q)^{-l(\pi)} \tilde{x}_{j_{\pi 1}} \otimes \tilde{x}_{j_{\pi 2}} \otimes \ldots \otimes \tilde{x}_{j_{\pi m}} \ ,$$

where $J = (j_1 < j_2 < \ldots < j_m)$ is an m-element subset of $\{1, \ldots, n\}$; for details, see the preprint by PHH&L.

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- Independent work by **Foata & Han** gives an alternative new proof of **qMMT** using combinatorics on words. They also analyze the algebra of right quantum matrices in detail and give various modifications of **qMMT**

  (3 preprints, December 2005, available on arXiv)